A philosophy of modelling and computing homotopy types

Ronnie Brown

June 17, 2015

CT2015, Aveiro

In homotopy theory, identifications in low dimensions have influence on high dimensional homotopical invariants.
The aim is to model this by using universal properties of algebraic objects with
strict interacting operations in a range of dimensions $0, \ldots, n$.
Roots in work 1941-1950 of Henry Whitehead.
Origin: 1965 with groupoids, and then with
Chris Spencer (1971-76), Philip Higgins (1974-2005),
crossed modules, crossed complexes, cubical higher groupoids,
Jean-Louis Loday (1981-1987)
cat^n-groups, crossed squares,
and many others, e.g. Graham Ellis, Richard Steiner, Andy Tonks.

Just as homotopy groups are defined only for
spaces with one base point,
these functors with more general values are defined only on
spaces with more general structural data.

We consider functors

$$\begin{pmatrix}\text{Topological}\\ \text{Data}\end{pmatrix} \underset{\mathbb{B}}{\overset{\mathbb{H}}{\rightleftarrows}} \begin{pmatrix}\text{Algebraic}\\ \text{Data}\end{pmatrix}$$

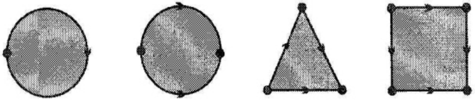
such that

1) $\mathbb{H}$ is homotopically defined.
2) $\mathbb{H}\mathbb{B}$ is equivalent to 1.
3) The Topological Data has a notion of connected.
4) For all Algebraic Data A, $\mathbb{B}A$ is connected.
5) "Nice" colimits of connected Topological Data are :
   (a) connected, and
   (b) preserved by $\mathbb{H}$.

The aim is precise algebraic colimit calculations of some homotopy types.

Broad and Narrow Algebraic Models

The modellizing is more complicated, since the Algebraic Data, and so the functors $\mathbb{H}, \mathbb{B}$, diversify in dimensions > 1, with various geometric models:



disk globe simplex cube

We have

Broad Algebraic Data for intuition, conjectures, proving theorems.

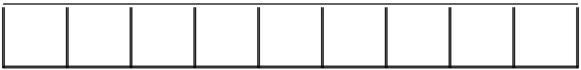
Narrow Algebraic Data for calculation, relation with classical invariants.

The algebraic equivalence between these, of Dold-Kan type, is then a key for results. The more complicated the proof the more useful it can be, once done.

I need to give an example of this distinction!
Consider crossed modules, and the functor Π_2 sending a based pair (X, A, a) to the crossed module

$$\pi_2(X, A, a) \to \pi_1(A, a).$$

Conjecturing that this functor satisfies a van Kampen type theorem could be, and was, regarded as ridiculous.
Note that in second relative homotopy group, all compositions are on a line, as in



in order to obtain a group.

Enter double groupoids

However there is another construction which is more symmetric, shown as a picture in dimension 2 as

$$
\begin{tikzcd}[row sep=small, column sep=small]
\cdot \arrow[r] \arrow[d] & 2 \\
1 &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
a \arrow[r, dash] \arrow[d, dash] & A \arrow[r, dash] & a \arrow[d, dash] \\
A \arrow[d, dash] & X & A \arrow[d, dash] \\
a \arrow[r, dash] & A \arrow[r, dash] & a
\end{tikzcd}
$$

With this model, one can see how to jack up the usual proof to dimension 2!
So crossed modules form a narrow model, and double groupoids with connections form a broad model.

Method

Two pushouts:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & G \end{tikzcd}$$

Algebraic Data

$$\begin{tikzcd} \mathbb{B}C \arrow[r] \arrow[d] & \mathbb{B}A \arrow[d] \\ \mathbb{B}B \arrow[r] & X \end{tikzcd}$$

Topological Data

By Properties 2), 4) and 5)

$\mathbb{H}X \cong G.$ Bingo!

Paradigmatic Example:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mathbb{I} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Groupoids

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ ([0,1],\{0,1\}) \arrow[r] & (S^1,\{0\}) \end{tikzcd}$$

So

$$\pi_1(S^1,0) \cong \mathbb{Z}$$

Dimension 1 Example:

- TopData = Pairs (X, C) of a space X with a set $C \cap X$ of base points.
- (X, C) is connected if C meets each path component of X.
- Alg Data = Groupoids
- $\mathbb{H}(X, C) = \pi_1(X, C)$, fundamental groupoid on $X \cap C$.
- $\mathbb{B}(G) = (BG, Ob(G))$.

Groupoid Seifert-van Kampen Theorem (RB, 1967):
If $X = U \cup V, W = U \cap V$, U, V are open, C meets each path component of U, V, W, then
(Con) (X, C) is connected, and
(Iso)

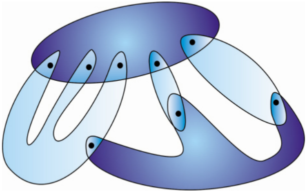
$$\begin{array}{ccc} \pi_1(W, C) & \to & \pi_1(V, C) \\ \downarrow & & \downarrow \\ \pi_1(U, C) & \to & \pi_1(X, C) \end{array}$$

is a pushout of groupoids.

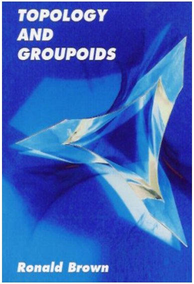
We are handling and computing the whole 1-type.

$$(X, C) = (\text{union}) \xrightarrow{\text{SvKT}} \pi_1(X, C) \xrightarrow{\text{combinatorics}} \pi_1(X, c).$$

Strange. One can completely determine $\pi_1(X, C)$ and so any $\pi_1(X, c)$! A new anomaly!



Try doing that with covering spaces!



Revised, extended, retitled 2006 edition of book published in 1968, 1988.

One (French) take-up of $\pi_1(X, C)$ in other topology texts..

Example, a key method in groupoids (Philip Higgins,1964):
$G =$ groupoid with object set C;
$f : C \to D$ is a function.
Form a pushout

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ G \arrow[r] & f_*(G) \end{tikzcd}$$

Let $X = BG$:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ \pi_1(X, C) \arrow[r] & \pi_1(D \cup_f X, D) \end{tikzcd}$$

The use of this "change of base" groupoid construction includes free groups, free products.

Alexander Grothendieck(1983, letter to RB) . .... both the choice of a base point, and the 0-connectedness assumption, however innocuous they may seem at first sight, seem to me of a very essential nature. To make an analogy, it would be just impossible to work at ease with algebraic varieties, say, if sticking from the outset (as had been customary for a long time) to varieties which are supposed to be connected. Fixing one point, in this respect (which wouldn't have occurred in the context of algebraic geometry) looks still worse, as far as limiting elbow-freedom goes!

Move one step higher: Let $f : G \to H$ be a morphism of groups. We want to compute the 2-type of the mapping cone of $Bf : BG \to BH$. Move to crossed modules.

$$\begin{tikzcd}
(1 \to G) \arrow[r, "f"] \arrow[d] & (1 \to H) \arrow[d] \\
(1 : G \to G) \arrow[r] & (\mu : f_*(G) \to H)
\end{tikzcd}$$

Pushout
Induced crossed module

Another change of base!

$$\begin{tikzcd}
(BG, BG) \arrow[r, "f"] \arrow[d] & (BH, BH) \arrow[d] \\
(B(G \to G), BG) \arrow[r] & (X, Y)
\end{tikzcd}$$

Homotopy pushout

Now $B(1 : G \to G)$ is clearly contractible. So $X \simeq C(Bf)$. Also $Y = BH$. So we have computed:

$$(\pi_2(BH \cup_{Bf} C(BG), BH) \to H) \cong (f_*(G) \to H)$$

Related methods give a description of

$$\pi_2(X \cup_g CA, X, x) \to \pi_1(X, x)$$

as induced from the crossed module
$1 : \pi_1(A, a) \to \pi_1(A, a)$ by $\pi_1(g) : \pi_1(A.a) \to \pi_1(X, x)$;
1941-49 theorem of J.H.C. Whitehead on free crossed modules is the case A is a wedge of circles.

Now we would like to compute the 3-type of the mapping cone of a morphism of crossed modules. So we have to move to crossed squares!No time to say exactly what these are but they certainly involve

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd}$$

This is essentially a crossed module of crossed modules.
Generalise a kernel of a morphism of crossed modules.

So there are actions of P on M, N, L and of M, N on each other, and on L, via P.
There is also a map $h : M \times N \to L$ which is a biderivation, i.e. rules analogous to those for a commutator.

Standard topological example: triad of based spaces $(X : Y, Z)$ with $W = Y \cap Z$:

$$\begin{tikzcd} \pi_3(X; Y, Z) \arrow[r] \arrow[d] & \pi_2(Z, W) \arrow[d] \\ \pi_2(Y, W) \arrow[r] & \pi_1(W) \end{tikzcd}$$

$h : \pi_2(Y, W) \times \pi_2(Z, W) \to \pi_3(X; Y, Z)$ is here the Generalized Whitehead Product.

Suppose given a pushout of crossed squares:

$$\begin{array}{ccc}
\begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} & \to & \begin{pmatrix} 1 & N \\ 1 & P \end{pmatrix} \\
\downarrow & & \downarrow \\
\begin{pmatrix} 1 & 1 \\ M & P \end{pmatrix} & \to & \begin{pmatrix} L & N \\ M & P \end{pmatrix}
\end{array}$$

Then we write $L = M \otimes N$.In particular if M, N are normal subgroups of P, we get the commutator map $[\ ;\ ] : M \times N \to P$ factors through a morphism of groups $\kappa : M \otimes N \to P$. (Loday/RB, 1984)

Current bibliography on this nonabelian tensor product has 131 items.

Now suppose given a morphism $(M \to P) \to (R \to Q)$ of crossed modules.
The 3-type of the mapping cone of $B(M \to P) \to B(R \to Q)$ is given by the pushout crossed square in

$$\begin{array}{ccc} \begin{pmatrix} 1 & 1 \\ M & P \end{pmatrix} & \xrightarrow{(f,g)} & \begin{pmatrix} 1 & 1 \\ R & Q \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} M & P \\ M & P \end{pmatrix} & \to & \begin{pmatrix} L & g_*(P) \\ R & Q \end{pmatrix} \end{array}$$

because

$$B\begin{pmatrix} M & P \\ M & P \end{pmatrix}$$

is contractible.

Then L is of the form

$$[(R \otimes g_*(P)) \circ g_*(M)]/ \sim$$

where the relations $\sim$ can be written down in detail.

Conclusion

These methods allow explicit nonabelian colimit calculations in higher homotopy theory, spreading over a range of dimensions,
and in so doing, generate new algebraic constructions.
Some aspects are dealt with in the book: