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ORIGINAL PAPER

A Conceptual Construction of Complexity Levels Theory in Spacetime Categorical Ontology: Non-Abelian Algebraic Topology, Many-Valued Logics and Dynamic Systems

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Abstract A novel conceptual framework is introduced for the Complexity Levels Theory in a Categorical Ontology of Space and Time. This conceptual and formal construction is intended for ontological studies of Emergent Biosystems, Super-complex Dynamics, Evolution and Human Consciousness. A claim is defended concerning the universal representation of an item's essence in categorical terms. As an essential example, relational structures of living organisms are well represented by applying the important categorical concept of natural transformations to bio-molecular reactions and relational structures that emerge from the latter in living systems. Thus, several relational theories of living systems can be represented by natural transformations of organismic, relational structures. The ascent of man and other living organisms through adaptation, is viewed in novel categorical terms, such as variable biogroupoid representations of evolving species. Such precise but flexible evolutionary concepts will allow the further development of the unifying theme of local-to-global approaches to highly complex systems in order to represent novel patterns of relations that emerge in super- and ultra-complex systems in terms of compositions of local procedures. Solutions to such local-to-global problems in highly complex systems with 'broken symmetry' might be possible to be reached with the help of higher homotopy theorems in algebraic topology such as the

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generalized van Kampen theorems (HHvKT). Categories of many-valued, Łukasiewicz-Moisil (LM) logic algebras provide useful concepts for representing the intrinsic dynamic 'asymmetry' of genetic networks in organismic development and evolution, as well as to derive novel results for (non-commutative) Quantum Logics. Furthermore, as recently pointed out by Baianu and Poli (Theory and applications of ontology, vol 1. Springer, Berlin, in press), LM-logic algebras may also provide the appropriate framework for future developments of the ontological theory of levels with its complex/entangled/intertwined ramifications in psychology, sociology and ecology. As shown in the preceding two papers in this issue, a paradigm shift towards *non-commutative*, or non-Abelian, theories of highly complex dynamics—which is presently unfolding in physics, mathematics, life and cognitive sciences—may be implemented through realizations of higher dimensional algebras in neurosciences and psychology, as well as in human genomics, bioinformatics and interactomics.

Keywords Categorical ontology and the theory of levels ·
Formal foundation and relational structure of categorical ontology and emergent complexity theories · Ontological essence and Universal properties of items ·
Mathematical categories, Groupoids, Locally Lie groupoids, Groupoid Atlas,
Stacks, Fibred categories · Relational biology principles ·
Higher homotopy—General Van Kampen Theorems (HHvKT) and Non-Abelian algebraic topology (NAAT) ·
Non-commutativity of diagrams and non-Abelian theories—Non-Abelian categorical ontology ·
Non-commutative topological invariants of complex dynamic state spaces ·
Natural transformations in molecular and relational biology: Molecular class variables (mcv) ·
Natural transformations and the Yoneda-Grothendieck lemma/construction ·
Variable groupoids, Variable categories, Variable topology and atlas structures ·
Biomolecular classes and Metabolic repair systems

1 Introduction

In the preceding contributions of Baianu et al. (2007a, b in this issue) the mathematical concepts of category theory were employed to provide a precise foundation for a categorical ontology theory of levels and ultra-complexity. Here, we explore a refined essence of ontology via mathematical relations that in part complements the scope of these other two contributions in this issue, and at the same time, may lead the reader to precise conceptual sources for a formal ontological theory of complexity levels.

A novel conceptual framework is introduced here for the Categorical Ontology of Space and Time in Emergent Biosystems, Super-complex Dynamics, Evolution and Human Consciousness. The ascent of man and other living organisms through adaptation was viewed in novel categorical terms, such as variable biogroupoid representations of evolving species (Baianu et al. 2007a). These precise but flexible evolutionary concepts allow the development of the unifying theme of local-to-global approaches to highly complex systems. In order to represent the basic patterns of relations that emerge in super- and ultra- complex systems one may utilize novel compositions of local procedures. Solutions to important local-to-global problems in highly complex systems with 'broken symmetry' might be attainable with the help of higher homotopy theorems in algebraic topology such as the generalized van Kampen theorems (HHvKT) presented here in Sect. 4.

Ontology has acquired over time several meanings and has been also approached in several different ways, however mostly connected to the concept of an '*objective existence*'; we shall consider here the noun 'existence' as a basic, or primitive, concept not definable in more fundamental terms. The attribute 'objective' will be assumed with the same meaning as in '*objective reality*', and reality is understood as whatever has an existence which can be rationally or empirically verified independently by human observers in a manner which is neither arbitrary nor counter-factual. Furthermore, the *meaningful classification of items* that belong to such an 'objective reality' is one of the major tasks of ontology. Whereas the existential essence of all items is dynamic, Science and Mathematics search for the abstract, unchanging and simplifying essence that underlies 'all' objective reality that we have encountered.

This approach is however in harmony with the theme and approach of the ontological theory of levels of reality (Poli 1998, 2001a, b) by considering categorical models of complex systems in terms of an evolutionary dynamic viewpoint. Thus our main descriptive approach involves the mathematical techniques of category theory which afford describing the characteristics and binding of levels, besides the links with other theories. Whereas Hartmann (1952) stratified levels in terms of the four frameworks: physical, 'organic'/biological, mental and spiritual, we restrict mainly to the first three.

It is clear we have made extensive use of categories of several types, in particular, the simpler subclass of categories known as 'groupoids' which are instrumental in describing processes of reciprocity within various types of cell networks, graphs, and other configurative schemes (with intrinsic reciprocity) which may feature in an ontological theory of levels. A broader picture is to look at *non-commutative local-to-global theorems* and the various generalizations of the van Kampen Theorem dealing with the fundamental group(oid) in relationship to the partitioning of a given topological space (see e.g., the Fields Institute survey of Brown 2004). *Multiple groupoids* provide a descriptive mechanism for groupoids that 'interact' in a certain way and lead to the formulation of *higher homotopy groupoids* with properties described by an *algebra of cubes* (Brown and Higgins 1981; Brown and Loday 1987). Thus over the years, the first author with his colleagues has paved a path towards a comprehensive theory of *non-commutative algebraic topology* (Brown et al. in preparation). The potential for applications outside of strictly mathematical fields remains high and in our previous contributions in this issue we have described this prospect in terms of the theories of Super/Ultra Complexity within both a microscopic and macroscopic context. Specifically, we devote further attention to the subject of natural transformations occurring within biomolecular classes and reactions. A significant point here is the notion of 'variable topology,' a concept which can be expanded to 'variable categories and colimits.' In this regard *Higher Dimensional Algebra* affords us the necessary techniques as provided by the first named author's work on such topics as *Higher Homotopy Groupoids*, the related *van Kampen Theorems* and the instrumentation of *crossed complexes*. Other key concepts involve those of *fibred categories* and *stacks*—concepts envisaged by Alexander Grothendieck as part of the golden thread towards understanding how algebra, geometry and topology are inter-related/interwoven. It is unfortunate that the founding father of relational biology, Nicholas Rashevsky, or his mathematical/categorical disciple, Robert Rosen, have not made contact with Grothendieck's 'Pursuing Stacks' program. If indeed this had been the case, then a distinguishing framework for 'variable categories' might have been uncovered and the entire field might have enjoyed sooner a renaissance beginning with the late 1970s. But advancements in science, just as for "Rome", are simply "not built in a day"; unlike the latter, however, that was burnt before being re-built, the edifice of mathematics and science, in general, is constantly being re-built upon its older foundation. Fortunately, the time seems to be just right for such a 'stacks re-construction', since any durable theory of complexity (and that of the corresponding higher levels) must incorporate a scope for which categories *can be allowed to vary within a given system*, and thus, as far as we are concerned, the risk of 'categorical error' is minimized, if not altogether removed.

On the other hand, it will also be shown that categories of many-valued, Łukasiewicz-Moisil (LM) logic algebras provide additional, very useful concepts for representing the intrinsic dynamic 'asymmetry' in organismic development, evolution, and the human mind, as well as to derive novel results for (non-commutative) Quantum Logics and non-Abelian Quantum Ontology.

2 Background to Category Theory: Categories, Functors and Natural Transformations

Our main references to categeory theory are Borceux (1994), Mac Lane (2000), Mitchell (1965) and Popescu (1973).

2.1 Definition of a Category

A *category* C consists of:

1. a class $\mathrm{Ob}(\mathsf{C})$ called the *objects of* C;
2. for each pair of objects a,b of $\mathrm{Ob}(\mathsf{C})$, a set of *arrows* or *morphisms* $f : a \longrightarrow b$. We sometimes denote this set by $\mathrm{Hom}_{\mathsf{C}}(a, b)$. Here we say that a is the *domain of* f, denoted $a = dom\, f$, and b is the *codomain of* f, denoted $b = cod\, f$;
3. given two arrows $f : a \longrightarrow b$ and $g : b \longrightarrow c$ with $dom\ g = cod\ f$, there exists a composite arrow $g \circ f : a \longrightarrow c$. Further
   (i) Composition is *associative*: given $f \in \mathrm{Hom}_{\mathsf{C}}(a, b), g \in \mathrm{Hom}_{\mathsf{C}}(b, c)$ and $h \in \mathrm{Hom}_{\mathsf{C}}(c, d)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) Each object admits an identity arrow $\mathrm{id}_a : a \longrightarrow a$, where for all $f \in \mathrm{Hom}_{\mathsf{C}}(a,c)$ and all $g \in \mathrm{Hom}_{\mathsf{C}}(b,a)$, we have $f \circ \mathrm{id}_a = f$, and $\mathrm{id}_a \circ g = g$.

Typical examples of a category are:

$\mathsf{C} = \mathsf{Set}$ where the objects of Set are sets and the arrows are simply set maps.
$\mathsf{C} = \mathsf{Top}$ where the objects of Top are topological spaces and the set of arrows $\mathrm{Hom}_{\mathsf{Top}}(X,Y)$ is the set of all continuous maps $f : X \longrightarrow Y$ between objects X and Y, and where the composition law in Top is the composition of continuous functions.
$\mathsf{C} = \mathsf{Group}$ where the objects are groups and the arrows $f : G \longrightarrow H$ are group homomorphisms between groups G and H.

Observe that $\mathrm{Ob}(\mathsf{C})$ need not be a set. When it is we shall say that C is a *small category*.

Let us say that an object $\mathbf{i}$ in any category is said to be *initial* if for every object a, there is exactly one arrow $f : \mathbf{i} \longrightarrow a$, whereas an object $\mathbf{t}$ in any category is said to be *terminal* if for every object a, there is exactly one arrow $f : a \longrightarrow \mathbf{t}$. Any two initial (resp. terminal) objects can be shown to be isomorphic.

Corresponding to each category C, is its *opposite category* C^{op} obtained by reversing the arrows. Specifically, C^{op} has the same objects as C, but to each arrow $f : a \longrightarrow b$ in C, there corresponds an arrow $f^- : b \longrightarrow a$ in C^{op}, so that $f^- \circ g^-$ is defined once $g \circ f$ is defined, and so $f^- \circ g^- = (g \circ f)^-$.

Let Q and C be categories. We say that Q is a *subcategory* of C if

1. (inclusion of object sets) each object of Q is an object of C;
2. (inclusion of arrow sets) for all objects a,b of Q, $\mathrm{Hom}_{\mathsf{Q}}(a,b) \subseteq \mathrm{Hom}_{\mathsf{C}}(a,b)$;
3. composition '$\circ$' is the same in both categories and the identity $\mathrm{id}_a : a \longrightarrow a$ in Q is the same as in C.

A morphism m with codomain x is called *monic* if for all objects y and pairs of morphisms u,v: $x \rightarrow y$, $um = vm$ implies $u = v$. One can then define a *subobject* of x as an equivalence class of monics. The category of sets has preferred monics, namely the inclusions of subsets.

Sometimes it is said that a subobject of any object x of $\mathrm{Ob}(\mathsf{C})$ is *monic* with codomain x, a notion that generalizes the concept of a subset $A \subseteq X$ has a preferred injective (i.e., a one-to-one map) $A \longrightarrow X$ which sends $x \in A$ to $x \in X$.

2.2 Abelian Categories

A category C is said to be *Abelian* if it satisfies the following axioms:

- C has a zero object (which is both initial and terminal).
- ***Ab1*** and ***Ab1****. For every pair of objects in C there is a product and a sum.
- ***Ab2*** and ***Ab2****. Every map has a kernel and cokernel.
- ***Ab3*** and ***Ab3****. Every monomorphism is a kernel of a map and every epimorphism is a cokernel of a map.

Abelian categories (under the name of 'Additive Categories') were introduced by Grothendieck (1957) in order to unify the cohomology theories of groups and sheaves using *derived functors*, as well as to implement the technical concept of *spectral sequences*. The above definition is due to Freyd (1964, 2003). The category of Abelian groups (as do many other algebraic structures consisting of certain rings, modules and vector spaces) constitutes an Abelian category (Popescu 1973; Freyd 1964, 2003). Furthermore, Abelian categories are the building blocks of *homological algebra* (cf. Dold and Puppe 1961 in the simplicial context). When commutativity is global in a structure, as in an Abelian (or commutative) group, commutative groupoid, commutative ring, etc., such a structure that is commutative throughout is usually called ***Abelian***. However, in the case of category theory, this concept of Abelian structure has been extended to a special class of categories that have meta-properties formally similar to those of the category of commutative groups, *Ab*-**G**; the necessary and sufficient conditions for such 'Abelianness' of categories other than that of Abelian groups were expressed as three axioms **Ab1** to **Ab3** and their duals (Freyd 2003; see also the additional details in Gabriel, 1962; Mitchell, 1965; Oort, 1970 and Popescu 1973). A first step towards re-gaining something like the 'global commutativity' of an Abelian group is to require that all classes of morphisms [A,B] or Hom(A,B) have the structure of commutative groups; subject to a few other general conditions such categories are called **additive**. Then, some kind of global commutativity is assured for all morphisms of *additive* categories. However, in order to ensure that an additive category is well 'modelled' by the category of Abelian groups, according to Mitchell (1965), it must also be exact and have finite products. The exactness condition amounts to requiring that each morphism in an additive category ***A*** can be decomposed into, or expressed as the composition of, an epimorphism and monomorphism, in addition to requiring that ***A*** has kernels, cokernels, and also that it is both normal and conormal; the requirement that ***A*** is **normal** expresses the condition that every monomorphism in ***A*** is a kernel, whereas the requirement that ***A*** is ***conormal*** means that every epimorphism of ***A*** must be a cokernel. Implicitly, ***A*** has a null object, **0**, the Ab1 axiom of Freyd (2003). Such Abelian extensions in categories allow an unified treatment of both (commutative) Homological Algebra (Mac Lane 1963; Grothendieck 1957) and Algebraic Geometry (Grothendieck and Dieudonné 1960).

Loosely speaking, we may consider 'non-Abelian categories' as the 'complement' of Abelian categories in the 'category of all categories'. The former consist of a wide class of *topological* groups and spaces. The non-Abelian homology theory of groups is studied by Innassaridze (2002).

2.3 Groupoids: Locally Lie Groupoids, Atlas of Groupoids and Free-Generated Groupoids

One main example of a category which figures extensively in our work is that of a *groupoid* G : a small category in which every morphism is invertible; we denote the set of objects by $X = \mathrm{Ob}(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to y. The standard mathematical notion of a *group* is that of simply a groupoid with a single object (the *identity*). In this respect groupoids may be loosely viewed as certain categorical structures admitting 'multiple identities' (Brown 1987, 2006; Weinstein 1996).

A *topological groupoid* is a groupoid internal to the category Top. More specifically this consists of a space G, a distinguished space $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s : \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{2.1}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ : \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{(\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G} : s(\gamma_1) = r(\gamma_2)\} \longrightarrow \mathsf{G} \tag{2.2}$$

such that the following hold:

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2), r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.
(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.
(3) $\gamma \circ s(\gamma) = \gamma, r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.
(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.
(5) The composition has a two-sided, continuous inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

For $u \in \mathrm{Ob}(\mathsf{G})$, the space of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G at u.

2.3.1 The Free-generated Groupoid of a Graph

Given a directed graph Γ, there is a way to construct an associated free groupoid $\mathsf{FreeCat}(\Gamma)$ of 'paths' in Γ. The elements of this groupoid from vertex v to vertex w of Γ consist either of the identity at v (if $v = w$) or 'paths' moving from v to w along the edges of Γ or their formal reverse in a composable way, allowing cancellation. For full details we refer to Brown (2006). Those familiar with the following language will recognize that a defining property of this construction is that it is left adjoint to the forgetful functor which to a groupoid assigns its underlying graph.

It should not take any leap of the reader's imagination to see that the concept is applicable to automata in relationship to arrows between machine states where the set of (finite) strings usually admits a *semigroup structure* (recall that a semigroup $(G,*)$ consists of a set G and an associate binary operation $*$ on that set). The well-known Krohn–Rhodes Theorem explains the algebraic decomposition of such (finite) semigroup automata in terms of the wreath products of certain finite groups and semigroups (see e.g., Arbib 1968; Eilenberg 1974, 1976).

2.3.2 Locally Lie Groupoids

We commence with the key concept of a locally Lie groupoid. '*A locally Lie groupoid* (Pradines 1966; Aof and Brown 1992) is a pair (G, W) consisting of a groupoid G with range and source maps denoted α, β respectively, (in keeping with the last quoted literature) together with a smooth manifold W, such that:

(1) $\mathrm{Ob}(\mathsf{G}) \subseteq \mathrm{W} \subseteq \mathsf{G}$.
(2) $W = W^{-1}$.
(3) The set $W_\delta = \{W \times_\alpha W\} \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W_δ of the difference map $\delta : \mathsf{G} \times_\alpha \mathsf{G} \longrightarrow \mathsf{G}$ given by $(g, h) \mapsto gh^{-1}$, is smooth.
(4) The restriction to W of the maps α, β are smooth and (α, β, W) admits enough smooth admissible local sections.
(5) W generates G as a groupoid.

Let us explain some relevant terms. A *smooth local admissible section* of (α, β, W) is a smooth function s from an open subset of U of $X = \mathrm{Ob}(\mathsf{G})$ to W such that $\alpha s = 1_U$ and βs maps U diffeomorphically to its image which is open in X. It is such a smooth local admissible section which is thought of as a *local procedure* (in the situation defined by the locally Lie groupoid (G, W)).

There is a *composition*, originally due to Charles Ehresmann, of these local procedures given by $s * t(x) = s(\beta t(x)) \circ t(x)$ where $\circ$ is the composition in the groupoid G. The domain of $s * ts * t$ is usually smaller than that of t and may even be empty. Furthermore, the codomain of $s * t$ may not be a subset of W: thus the notion of smoothness (i.e., differentiability) of $s*t$ may not make sense. In other words, the composition of local procedures may not be a local procedure. Nonetheless, the set $\Gamma^\omega(\mathsf{G}, W)$ of all compositions of local procedures with its composition * has the structure of an *inverse semigroup*, and it is from this that the Holonomy Groupoid, $\mathbf{Hol}(\mathsf{G}, W)$ is constructed as a Lie groupoid in Aof and Brown (1992), following details given personally by Pradines to Brown concerning the work of Pradines (1966), with further details reported in Brown (1981).

The motivation for this construction, due to Pradines (1966), was to construct the *monodromy groupoid* Mon(G) of a Lie groupoid G. The details are given in Brown and Mucuk (1995, 1996). The monodromy groupoid has this name because of the *monodromy principle* on the extendability of local morphisms. It is a *local-to-global* construction having a kind of *left adjoint* property given in detail in Brown and Mucuk (1996). So it has certain properties that are analogous to a van Kampen theorem (to be discussed later). Further developments of this topic are presented in Brown and Mucuk (1995, 1996), Brown and İçen (2003).

The holonomy construction is applied to give a Lie structure to Mon(G). When G is the pair groupoid $X \times X$ of a manifold X, then Mon(G) is the *fundamental groupoid* $\pi_1 X$. It is crucial that this construction of Mon(X) is independent of paths in X, but is defined by a suitable neighbourhood of the diagonal in $X \times X$, which is in the spirit of synthetic differential geometry, and so has the possibility of being applicable in wider situations. Further work will be necessary to extend this construction to define *higher homotopy groupoids* with useful properties, but in all events, how do these concepts figure in the descriptive mechanisms for levels of complexity? First, in a real quantum system, a *unique* holonomy groupoid may represent *parallel transport* processes and the '*phase-memorizing*' properties of such remarkable systems (cf. the Berry phase and Hannay angle, Anandan 1992). This theme could be then further pursued by employing *locally Lie groupoids in* *local-to-global procedures* (cf. Aof and Brown 1992) for the construction in Quantum Spacetime of the *Holonomy Groupoid* (which is *unique*, according to the Globalization Theorem). The 'Lie' property suggests that the ensuing phase transition, possibly realized in terms of 'symmetry breaking', occurs through a degree of differentiability. In this respect the double groupoid approach to holonomy appears strikingly relevant (Brown and Spencer 1976; Brown and İçen 2001).

In a not unrelated sense, the notion of a *generalized* van Kampen theorem has many suggestive possibilities for both extensions and applications, and it should provide a basis for *higher dimensional, non-Abelian* methods in *local-to-global* questions in theoretical physics and Categorical Ontology, and therefore opens up completely new fields.

2.3.3 The Concept of a Groupoid Atlas

Motivation for the notion of groupoid atlas comes from considering families of group actions on a given set. As a notable instance, a subgroup H of a group G gives rise to a group action of H on G whose orbits are the cosets of H in G. However, a common situation is to have more than one subgroup of G, and then the various actions of these subgroups on G are related to the actions of the intersections of the subgroups themselves. This situation is handled by the notion of *Global Action*, as defined in Bak et al. (2006). A *global action* $\mathcal{A}$ consists of the following data:

(a) an indexing set $\Psi_{\mathcal{A}}$ called *the coordinate system of* $\mathcal{A}$, together with a reflexive relation $\leq$ on $\Psi_{\mathcal{A}}$;
(b) a set $X_{\mathcal{A}}$ and a family of subsets $(X_{\mathcal{A}})_\alpha$ of $X_{\mathcal{A}}$ for $a \in \Psi_{\mathcal{A}}$;
(c) a family of group actions $(G_{\mathcal{A}})_\alpha \curvearrowright (X_{\mathcal{A}})_\alpha$, i.e., maps $(G_{\mathcal{A}})_\alpha \times (X_{\mathcal{A}})_\alpha \longrightarrow (X_{\mathcal{A}})_\alpha$, with the usual group action axioms, for all $a \in \Psi_{\mathcal{A}}$;
(d) for each pair $\alpha \leq \beta$ in $\Psi_{\mathcal{A}}$, a group homomorphism

$$(G_{\mathcal{A}})_{\alpha \leq \beta} : (G_{\mathcal{A}})_\alpha \longrightarrow (G_{\mathcal{A}})_\beta.$$

This data must satisfy the following axioms:

(a) If $\alpha \leq \beta$ in Ψ_A, then $(G_{\mathcal{A}})_\alpha$ leaves $(X_{\mathcal{A}})_\alpha \cap (X_{\mathcal{A}})_\beta$ invariant.
(b) For each pair $\alpha \leq \beta$, if $\sigma \in (G_{\mathcal{A}})_\alpha$, and $x \in (X_{\mathcal{A}})_\alpha \cap (X_{\mathcal{A}})_\beta$, then $\sigma x = (G_{\mathcal{A}})_{\alpha \leq \beta}(\sigma)x$.

The diagram $G_{\mathcal{A}} : \Psi_{\mathcal{A}} \longrightarrow \mathsf{Groups}$, is called the *global group of* $\mathcal{A}$, and the set $X_{\mathcal{A}}$ is called the *enveloping set* or the *underlying set of* $\mathcal{A}$.

Suppose we have a group action $G \curvearrowright X$. Then we have a category $\mathsf{Act}(G, X)$ with object set X and $G \times X$ its arrow set. It is straightforward to show that $\mathsf{Act}(G, X)$ is actually a groupoid (Bak et al. 2006). Effectively, given an arrow (g,x), we have the source and target defined, respectively, by $s(g,x) = x$, and $t(g, x) = g \cdot x$, represented by $(g, x) : x \rightarrow g \cdot x$. The composition of (g,x) and (g', x') is defined when the target of (g,x) is the source of (g', x'), i.e., $x' = g \cdot x$. This yields a composition $(g'g,x)$ as shown in:

$$x \xrightarrow{(g,x)} g \cdot x \xrightarrow{(g',gx)} g'g \cdot x \tag{2.3}$$

We have an identity at x given by $(1,x)$, and for any element (g,x) its inverse is $(g^{-1}, g \cdot x)$. A key point in this construction is that the orbits of a group action then become the connected components of a groupoid. Also this enables relations with other uses of groupoids.

The above account motivates the following. A *groupoid atlas* $\mathcal{A}$ on a set $X_{\mathcal{A}}$ consists of a family of 'local groupoids' $(\mathsf{G}_{\mathcal{A}})$ defined with respective object sets $(X_{\mathcal{A}})_{\alpha}$ taken to be subsets of $X_{\mathcal{A}}$. These local groupoids are indexed by a set $\Psi_{\mathcal{A}}$, again called the *coordinate system of* $\mathcal{A}$ which is equipped with a reflexive relation denoted by $\leq$. This data is to satisfy the following conditions (Bak et al. 2006):

(1) If $\alpha \leq \beta$ in $\Psi_{\mathcal{A}}$, then $(X_{\mathcal{A}})_{\alpha} \cap (X_{\mathcal{A}})_{\beta}$ is a union of components of $(\mathsf{G}_{\mathcal{A}})$, that is, if $x \in (X_{\mathcal{A}})_{\alpha} \cap (X_{\mathcal{A}})_{\beta}$ and $g \in (\mathsf{G}_{\mathcal{A}})_{\alpha}$ acts as $g : x \longrightarrow y$, then $y \in (X_{\mathcal{A}})_{\alpha} \cap (X_{\mathcal{A}})_{\beta}$.

(2) If $\alpha \leq \beta$ in $\Psi_{\mathcal{A}}$, there is given a groupoid morphism defined between the restrictions of the local groupoids to intersections

$$(\mathsf{G}_{\mathcal{A}})_{\alpha}|(X_{\mathcal{A}})_{\alpha} \cap (X_{\mathcal{A}})_{\beta} \longrightarrow (\mathsf{G}_{\mathcal{A}})_{\beta}|(X_{\mathcal{A}})_{\alpha} \cap (X_{\mathcal{A}})_{\beta},$$

and which is the identity morphism on objects.

2.4 Functors

Let C and Q be two categories. *A covariant functor* $\mathbf{F} : \mathsf{Q} \longrightarrow \mathsf{C}$ maps objects of $\mathbf{Q}$ onto objects of $\mathbf{C}$, and morphisms of $\mathbf{Q}$ onto morphisms of $\mathbf{C}$, so that:

1. for each object a of Q, there is an object $\mathbf{F}(a)$ of C;
2. to each arrow $f \in \mathrm{Hom}_{\mathsf{Q}}(a, b)$, there is assigned an arrow $\mathbf{F}(f) : \mathbf{F}(a) \longrightarrow \mathbf{F}(b)$, such that $\mathbf{F}(\mathrm{id}_a) = \mathrm{id}_{\mathbf{F}(a)}$, and if $g \in \mathrm{Hom}_{\mathsf{C}}(b, c)$, then $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$.

Likewise one can define a *contravariant functor* by standard modifications to the previous definition: $\mathbf{F}(f) : \mathbf{F}(b) \longrightarrow \mathbf{F}(a), \mathbf{F}(g \circ f) = \mathbf{F}(f) \circ \mathbf{F}(g)$, etc.

A basic example is the (covariant) *forgetful functor* $\mathbf{F} : \mathsf{Top} \longrightarrow \mathsf{Set}$, which for any topological space X, $\mathbf{F}(X)$ is just the underlying set, and for a continuous map f, $\mathbf{F}(f)$ is the corresponding set map.2

2.5 Adjoint Functors and A Foundation for Semantics

We shall illustrate in subsequent Sects. 4–7 several applications to bionetworks of another very important type of functorial construction which *preserves colimits* (and/or *limits*); this construction is only possible for those pairs of categories which exhibit certain important similarities represented by an *adjointness relation*. Therefore, *adjoint functor* pairs are here defined with the aim of utilizing their properties in representing certain *similarities* between categories of bionetworks, as well as preserving, respectively, their limits and colimits.

Definition 2.1 Let us consider two covariant functors F and G between two categories C and C' arranged as follows:

$$\mathsf{C} \xrightarrow{F} \mathsf{C}' \xrightarrow{G} \mathsf{C} \tag{2.4}$$

We shall define F to be a *left adjoint functor* of G, and we define G to be a *right adjoint functor* of F, if for any X an object of category C, and any object X' of C', there exists a *bijection*

$$t(X, X') : \mathrm{Hom}_{\mathsf{C}}(X, G(X')) \longrightarrow \mathrm{Hom}_{\mathsf{C}'}(F(X), X'),$$

such that for any morphism $f : X \longrightarrow Y$ of C and morphism $f' : X' \longrightarrow Y'$ of C', the following diagrams of sets and canonically constructed mappings are *natural* (or *commutative*):

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{C}}(Y, G(X')) & \xrightarrow{t(Y,X')} & \mathrm{Hom}_{\mathsf{C}'}(F(Y), X') \\
{\scriptstyle h_{G(X')}(f)}\downarrow & & \downarrow{\scriptstyle h_{X'}(F(f))} \\
\mathrm{Hom}_{\mathsf{C}}(X, G(X')) & \xrightarrow{t(X,X')} & \mathrm{Hom}_{\mathsf{C}'}(F(X), X')
\end{array} \tag{2.5}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{C}}(X, G(X')) & \xrightarrow{t(X,X')} & \mathrm{Hom}_{\mathsf{C}'}(F(X), X') \\
{\scriptstyle h_{G(X')}(f)}\downarrow & & \downarrow{\scriptstyle h_{X'}(F(f))} \\
\mathrm{Hom}_{\mathsf{C}}(X, G(Y')) & \xrightarrow{t(X,Y')} & \mathrm{Hom}_{\mathsf{C}'}(F(X), Y')
\end{array} \tag{2.6}$$

In particular, we shall denote by $\eta_X : X \longrightarrow GF(X)$, the morphism $t^{-1}(X, F(X))(\mathbf{1}_{F(X)})$. Also, we shall denote by

$$\varepsilon_{X'} : FG(X') \longrightarrow X',$$

the morphism $\varepsilon(G(X'), X')(\mathbf{1}_{G(X')})$,(Popescu 1973, p.11).

One can easily verify that the following diagrams, which are canonically constructed, are also *natural* in C and C' for any morphism $f : X \longrightarrow Y$ in C, and for any morphism $f' : X' \longrightarrow Y'$ in C', respectively.

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & GF(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF(f)} \\
Y & \xrightarrow{\eta_Y} & GF(Y)
\end{array} \tag{2.7}$$

and

$$\begin{array}{ccc}
FG(X') & \xrightarrow{\varepsilon_{X'}} & X' \\
{\scriptstyle FG(f')}\downarrow & & \downarrow{\scriptstyle GF(f)} \\
FG(Y') & \xrightarrow{\varepsilon_{Y'}} & Y'
\end{array} \tag{2.8}$$

Such adjoint functors *commute*, respectively, with either *limits* or *colimits* as specified by the following theorem (Theorem 5.4 on p. 17 of Popescu 1973).

Theorem 2.1 *Given categories* C *and* D *, let* $F : \mathsf{C} \longrightarrow \mathsf{D}$ *be the left adjoint of the functor* $G : \mathsf{D} \rightarrow \mathsf{C}$*. Then, one has:*

(1) *F commutes with the colimit in* C *of any functor;*
(2) *G commutes with the limit in* D *of any functor.*

One also has the following important theorem (Popescu 1973, Theorem 5.3, p. 13).

Theorem 2.2 *Let* $F : \mathsf{C} \longrightarrow \mathsf{C}'$ *be a covariant functor. The following assertions are equivalent:*

(1) *F is full and faithful and any object* X' *of* C' *is isomorphic to an object* $F(X)$*, with X being an object of* C;
(2) *F is full and faithful, and has a full and faithful left adjoint;*
(3) *F is full and faithful, and has a full and faithful right adjoint.*

Definition 2.2 Two categories C and C' will be called *equivalent* if there is a covariant functor $F : \mathsf{C} \longrightarrow \mathsf{C}'$ which satisfies any of the three assertions in Theorem 2.2. The functor F will be called an *equivalence* from C to C'.

Note also the use of adjoint functors as a foundation for semantics, and the category of categories as a foundation for mathematics (Lawvere 1963, 1966, 1969).

2.6 Natural Transformations and Functorial Constructions in Categories

Categorical constructions make use of *functors* between categories as well as the higher order 'morphisms' between such functors called *natural transformations* that belong to a '*2-category*' (see for example Lawvere 1966). Such constructions also pave the way to *Higher Dimensional Algebra* which will be introduced in the next section. Especially effective are the *functorial constructions* which employ the '*hom*' *functors* defined next; this construction will then allow one to prove a very useful categorical result—*the Yoneda-Grothendieck Lemma*.

Let C be any category and let X be an object of C. We denote by $h^X : \mathsf{C} \longrightarrow \mathsf{Set}$ the functor obtained as follows: for any $Y \in \mathrm{Ob}(\mathsf{C})$ and any $f : X \longrightarrow Y$, $h^X(Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$; if $g : Y \longrightarrow Y'$ is a morphism of C then $h^X(f) : \mathrm{Hom}_{\mathsf{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(X, Y')$ is the map $h^X(f)(g) = fg$. One can also denote h^X as $\mathrm{Hom}_{\mathsf{C}}(X, -)$. Let us define now the very important concept of *natural transformation* which was first introduced by Eilenberg and Mac Lane (1945). Let $X \in \mathrm{Ob}(\mathsf{C})$ and let $F : \mathsf{C} \longrightarrow \mathsf{Set}$ be a covariant functor. Also, let $x \in F(X)$. We shall denote by $\eta_X : h^X \longrightarrow F$ the *natural transformation* (or *functorial morphism*) defined as follows: if $Y \in \mathrm{Ob}(\mathsf{C})$ then $(\eta_x)_Y : h^X(Y) \longrightarrow F(Y)$ is the mapping defined by the equality $(\eta_x)_Y(f) = F(f)(x)$; furthermore, one imposes the (*commutativity*) or naturality conditions on the following diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & F(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ G(X) & \xrightarrow[\eta_Y]{} & G(Y) \end{array} \tag{2.9}$$

Lemma 2.1 (The Yoneda-Grothendieck Lemma) *Let* $X \in \mathrm{Ob}(\mathsf{C})$ *and let* $F : \mathsf{C} \longrightarrow \mathsf{Set}$ *be a covariant functor. The assignment* $x \in F(X) \mapsto \eta_x$ *defines a bijection, or one-to-one correspondence, between the set F(X) and the set of natural transformations (or functorial morphisms) from* h^X *to F.*

This important lemma can be interpreted as stating that any category can be realized as a category of family of 'sets with structure' and structure preserving families of functions between sets (see also Sects. 7 and 8, and the references cited therein for its applications to the construction of categories of genetic networks or (**M,R**)-systems). Note also that the Yoneda-Grothendieck Lemma was previously employed to construct *generalized* Metabolic-Replication, or (**M,R**)-systems (Baianu 1973; Baianu and Marinescu 1974), which are categorical representations of the *simplest* enzymatic (metabolic) and genetic networks (Rosen 1958a).

2.7 Natural Transformations of Organismic Structures: Biomolecular Reaction Models in Categories

A simple introduction of molecular models in categories is based here on set-theoretical models of chemical transformations. Consider the very simple case of unimolecular chemical transformations (Bartholomay 1971):

$$T : A \times I \longrightarrow B \times I \tag{2.10}$$

with A being the original sample set of molecules and $I = [0, t]$ being defined as a finite segment of the real time axis; thus, $A \times I$ allows the indexing of each A-type molecule by the instant of time at which each molecule $a \varepsilon A$ is actually transforming into a B-type molecule (see also Eq. 3 of Bartholomay 1971). $B \times I$ then denotes the set of the newly formed B-type molecules which are indexed by their corresponding instance of birth.

A molecular class, denoted as A, is specified along with $f : A \longrightarrow A$, the *endomorphisms* that form the set $H(A,A)$.

One can then consider the category, $\underline{\mathsf{M}}$, of these molecular classes and their chemical transformations and also introduce natural transformations between certain canonical (hom) functors, as shown explicitly in Sect. 2.7.2. A hom-functor, h^A, indexed by a specified object A, is defined as:

$$h^A : \underline{\mathsf{M}} \longrightarrow \underline{\mathsf{Set}}$$

with its action determined by

$$h^A(X) = H(A,X) \text{ for any } X \in \underline{\mathsf{M}}$$

and

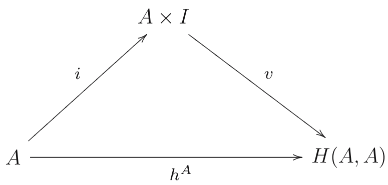
$$h^A(t) = m : H(A,A) \longrightarrow H(A,B) \text{ for any } t : A \longrightarrow B$$

where **A**= ***Molecular Class of type A−molecules*** and **B** = ***Molecular class of reaction products or type B−molecules***

Such hom-functors—which provide representations of chemical or biochemical reactions, (that is quantum molecular transformations of **molecular class A** into **molecular class B** of reaction B-products, or molecules of type "B")—thus allow the *emergence* of the next level of organization—the natural transformations obtained through the canonical Yoneda-Grothendieck construction.

*2.7.1 Definition of the Molecular Class (or set) Variable (**mcv**)*

The flexible notion of a *molecular class variable* (***mcv***) is precisely represented by the morphisms **v** in the following diagram:

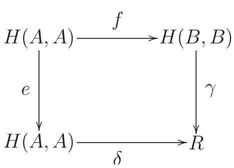


where morphisms v are induced by the inclusion mappings $i : A \longrightarrow A \times I$ and the commutativity conditions $h^A = v \circ i$. The naturality of this diagram simply means that such commutativity conditions hold for any functor h^A defined as above. Note also that one can define a (non-commutative) Clifford algebra (see e.g., Plymen and Robinson 1994) for the mcv-observables by endowing $A \times I$ and A with the appropriate non-commutative structures, thus generating an mcv-quantum space that is its own dual!

Simply stated, the *observable of an mcv* **B**, characterizing the chemical reaction product molecules "*B*" is defined as a morphism:

$$\gamma : H(B,B) \longrightarrow R$$

with R being the set of real numbers. This *mcv-observable* is subject to the following commutativity conditions:



(2.11)

with $c : A_u^* \longrightarrow B_u^*$, and A_u*, B_u* being specially prepared *fields of states*, within a measurement uncertainty range, Δ.

On the other hand, by endowing various classes A with different Hilbert space (topological) structures one obtains mcv's that are also endowed with variable topologies determined by such 'indexing' Hilbert spaces.

The next level of complexity emerges then by extending the above representations to *multi-molecular* reactions, coupled reactions,…, stable biochemical hypercycles—as in living organisms, and also perhaps in the now extinct primeval, single-cell organism. As we had shown previously, this extended representation then involves the *canonical functor* of category theory:

$$h : M \longrightarrow [M, \mathsf{Set}]$$

that assigns to each molecular set A the functor h^A, and to each chemical transformation $t : A \longrightarrow B$, the natural transformation $h^A \longrightarrow h^B$.

2.7.2 Natural Transformations as Representations of Emergent Biomolecular Reactions: A Category of Molecular Classes and their Chemical Transformations, *M*

Let **C** be any category and X an object of C. We denote by $h^X : \mathsf{C} \longrightarrow \mathsf{Set}$ the functor obtained as follows: for any $Y \in \mathrm{Ob}(\mathsf{C})$ and any $f : X \longrightarrow Y$, $h^X(Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$; if $g : Y \longrightarrow Y'$ is a morphism of C then $h^X(f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(X, Y')$ is the map $h^X(f)(g) = fg$. One can also denote h^X as $\mathrm{Hom}_{\mathsf{C}}(X, -)$. Let us define now the very important concept of *natural transformation* which was first introduced by Eilenberg and Mac Lane (1945). Let $X \in \mathrm{Ob}(\mathsf{C})$ and let $F : \mathsf{C} \longrightarrow \mathsf{Set}$ be a covariant functor. Also, let $x \in F(X)$. We shall denote by $\eta_X : h^X \longrightarrow F$ the *natural transformation* (or *functorial morphism*) defined as follows: if $Y \in \mathrm{Ob}(\mathsf{C})$ then $(\eta_x)_Y : h^X(Y) \longrightarrow F(Y)$ is the mapping defined by the equality $(\eta_x)_Y(f) = F(f)(x)$; furthermore, one imposes the *naturality* (or *commutativity*) condition on the following diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & F(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ G(X) & \xrightarrow[\eta_Y]{} & G(Y) \end{array} \qquad (2.12)$$

Recall that the hom-functor, h^A, indexed by a specified object A is defined as:

$$h^A : \underline{\mathsf{M}} \longrightarrow \underline{\mathsf{Set}}$$

with its action defined as:

$$h^A(X) = H(A, X) \text{ for any } X \in \underline{\mathsf{M}}$$

and

$$h^A(t) = m : H(A, A) \longrightarrow H(A, B) \text{ for any } t : A \longrightarrow B$$

where **A = Molecular Class** and **B = Molecular class of reaction products of type "B"**, resulting from a chemical reaction.

2.7.3 The Representation of Unimolecular, Biochemical Reactions as Natural Transformations

The *unimolecular chemical reaction* is here represented by the natural transformations $\eta : h^A \longrightarrow h^B$, through the following commutative diagram:

$$\begin{array}{ccc} h^A(A) = H(A,A) & \xrightarrow{\eta_A} & h^B(A) = H(B,A) \\ {\scriptstyle h^A(t)}\downarrow & & \downarrow{\scriptstyle h^B(t)} \\ h^A(B) = H(A,B) & \xrightarrow[\eta_B]{} & h^B(B) = H(B,B) \end{array} \tag{2.13}$$

with the states of the molecular sets $Au = a_1, \ldots, a_n$ and $Bu = b_1, \ldots b_n$ being represented by certain endomorphisms, respectively from $H(A,\ A)$ and $H(B,\ B)$.

2.7.4 A Simple Metabolic-Repair (M,R)-System with Reverse Transcription as an example of Multi-molecular Reactions Represented by Natural Transformations

We shall consider again the diagram corresponding to the simplest (**M,R**)-system realization of a Primordial Organism, PO.

The RNA and/or DNA duplication and cell divisions would occur by extension to the right of the simplest MR-system, (f, Φ), through the $\beta : H(A,B) \rightarrow H(B,H(A,B))$ and γ:$H(B,\ H(A,\ B)) \rightarrow\ H(H(A,\ B), H(B,\ H(A,\ B)))$ morphism. Note in this case, the 'closure' entailed by the functional mapping, γ, that physically represents the regeneration of the cell's *telomere* thus closing the DNA-loop at the end of the chromosome in germ cells of eukaryotes. Thus γ represents the activity of a *reverse transcriptase*. Adding to this diagram an hTERT suppressor gene would provide a *feedback* mechanism for an effective control of the cell division and the possibility of cell cycle arrest in higher, multi-cellular organisms (which is present only in *somatic* cells). The other alternative-which is preferred here-is the addition of an hTERT *promoter gene* that may require to be activated in order to begin cell cycling. This also allows one to introduce simple models of carcinogenesis or cancer cells.

Rashevsky's hierarchical theory of organismic sets can also be constructed by employing mcv's with their observables and natural transformations as it was shown in a previous report (Baianu 1980a, b, 1983).

Thus, one obtains by means of natural transformations and the Yoneda-Grothendieck construction a unified, categorical-relational theory of organismic structures that encompasses those of organismic sets, biomolecular sets, as well as the general $(\boldsymbol{M},\boldsymbol{R})$*-systems/autopoietic systems which takes explicitly into account both the molecular and quantum levels in terms of molecular class variables* (Baianu 1980a, b, 1983, 1984, 1987a, b, 2007).

2.8 Higher Order Categories and Cobordism

In higher dimensional algebra the concept of an abstract category generalizes to that of an n-category. We list here a short (but tentative) dictionary of analogies between general relativity theory (GR) and quantum theory (QT):

(1) (GR) pairs of spatial (n–1)-manifolds (M_1, M_2)-(QT) assigned Hilbert spaces H_1, H_2, respectively
(2) (GR) cobordism leading to a spacetime n-manifold M—(QT) (unitary) operator $T : H_1 \longrightarrow H_2$
(3) (GR) composition of cobordisms—(QT) composition of operators
(4) (GR) identity cobordism—(QT) identity operator.

The next step is to re-phrase this interplay of ideas categorically. So let Hilb denote the category whose objects are Hilbert spaces H with arrows the bounded linear operators on H. Let nCob denote the category whose objects are (n–1)-dimensional manifolds as above, and whose arrows are cobordisms between objects. Next we define a functor

$$Z : \mathsf{nCob} \longrightarrow \mathsf{Hilb}, \tag{2.14}$$

which assigns to any (n–1)-manifold M_1, a Hilbert space of states $Z(H_1)$, and to any n-dimensional cobordism $M : M_1 \longrightarrow M_2$, a (bounded) linear operator $Z(M) : Z(M_1) \longrightarrow Z(M_2)$, satisfying:

(i) given n-cobordisms $M : M_1 \longrightarrow M_2$ and $\check{M} : \check{M}_1 \longrightarrow \check{M}_2$, we have $Z(M\check{M}) = Z(\check{M})Z(M)$.
(ii) $Z(\mathrm{id}_{M_1}) = \mathrm{id}_{Z(M_1)}$.

Observe that (i) means the duration of time corresponding to the cobordism M followed by that of the cobordism $\check{M}$, is the same as the combined duration for that of $M, \check{M}$. Condition (ii) is the standard functorial condition for mapping identities by functors in Category Theory as already specified in Sect. 2.4. In the special case of the functor Z defined above, the identity id_{Mi} of a manifold M_i is uniquely mapped by Z onto the identity $\mathrm{id}_{Hi} = Z(Mi)$; this standard categorical condition for defining Z as a functor, may be neither physically satisfied nor physically meaningful because not every quantum observable has its classical analogue in GR. Such a theory thus necessitates further development, on the one hand, the relationship between nCob and n-categories (cf. Baez and Dolan 1995; Baez 2001), and on the other, that of a (non-commutative) theory of presheaves of Hilbert spaces/C^*-algebras which can be fitted into some quantum logical mechanism. Further, there is a necessity to realize the Grothendieck (1971) idea of *fibrations of n-categories over n-categories* as a possible unifying model for these theories. Naturally enough this pronouncement leads to the topic of the following subsection.

2.9 Fibred Categories: Pursuing Stacks

In the spirit of creating variable groupoids and more generally, variable categories, we introduce a fundamental notion of *fibred categories* quite at the heart of the

Grothendieck program of 'pursuing stacks' (see Grothendieck, 2007). For now, the references to this subsection are mainly Mac Lane and Moerdijk (1992) and Moerdijk (2002), and here we assume some familiarity with the theory of sheaves. Let X be a topological space and let $\mathcal{O}(X)$ be its category of open sets and inclusions.

2.9.1 The Notion of a Torsor

Let $\mathcal{G}$ be a sheaf of groups on X and let $\mathcal{S}$ be a sheaf on X. An action of $\mathcal{G}$ on $\mathcal{S}$ is a map of sheaves

$$\mu : \mathcal{G} \times \mathcal{S} \longrightarrow \mathcal{S},$$

(with components $\mu_U : \mathcal{G}(U) \times \mathcal{S}(U) \longrightarrow \mathcal{S}(U)$ denoted by $\mu_U(g, a) = g \cdot a$) satisfying the condition for a (left) action. We call $\mathcal{S}$ a $\mathcal{G}$*–torsor* if:

(1) $X = \bigcup\{U : \mathcal{S}(U) \neq \emptyset\}$.
(2) For each open set $U \subseteq X$, the action of $\mathcal{G}(U)$ on $\mathcal{S}(U)$ is free and transitive.

A morphism $\mathcal{S} \longrightarrow \mathcal{S}'$ of $\mathcal{G}$-torsors is a morphism of sheaves which commutes with the action. For further properties, see Mac Lane and Moerdijk (1992).

2.9.2 Fibred Categories and Descent Data

If

$$\mathbb{A} \xrightarrow{F} \mathbb{B} \underset{H}{\overset{G}{\rightrightarrows}} \mathbb{C} \xrightarrow{K} \mathbb{D}$$

are functors between categories, then any natural transformation $\tau : G \longrightarrow H$ induces natural transformations $K\tau : KG \longrightarrow KH$ and $\tau F : GF \longrightarrow HF$.

A fibred category $\mathbb{F}$ over X consists of:

- a category $\mathbb{F}(U)$ for each open set $U \subseteq X$,
- a functor $i^* : \mathbb{F}(U) \longrightarrow \mathbb{F}(V)$, for each inclusion $i : V \hookrightarrow U$ in $\mathcal{O}(X)$,
- a natural isomorphism

$$\tau = \tau_{i,j} : (ij)^* \longrightarrow j^* i^*,$$

for each pair of inclusions $j \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$.

Also, given composable inclusions $N \overset{k}{\hookrightarrow} W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$, the following diagram is commutative:

$$\begin{array}{ccc} (ijk)^* & \xrightarrow{\tau_{ij,k}} & k^*(ij)^* \\ {\scriptstyle \tau_{i,jk}}\downarrow & & \downarrow{\scriptstyle k^*\tau_{i,j}} \\ (jk)^* i^* & \xrightarrow{\tau_{j,k} i^*} & k^* j^* i^* \end{array}$$

A fibred category $(\mathbb{F}, \tau)$ in which all $\tau_{i,j}$ are identity transformations is usually called a *presheaf of categories*; thus, a fibred category is a presheaf up to isomorphism. Morphisms of fibred categories are defined accordingly.

- Let $\mathbb{F}$ be a fibred category over X and let a,b be objects in $\mathbb{F}(U)$. Then the assignment

$$V \mapsto \mathrm{Hom}_{\mathbb{F}(V)}(i^*a, i^*b),$$

for the inclusion $i : V \hookrightarrow U$, defines a presheaf denoted $\underline{\mathrm{Hom}}_{\mathbb{F}}(a, b)$ on U.

- Any morphism $\phi : \mathbb{F} \longrightarrow \mathbb{G}$ of fibred categories induces a morphism of presheaves on U:

$$\phi_{a,b} : \underline{\mathrm{Hom}}_{\mathbb{F}}(a, b) \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{G}}((\phi_U(a), \phi_U(b)).$$

2.9.3 Pre-stacks

A fibred category $\mathbb{F}$ over X is called a *pre-stack* if for any objects $a, b \in \mathbb{F}(U)$, the presheaf $\underline{\mathrm{Hom}}_{\mathbb{F}}(a, b)$, is a sheaf. For any space X, the inclusion

$$\text{Prestacks} \hookrightarrow \text{Fibred categories over } X$$

admits a left adjoint functor denoted here as $\mathbb{F} \mapsto \bar{\mathbb{F}}$.

2.10 Descent Data and Stacks of Groupoids

Let $\mathbb{F}$ be a fibred category over X and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of an open set $U \subseteq X$. The category $\mathrm{DES}(\mathcal{U}, \mathbb{F})$ of *descent data* consists of:

- Objects: systems $(a, \theta) = (\{a_\alpha, \{\theta_{\alpha\ \beta}\})$, where each a_α is an object of $\mathbb{F}(U_\alpha)$ and

$$\theta_{\alpha\beta} : a_{\alpha\beta} : a_\beta | U_{\alpha\beta} \xrightarrow{\cong} a_\alpha | U_{\alpha\beta},$$

is an isomorphism (where $a_\beta | U_{\alpha\beta} = i^*(a_\beta)$ relative to the inclusion $i : U_{\alpha\beta} \hookrightarrow U_\beta$, etc.) These isomorphisms are subject to the cocycle condition in $\mathbb{F}(U_{\alpha\beta\gamma})$:

$$\theta_{\alpha\alpha} = 1, \quad \theta_{\alpha\beta} \circ \theta_{\beta\gamma} = \theta_{\alpha\gamma}.$$

- Arrows: $(a, \theta) \xrightarrow{f} (b, \rho)$, families of arrows $f_\alpha : a_\alpha \longrightarrow b_\alpha$ in $\mathbb{F}(U_\alpha)$ satisfying $\rho_{\alpha\beta} f_\beta = f_\alpha \theta_{\alpha\beta}$, that is, the following diagram commutes:

$$\begin{array}{ccc} a_\beta|U_{\alpha\beta} & \xrightarrow{f_\beta} & b_\beta|U_{\alpha\beta} \\ {\scriptstyle \theta_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \rho_{\alpha\beta}} \\ a_\alpha|U_{\alpha\beta} & \xrightarrow{f_\alpha} & b_\beta|U_{\alpha\beta} \end{array}$$

The fibred category $\mathbb{F}$ is said to be a *stack* if each such functor $D : \mathbb{F}(U) \longrightarrow \mathrm{DES}(\mathcal{U}, \mathbb{F})$ is an equivalence of categories. Any sheaf of categories on X defines a prestack, but in general it might not define a stack. In any case, in our quest for a unifying concept of 'variable groupoids' we have reached here an essential framework since when each category $\mathbb{F}(U)$ is a groupoid, we then have a *stack of groupoids*.

3 Non-Abelian Concepts and Theories

The recent developments considered in this section point towards a paradigm shift in Categorical Ontology and to its extension to more general, *Non-Abelian theories*, well beyond the bounds of commutative structures/spaces and also free from the *logical* restrictions and limitations imposed by the Axiom of Choice to Set Theory. Additional restrictions imposed by representations using set theory also occur as a result of the 'primitive' notion of set membership, and also because of the 'discrete topology', the consequent impoverished structure of simple sets. It is interesting that D'Arcy W. Thompson also arrived in 1941 at an ontologic "*principle of discontinuity*" which "is inherent in all our classifications, whether mathematical, physical or biological… In short, nature proceeds *from one type to another* among organic as well as inorganic forms… and to seek for stepping stones across the gaps between is to seek in vain, for ever. Our geometrical analogies weigh heavily against Darwin's conception of endless small variations; they help to show that discontinuous variations are a natural thing, that "mutations"—or sudden changes, greater or less—are bound to take place, and new "types" to have arisen, now and then." (p. 1094 of Thompson 1994, re-printed edition).

Classic examples of non-Abelian concepts in Quantum Mechanics are the algebra of quantum observables (Dirac 1962) and Clifford algebra.

3.1 Definition of a Clifford Algebra

Consider a pair (V, Q), where V denotes a real vector space and Q is a quadratic form on V. *The Clifford algebra associated to V* denoted Cl(V) = Cl(V, Q), is the algebra over $\mathbb{R}$ generated by V, where for all v, $w \in V$, the relations

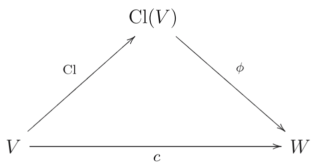
$$v \cdot w + w \cdot v = -2Q(v, w), \tag{3.1}$$

are satisfied; in particular, we have $v^2 = -2Q(v, v)$.

If W is an algebra and $c : V \longrightarrow W$ is a linear map satisfying

$$c(w)c(v) + c(v)c(w) = -2Q(v, w), \tag{3.2}$$

then there exists a unique algebra homomorphism $\phi : \mathrm{Cl}(V) \longrightarrow W$ such that the diagram



commutes. It is in this sense that Cl(V) is considered to be 'universal'.

For a given Hilbert space H, there is an associated *C*-Clifford algebra* $\mathrm{Cl}[H]$ which admits a canonical representation on $\mathcal{L}(\mathbb{F}(H))$ the bounded linear operators on the fermionic, free-field Fock space $\mathbb{F}(H)$ of H as in Plymen and Robinson (1994), and hence we a have a natural sequence of maps

$$H \longrightarrow \mathrm{Cl}[H] \longrightarrow \mathcal{L}(\mathbb{F}(H)). \tag{3.3}$$

3.2 Quantization, Space 'Deformation' and Non-Abelian Structures

There are several interesting mathematical constructions of non-commutative 'geometric spaces' obtained by 'deformation', such as those introduced by Connes (1994) as possible models for the physical, quantum spacetime which were discussed in detail in our previous paper in this issue (Baianu et al. 2007b). Thus, the microscopic, or quantum, 'first' level of physical reality does *not* appear to be subject to the categorical naturality conditions of Abelian TC-FNT—the 'standard' mathematical theory of categories (functors and natural transformations). It would seem therefore that the commutative hierarchy discussed above is not sufficient for the purpose of a General, Categorical Ontology which considers all items, at all levels of reality, including those on the 'first', quantum level, which is non-commutative. On the other hand, the mathematical, Non-Abelian Algebraic Topology (Brown et al. 2008), the Non-Abelian Quantum Algebraic Topology (NA-QAT; Baianu et al. 2004), and the physical, non-Abelian Gauge theories (NAGT) may provide the ingredients for a proper foundation for non-Abelian, hierarchical multi-level theories of a super-complex system dynamics in a General Categorical Ontology (GCO). Furthermore, it was recently pointed out (Baianu et al. 2004, 2006) that the current and future development of both NA-QAT and of a quantum-based Complex Systems Biology, *a fortiori*, involve *non-commutative*, many-valued logics of quantum events, such as a modified Łukasiewicz-Moisil (LMQ) logic algebra (Baianu et al. 2006), complete with a fully developed, novel probability measure theory grounded in the LM-logic algebra (Georgescu 2006). The latter paves the way to a new projection operator theory founded upon the *non-commutative quantum logic of events*, or dynamic processes, thus opening the possibility of a complete, *Non-Abelian Quantum theory* that can incorporate various themes such as:

- A Quantum-Algebraic Theory of operators, states and (quantum) groups (Alfsen and Schultz 2003; Majid 1995, 2002; Roberts 2004);
- Topological (and higher homotopy) Quantum Field Theories (Porter 1998; Martins and Porter 2004; Porter and Turaev 2005);
- Derivation of potential quantum invariants through Local-to-Global Procedures, novel non-Abelian concepts and the results presented in the next section, especially HHvKT;

We shall briefly consider next such fundamental/logical, quantum concepts in relation to quantization and the underlying non-Abelian structures.

3.3 An Example of Quantization: the Wigner–Weyl–Moyal Quantization Procedure

We have mentioned that a governing principle of quantization involves 'deforming', in a certain way, an algebra of functions on a phase space to an algebra of operator kernels. The more general techniques revolve around using such kernels in representing asymptotic morphisms. A fundamental example is an asymptotic morphism $C_0(T^*\mathbb{R}^n) \longrightarrow \mathcal{K}(L^2(\mathbb{R}^n))$ as expressed by the *Moyal deformation*:

$$[T_\hbar(a)f](x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a\Big(\frac{x+y}{2}, \xi\Big) \exp\Big[\frac{\imath}{\hbar}\Big] f(y)\, dy\, d\xi, \tag{3.4}$$

where $a \in C_0(T^*\mathbb{R}^n)$ and the operators $T_\hbar(a)$ are of trace class. In Connes (1994), it is called the *Heisenberg deformation*. Such 'quantizing deformations' can be thought to generate *non-commutative 'spaces'*, or *non-commutative 'geometry'*, *loc.cit*.

An elegant way of generalizing this construction entails introducing *the tangent groupoid* $\mathcal{T}X$ of a suitable space X and using asymptotic morphisms. Putting aside a number of technical details which can be found in Connes (1994) or Landsman (1998), the tangent groupoid $\mathcal{T}X$ is defined as the normal groupoid of a pair Lie groupoid $X \times X \rightrightarrows X$ obtained by 'blowing up' the diagonal $diag(X)$ in X. More specifically, if X is a (smooth) manifold let $G' = X \times X \times (0, 1]$ and $G'' = TX$, from which it can be seen $diag(G') = X \times (0, 1]$ and $diag(G'') = X$. Then in terms of disjoint unions we have

$$\begin{array}{rcl} \mathcal{T}X & = & G' \bigvee G'' \\ diag(\mathcal{T}\mathcal{X}) & = & diag(G') \bigvee diag(G''). \end{array} \tag{3.5}$$

In this way $\mathcal{T}X$ shapes up both as a smooth groupoid, as well as a manifold with boundary.

Quantization relative to $\mathcal{T}X$ is outlined by Várilly (1997) to which we refer for details. The procedure entails characterizing a function on $\mathcal{T}X$ in terms of a pair of functions on G' and G'', respectively, the first of which will be a kernel and the second will be the inverse Fourier transform of a function defined on T^*X. It will be instructive to consider the case $X = \mathbb{R}^n$ as a suitable example. So we take a function $a(x,\xi)$ on $T^*\mathbb{R}^n$ whose inverse Fourier transform

$$\mathcal{F}^{-1}(a(u, v)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp[\imath\xi v] a(u, \xi) d\xi, \tag{3.6}$$

thus yields a function on $T\mathbb{R}^n$. Consider next the terms

$$x := \exp_u\Big[\frac{1}{2}\hbar v\Big] = u + \frac{1}{2}\hbar v, y := \exp_u\Big[-\frac{1}{2}\hbar v\Big] = u - \frac{1}{2}\hbar v, \tag{3.7}$$

which on solving leads to $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{\hbar}(x-y)$. Then the following family of operator kernels

$$k_a(x,y,\hbar) := \hbar^{-n}\mathcal{F}^{-1}a(u,v) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right)\exp\left[\frac{\iota}{\hbar}(x-y)\xi\right]a(u,\xi)d\xi, \tag{3.8}$$

realize the Moyal quantization.

3.4 Quantum Logics: Definitions and Essential Properties. Lattices and Von Neumann–Birkhoff (VNB) Quantum Logic

The development of Quantum Mechanics from its very beginnings both inspired and required the consideration of specialized logics compatible with a new theory of measurements for microphysical systems. Such a specialized logic was initially formulated by von Neumann and Birkhoff (1932) and called 'Quantum Logic'.

We commence here by giving the *set-based definition of a Lattice*. An *s-lattice* **L**, or a 'set-based' lattice, is defined as a *partially ordered set* that has all binary products (defined by the *s*-lattice operation "$\bigwedge$") and coproducts (defined by the *s*-lattice operation "$\bigvee$"), with the "partial ordering" between two elements X and Y belonging to the *s*-lattice being written as "$X \preceq Y$". The partial order defined by $\preceq$ holds in **L** as $X \preceq Y$ if and only if $X = X \bigwedge Y$ (or equivalently, $Y = X \bigvee Y$, Eq. 3.1 (p. 49 of Mac Lane and Moerdijk 1992).

3.5 Categorical Definition of a Lattice

Utilizing the category theory concepts defined above, we now introduce a categorical definition of the concept of lattice that need be '*set-free*' in order to maintain logical consistency with the algebraic foundation of Quantum Logics and relativistic spacetime geometry. Such category-theoretical concepts unavoidably appear also in several sections of this paper as they provide the tools for deriving very important, general results that link Quantum Logics and classical (Boolean) Logic, as well as pave the way towards a universal theory applicable also to semi-classical, or mixed, systems. Furthermore, such concepts are indeed applicable to measurements in complex biological networks, as it will be shown in considerable detail in a subsequent paper in this volume (Baianu and Poli 2008).

A *lattice* is defined as a category (see, for example: Lawvere 1966; Baianu 1970; Baianu et al. 2004b) subject to all ETAC axioms, (but not subject, in general, to the Axiom of Choice usually encountered with sets relying on (distributive) Boolean Logic), that has all binary products and all binary coproducts, as well as the following 'partial ordering' properties:

(i) when unique arrows $X \longrightarrow Y$ exist between objects X and Y in **L** such arrows will be labelled by "$\preceq$", as in "s";
(ii) *the coproduct* of X and Y, written as "$X \bigvee Y$" will be called the "*sup object*, or "*the least upper bound*", whereas the product of X and Y will be written as "$X \bigwedge Y$", and it will be called an *inf object*, or "*the greatest lower bound*";

(iii) *the partial order* defined by $\preceq$ holds in **L**, as $X \preceq Y$ if and only if $X = X \bigwedge Y$ (or equivalently, $Y = X \bigvee Y$ (p. 49 of Mac Lane and Moerdijk 1992).

If a lattice **L** has **0** and **1** as objects, such that $0 \longrightarrow X \longrightarrow 1$ (or equivalently, such that $0 \preceq X \preceq 1$) for all objects X in the lattice **L** viewed as a category, then **0** and **1** are the unique, initial, and respectively, terminal objects of this concrete category **L**. Therefore, **L** has all finite limits and all finite colimits (p. 49 of Mac Lane and Moerdijk 1992), and is said to be *finitely complete and co-complete*. Alternatively, the lattice 'operations' can be defined via functors in a 2-category (for definitions of functors and 2-categories (see, for example Borceux (1994), Mac Lane (2000), Brown (2006) or Sect. 9 of Baianu et al. 2004b)) as follows:

$$\bigwedge : L \times L \longrightarrow L, \quad \bigvee : L \times L \to L \tag{3.9}$$

and 0,1: $1 \to L$ as a "lattice object" in a 2-category with finite products.

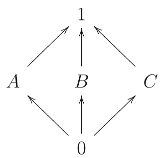
A lattice is called *distributive* if the following identity:

$$X \bigwedge (Y \bigvee Z) = (X \bigwedge Y) \bigvee (X \bigwedge Z). \tag{3.10}$$

holds for all X, Y, and Z objects in **L**. Such an identity also implies the dual distributive lattice law:

$$X \bigvee (Y \bigwedge Z) = (X \bigvee Y) \bigwedge (X \bigvee Z). \tag{3.11}$$

(Note how the lattice operators are 'distributed' symmetrically around each other when they appear in front of a parenthesis.) A *non-distributive* lattice is not subject to either restriction (13.13) or (13.14). An example of a non-distributive lattice is (cf. Pedicchio and Tholen, 2004):



(3.12)

3.6 Quantum Logics versus Crysippian Logic in Categorical Ontology

Quantum logics, and more generally, many-valued logics, may play greater roles in non-Abelian Ontology than Boolean logic as one needs to include the fundamental levels of reality in order to provide a conceptual 'closure' or complete theoretical framework.

3.6.1 Quantum Logics (QL) and Logical Algebras (LA)

As pointed out by Birkhoff and von Neumann (1936), a logical foundation of quantum mechanics consistent with quantum algebra is essential for both the completeness and mathematical validity of the theory. With the exception of a non-commutative geometry approach to unified quantum field theories (Connes 1994), the Butterfield and Isham framework (Butterfield and Isham 1998–2002) in terms of the 'standard' Topos (Mac Lane and Moerdijk 2000), and the 2-category approach by Baez (2001); other quantum algebra and topological approaches are ultimately based on set-theoretical concepts and differentiable spaces (manifolds). Since it has been shown that standard set theory which is subject to the axiom of choice relies on Boolean logic (Diaconescu 1976, cited in Mac Lane and Moerdijk 1992), there appears to exist a basic logical inconsistency between the quantum logic—which is not Boolean—and the Boolean logic underlying all differentiable manifold approaches that rely on continuous spaces of points, or certain specialized sets of elements. A possible solution to such inconsistencies is the definition of a generalized Topos concept, and more specifically, of a Quantum Topos concept which is consistent with both Quantum Logic and Quantum Algebras, being thus suitable as a framework for unifying quantum field theories and physical modelling of complex systems and systems biology.

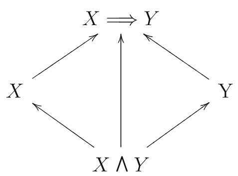
The problem of logical consistency between the quantum algebra and the Heyting logic algebra as a candidate for quantum logic is here discussed next.

3.7 Definition of an Intuitionistic Logic Lattice

A *Heyting algebra*, or *Brouwerian lattice*, H, is a *distributive lattice* with all finite products and coproducts, and which is also *Cartesian closed*. Equivalently, a Heyting algebra can be defined as a distributive lattice with both initial (0) and terminal (1) objects which has an "exponential" object defined for each pair of objects X, Y, written as: "$X \Rightarrow Y$" or X^Y, such that:

$$Z = (X \Longrightarrow Y) \Longleftarrow Z = X^Y, \tag{3.13}$$

In the Heyting algebra, $X \Rightarrow Y$ is a least upper bound for all objects Z that satisfy the condition $Z = X^Y$. Thus, in terms of a categorical diagram, the partial order in a Heyting algebra can be represented as



(3.14)

A lattice will be called complete when it has all small limits and small colimits (e.g., small products and coproducts, respectively). It can be shown (p. 51 of Mac Lane and Moerdijk 1992) that any complete and infinitely distributive lattice is a Heyting algebra.

3.8 Heyting–Brouwer Intuitionistic Foundations of Categories and Toposes

3.8.1 Subobject Classifier and the notion of a Topos

One of our main interests is in the notion of *topos*, a special type of category for which several (equivalent) definitions can be found in the literature. An important standard example is the category of (pre) sheaves on a small category C. We will need an essential component of the topos concept called a *subobject classifier*. In order to motivate the discussion, suppose we take a set X and a subset $A \subseteq X$. A characteristic function $\chi_A : X \longrightarrow \{0, 1\}$ specifies 'truth values' in the sense that one defines

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \neq A \end{cases}. \tag{3.15}$$

A topos C is required to possess an analog of the truth-value sets {0,1}. In order to specify this particular property, we consider a category C with a covariant functor $\mathsf{C} \longrightarrow \mathsf{Set}$, called a *presheaf*. The collection of presheaves on C forms a category in its own right, once we have specified the arrows. If $\mathcal{E}$ and $\mathcal{F}$ are two presheaves, then an arrow is a natural transformation $N : \mathsf{C} \longrightarrow \mathcal{F}$, defined in the following way. Given $a \in \mathrm{Ob}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(a, b)$, then there is a family of maps $N_a : \mathcal{E}(a) \longrightarrow \mathcal{F}(a)$, such that the diagram

$$\begin{array}{ccc} \mathcal{E}(a) & \xrightarrow{\mathcal{E}(f)} & \mathcal{E}(b) \\ {\scriptstyle N_a}\downarrow & & \downarrow{\scriptstyle N_b} \\ \mathcal{F}(a) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(b) \end{array} \tag{3.16}$$

commutes. Intuitively, an arrow between $\mathcal{E}$ and $\mathcal{F}$ serves to replicate $\mathcal{E}$ inside of $\mathcal{F}$.

Towards classifying subobjects we need the notion of a *sieve* on an object a of $\mathrm{Ob}(\mathsf{C})$. This is a collection S of arrows f in C such that if $f : a \longrightarrow b$ is in S and $g \in \mathrm{Hom}_{\mathsf{C}}(b, c)$ is any arrow, then the composition $f \circ g$ is in S.

We define a presheaf $\Omega : \mathsf{C} \longrightarrow \mathsf{Set}$, as follows. Let $a \in \mathrm{Ob}(\mathsf{C})$, then $\Omega(a)$ is defined as the set of all sieves on a. Given an arrow $f : a \longrightarrow b$, then $\Omega(f) : \Omega(a) \longrightarrow \Omega(b)$, is defined as

$$\Omega(f)(S) := \{g : b \longrightarrow c : g \circ f \in S\}, \tag{3.17}$$

for all $S \in \Omega(a)$. Let $\uparrow b$ denote the set of all arrows having domain the object b. We say that $\uparrow b$ is the *principal sieve on* b, and from the above definition, if $f : a \longrightarrow b$ is in S, then

$$\Omega(f)(S) = \{g : b \longrightarrow c : g \circ f \in S\} = \{g : b \longrightarrow c\} = \uparrow b. \tag{3.18}$$

Let us return for the moment to our motivation for defining Ω. The set of truth values {0,1} is itself a set and therefore an object in Set, furthermore, the set of subsets of a given set X corresponds to the set of characteristic functions χ_A as above. Likewise if C is a topos, Ω is an object of C, and there exists a bijective correspondence between subobjects of an object a and arrows $a \longrightarrow \Omega$, leading to the nomenclature *subobject classifier*. In this respect, a typical element of Ω relays a string of answers about the status of a given object in the topos. Furthermore, for a given object a, the set $\Omega(a)$ is endowed with the structure of a Heyting algebra (a distributive lattice with null and unit elements, that is relatively complemented).

The first choice of logic in a broad framework for quantum gravity and context-dependent measurement theories was intuitionistic related to the set-theoretic and presheaf constructions utilized for a context-dependent valuation theory (see Butterfield and Isham 1998–2002). The attraction, of course, comes from the fact that a topos is arguably a very general, mathematical model of a 'generalized space' that involves an intuitionistic logic algebra in the form of a special distributive lattice called a *Heyting Logic Algebra*, as was discussed earlier.

Subsequent research on Quantum Logics (Genoutti 1968; Dalla Chiara et al. 1968, 2004) resulted in several approaches that involve several types of non-distributive lattice (algebra) for n-valued quantum logics. Thus, modifications of the Łukasiewicz Logic Algebras that were introduced in the context of algebraic categories by Georgescu and Vraciu (1970), can provide an appropriate framework for representing quantum systems, or—in their unmodified form- for describing the activities of complex networks in categories of Łukasiewicz Logic Algebras (Baianu 1977). Such recent developments will be discussed next.

3.9 Łukasiewicz Quantum Logic (LQL)

With all assertions of the type *system A* is 'excitable' to the *i*th level and system B is excitable to the *j*th level on *e* can form a distributive lattice, $\mathbf{L}$ (as defined above in Sect. 3.1). The composition laws for the lattice will be denoted by $\bigcup$ and $\bigcap$. The symbol $\bigcup$ will stand for the logical non-exclusive 'or', and $\bigcap$ will stand for the logical conjunction 'and'. Another symbol "$\preceq$" allows for the ordering of the levels and is defined as *the canonical ordering* of the lattice. Then, one is able to give a symbolic characterization of the system dynamics with respect to each 'truth' level i. This is achieved by means of the maps $\delta_t : L \rightarrow L$ and N: $L \rightarrow L$, (with N being the negation). The necessary logical restrictions on the actions of these maps lead to an *n-valued Łukasiewicz*: Algebra of logical 'truth values' or nuances and operands.

(I) There is a map $N : L \longrightarrow L$, so that

$$N(N(X)) = X, \tag{3.19}$$

$$N\left(X \bigcup Y\right) = N(X) \bigcap N(Y) \tag{3.20}$$

and

$$N\left(X \bigcap Y\right) = N(X) \bigcup N(Y), \tag{3.21}$$

for any $X, Y \in \mathbf{L}$.

(II) There are (n–1) maps $\delta_i : L \longrightarrow L$ which have the following properties:
(a) $\delta_i\ (0) = 0$, $\delta_i(1) = 1$, for any $1 \leq i \leq n$–1;
(b) $\delta_i(X \bigcup Y) = \delta_i(X) \bigcup \delta_i(Y), \delta_i(X \bigcap Y) = \delta_i(X) \bigcap \delta_i(Y)$, for any $X, Y \in \mathbf{L}$, and $1 \leq i \leq n$–1;
(c) $\delta_i(X) \bigcup N(\delta_i(X)) = 1, \delta_i(X) \bigcap N(\delta_i(X)) = 0$, for any $X \in \mathbf{L}$;
(d) $\delta_i(X) \subset \delta_2(X) \subset \ldots \subset \delta_{(n-1)}(X)$, for any $X \in \mathbf{L}$;
(e) $\delta_i * \delta_j = \delta_i$ for any $1 \leq i, j \leq n$–1;
(f) If $\delta_i\ (X) = \delta_i\ (Y)$ for any $1 \leq i \leq n$–1, then $X = Y$;
(g) $\delta_i\ (N(X)) = N(\delta_j\ (X))$, for $i + j = n$. .

(Georgescu and Vraciu 1970).

The first axiom states that the double negation has no effect on any assertion concerning any level, and that a simple negation changes the disjunction into conjunction and conversely. The second axiom presents ten sub-cases that are summarized in equations (a)–(g). Sub-case (IIa) states that the dynamics of the system is such that it maintains the structural integrity of the system. It does not allow for structural changes that would alter the lowest and the highest 'energy' or 'truth' levels of the system. Thus, maps $\delta : L \longrightarrow L$ are here chosen to represent the dynamic behaviour of the quantum or classical systems in the absence of structural changes. Equation (IIb) shows that the maps (d) maintain the type of conjunction and disjunction. Equations (IIc) are chosen to represent assertions of the following type: ⟨the sentence ''a system component is excited to the th level or it is not excited to the same level⟩ "is true, and ⟨the sentence ''a system component is excited to the i-th level and it is not excited to the same level, at the same time'' is always false⟩.

Equation (IId) actually defines the actions of maps δ_t. Thus, Eq. (I) is chosen to represent a change from a certain level to another level as low as possible, just above the zero level of **L**. δ_2 carries a certain level x in assertion X just above the same level in δ_1 (X), δ_3 carries the level x-which is present in assertion X-just above the corresponding level in δ_2 (X), and so on. Equation (IIe) gives the rule of composition for the maps δ_t. Equation (IIf) states that any two assertions that have equal images under all maps δ_t, are equal. Equation (IIg) states that the application of δ to the negation of proposition X leads to the negation of proposition $\delta(X)$, if $i + j = n$.

In order to have the n-valued Łukasiewicz Logic Algebra represent correctly the basic behaviour of quantum systems (observed through measurements that involve a quantum system interactions with a measuring instrument—which is a macroscopic object, several of these axioms have to be significantly changed so that the resulting lattice becomes non-distributive and also (possibly) non-associative (Dalla Chiara et al. 2004).

On the other hand, for classical systems, modelling with the unmodified Łukasiewicz Logic Algebra can also include both stochastic and fuzzy behaviours. For an example of such models the reader is referred to a previous publication (Baianu 1977) modelling the activities of complex genetic networks from a classical standpoint. One can also define as in (Georgescu and Vraciu 1970) the 'centres' of certain types of Łukasiewicz n-Logic Algebras; then one has the following important theorem for such Centered Łukasiewicz n-Logic Algebras which actually defines an equivalence relation.

Theorem 3.1 The Adjointness Theorem (Georgescu and Vraciu 1970) *There exists an Adjointness between the Category of Centered Łukasiewicz n-Logic Algebras, **CLuk**−n , and the Category of Boolean Logic Algebras (**Bl**).*

Note: this adjointness (actually, equivalence) relation between the Centered Łukasiewicz n-Logic Algebra Category and **Bl** has a logical basis: *non* (*non*(*A*)) = *A* in both **Bl** and **CLuk**-*n*.

Conjecture 3.1 *There exist adjointness relationships, respectively, between each pair of the Centered Heyting Logic Algebra, **Bl** , and the Centered **CLuk**−n Categories.*

Remark 3.1 R1. Both a Boolean Logic Algebra and a Centered Łukasiewicz Logic Algebra can be represented as(are) Heyting Logic algebras (the converse is, of course, generally false!).

R2. The natural equivalence logic classes defined by the adjointness relationships in the above Adjointness Theorem define a fundamental, *logical groupoid* structure.

Note also that the above Łukasiewicz Logic Algebra is *distributive* whereas the quantum logic requires a *non-distributive* lattice of quantum 'events'. Therefore, in order to generalize the standard Łukasiewicz Logic Algebra to the appropriate Quantum Łukasiewicz Logic Algebra, *LQL*, axiom I needs modifications, such as: $N(N(X)) = Y \neq X$ (instead of the restrictive identity $N(N(X)) = X$, and, in general, giving up its 'distributive' restrictions, such as

$$N\left(X\bigcup Y\right) = N(X)\bigcap N(Y) \text{ and } N\left(X\bigcap Y\right) = N(X)\bigcup N(Y), \tag{3.22}$$

for any *X*, *Y* in the Łukasiewicz Quantum Logic Algebra, *LQL*, whenever the context, 'reference frame for the measurements', or 'measurement preparation' interaction conditions for quantum systems are incompatible with the standard 'negation' operation *N* of the Łukasiewicz Logic Algebra that remains however valid for classical systems, such as various complex networks with *n*-states (cf. Baianu 1977).

4 Local-to-Global Problems

Related to the local-to-global problems considered here and in the previous two articles in this issue, in Mathematics, Ehresmann previously developed many new themes in category theory. One example is *structured categories* with principal examples those of differentiable categories, groupoids, and multiple categories. His work on these is quite distinct from the general development of the mathematical theory of categories in the 20th century, and it is interesting to search for reasons for this distinction. One must be the fact that he used his own language and notation, which has not helped with the objectivation by several other, perhaps 'competing', mathematical schools. Another is surely that his early training and motivation came from analysis, rather than from algebra, in contrast to the origins of category theory in the work of Eilenberg and Mac Lane (including Steenrod and others) centered on homology theory and algebraic topology. Part of the developing language of category theory became essential in those areas, but other parts, such as those of algebraic theories, groupoids, multiple categories, were not used till fairly recently. It seems likely that Ehresmann's experience in analysis led him to the major theme of *local-to-global* questions. The author Brown first learned of this term from R. Swan in Oxford in 1957–1958, when as a research student Brown was writing up notes of his Lectures on the *Theory of Sheaves*. Swan explained to him that two important methods for local-to-global problems were *sheaves and spectral sequences*. But in fact, such problems are central in mathematics, science and technology. They are fundamental, for example, to the theories of *differential equations and dynamical systems*. Even deducing consequences of a set of rules is a local-to-global problem: the rules are applied *locally*, but we are interested in their *global* consequences.

Brown's work on local-to-global problems arose from writing an account of the Seifert-van Kampen theorem on the fundamental group. This theorem can be given as follows, as first shown by Crowell (1959):

Theorem 4.1 Crowell (1959). *Let the space X be the union of open sets U,V with intersection W, and suppose W,U,V are path connected. Let* $x_0 \in W$. *Then the diagram of* fundamental group *morphisms induced by inclusions:*

$$\begin{array}{ccc} \pi_1(W,x_0) & \xrightarrow{i} & \pi_1(U,x_0) \\ {\scriptstyle j}\downarrow & & \downarrow \\ \pi_1(V,x_0) & \longrightarrow & \pi_1(X,x_0) \end{array} \tag{4.1}$$

is a pushout of groups.

Here the 'local parts' are of course *U*, *V* put together with intersection *W* and the result describes completely, under the open set and connectivity conditions, the (*non-Abelian*) *fundamental group* of the global space *X*. This theorem is usually seen as a necessary part of basic algebraic topology, but one without higher dimensional analogues. On the other hand, the generalization of the van Kampen theorem to groupoids, and subsequently, indeed to the most general case of higher homotopy/higher dimensions—as well as non-Abelian cases—was carried out by author Brown and his research group.

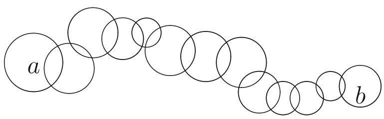
4.1 Iterates of Local Procedures using Groupoid Structures

Often we will look for the modelling of highly complex systems with various levels by specific functors into categories of categorical structures, and use natural transformations to compare such models. We have seen that the subclass of groupoids is essential for creating descriptive models for system reciprocity (i.e., *morphism invertibility, or isomorphism*) in the relay of signalling that occurs in various classes of genetic, neural and bionetworks, besides providing descriptive mechanisms for *local-to-global* properties within the latter, the collection of objects of which can comprise various genera of organismic sets. Groupoid actions and certain convolution algebras of groupoids (cf. Connes 1994) were suggested to be the main carriers of *non-commutative processes*. Many types of cell systems such as those representative of neural networks or physiological locomotion, can be described in terms of equivalence classes of cells, links and inputs, etc. leading to the notion of a system's *symmetry groupoid* the breaking of which can induce a transition from one state to another (Golubitsky and Stewart 2006). This notion of classification involves equivalence relations, but the groupoid point of view extends this notion not only to say that two elements are equivalent but also to label the proofs that they are equivalent.

The notion of *holonomy* occurs in many situations, both in physics and differential geometry. Non-trivial holonomy occurs when an iteration of local procedures which returns to the starting point can yield a change of phase, or of other more general values. Charles Ehresmann realized the notion of *local procedure* formalised by the notion of *local smooth admissible section of a smooth groupoid*, and Pradines (1966) generalised this to obtain a *global holonomy Lie groupoid* from a *locally* Lie groupoid: the details were presented in Aof and Brown (1992).

This concept of *local procedure* may be applicable to the evolution of super-complex systems/organisms for which there are apparently "missing links"—ancestors whose fossils cannot be found; when such links are genuinely missing, the evolution process can be viewed as maintaining an evolutionary trend not by virtue of analytical continuity, from point to point, but through overlapping regions from networks of genes and their expressed phenotype clusters. This idea of a local procedure applied to organismic speciation is illustrated below, with the intermediate circles representing such possible missing links, without the need to appeal to 'catastrophes'.

In this speciation example, the following picture illustrates a chain of local procedures (COLP) leading from species *a* to species *b via* intermediates that are not 'continuous' in the analytical sense discussed above:



One would like to be able to define such a chain, and equivalences of such chains, without resource to the notion of 'path' between points. The claim is that a candidate for this lies in the constructions of Charles Ehresmann and Jean Pradines for the *holonomy groupoid*. The globalization of structure can be thus encoded in terms of the *holonomy groupoid* which for any groupoid-related system encodes the notion of the subsequent *phase transition* (and its amplitude) of the latter phase towards a new phase (Aof and Brown 1992).

One question is whether a COLP is either a fact or a description. Things evolve and change in time. We think usually of this change as a real number modelling of time. But it may be easier to see what is happening as a COLP, since each moment of time has an environment, which is carried along as things evolve. The Aof-Brown paper, based on certain ideas of Charles Ehresmann and Jean Pradines, shows that such ideas have a mathematical reality, and that some forms of holonomy are nicely described in this framework of the globalisation theorem for a locally Lie groupoid.

The generalization of the manifold/atlas structure (Brown 2006) is that of the *groupoid atlas* (Bak 2006) which is already relevant in 'concurrent' and 'multi-agent systems' (Porter 2002). However, concurrent and multi-agent systems are distinct, though they may be somehow related. Concurrency itself is a theory about many processes occurring at the same time, or, equivalently, about processes taking place in multiple times. Since time has a direction, *multiple times* have a 'multiple direction', hence the *directed spaces*. This leads to a novel descriptive and computational technique for charting informational flow and management in terms of *directed spaces, dimaps and dihomotopies* (see e.g., Goubault 2003). These may provide alternative approaches to '*iterates of local procedures'* along with key concepts such as the notion of 'scheduling of paths' with respect to a cover that can be used as a globalization technique, for instance, to recover the Hurewicz continuous fibration theorem (Hurewicz 1955) as in Dyer and Eilenberg (1988).

Ontological levels themselves will entail '*processes* of processes' for which HDA seeks to provide the general theories of transport along n-paths and subsequent n-holonomy (cf. Brown and İçen 2001 for the two-dimensional case), thus leading to a globalization of the dynamics of local networks of organisms across which multiple morphisms interact, and which are multiply-observable. This representation, unless further specified, may not be able, however, to distinguish between levels and multiple processes occurring on the same level.

4.2 The van Kampen Theorem and its Generalizations to Groupoids and Higher Homotopy

The van Kampen Theorem 4.1 has an important and also anomalous role in algebraic topology. It allows computation of an important invariant for spaces built up out of simpler ones. It is anomalous because it deals with a non-Abelian invariant, and has not been seen as having higher dimensional analogues.

However Brown 1967, found a generalisation of this theorem to groupoids, as follows. In this, $\pi_1(X,X_0)$ is the fundamental *groupoid* of X on a set X_0 of base points: so it consists of homotopy classes relative to end points of paths in X joining points of $X_0 \cap X$.

Theorem 4.2 (The Van Kampen Theorem for the Fundamental Groupoid, (Brown 1967)) *Let the space X be the union of open sets U,V with intersection W, and let X_0 be a subset of X meeting each path component of U,V,W. Then*

(C) (connectivity) X_0 *meets each path component of X, and*
(I) (isomorphism) the diagram of groupoid morphisms induced by inclusions:

$$\begin{array}{ccc} \pi_1(W, X_0) & \xrightarrow{i} & \pi_1(U, X_0) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ \pi_1(V, X_0) & \xrightarrow[l]{} & \pi_1(X, X_0) \end{array} \tag{4.2}$$

is a pushout of groupoids.

Theorem 4.1 discussed in Sect. 4.1 is the special case when $X_0 = \{x_o\}$. From Theorem 4.2 one can compute a particular fundamental group $\pi_1(X, x_0)$ using combinatorial information on the graph of intersections of path components of *U*,*V*,*W*. For this it is useful to develop some combinatorial groupoid theory, as in Brown (2006) and Higgins (2005). Notice two special features of this method:

(i) The computation of the *invariant* one wants to obtain, *the fundamental group*, is obtained from the computation of a larger structure, and so part of the work is to give methods for computing the smaller structure from the larger one. This usually involves noncanonical choices, such as that of a maximal tree in a connected graph.
(ii) The fact that the computation can be done is surprising in two ways: (a) The fundamental group is computed *precisely*, even though the information for it uses input in two dimensions, namely 0 and 1. This is contrary to the experience in homological algebra and algebraic topology, where the interaction of several dimensions involves exact sequences or spectral sequences, which give information only up to extension, and (b) the result is a *non-commutative invariant*, which is usually even more difficult to compute precisely. Thus, exact sequences by themselves cannot show that a group is given as an HNN-extension: however such a description may be obtained from a pushout of groupoids (Sect. 8.4 in Brown 2006).

The reason for this success seems to be that the fundamental groupoid $\pi_1(X, X_0)$ contains information in *dimensions 0 and 1*, and therefore it can adequately reflect the geometry of the intersections of the path components of *U*,*V*,*W* and the morphisms induced by the inclusions of *W* in *U* and *V*. This fact also suggested the question of whether such methods could be extended successfully to *higher dimensions*.

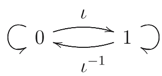
The following special case shows how the groupoid van Kampen Theorem gives an analogy between geometry and algebra. Let *X* be the circle S^1; choose *U*,*V* to be slightly extended semicircles including $X_0 = \{+1,-1\}$. Then $W = U \cap V$ is not path connected and so it is not clear where to choose a single base point. The day is saved by hedging one's bets, and using the two base points $\{+1,-1\}$. Now a key feature of groupoid theory is the groupoid I, the indiscrete groupoid on two objects 0,1, which acts as a unit interval object in the category of groupoids. One then compares the pushout diagrams, the first in spaces, the second in groupoids.

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & \mathbb{S}^1 \end{array} \qquad \begin{array}{ccc} \{0,1\} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \mathsf{I} & \longrightarrow & \mathbb{Z} \end{array}$$

spaces groupoids

The left hand diagram shows the circle as obtained from the unit interval [0,1] by identifying, in the category of spaces, the two end points 0,1. The right hand diagram shows the infinite group of integers $\mathbb{Z}$ as obtained from the finite groupoid I, again by identifying 0,1, but this time in the category of groupoids. Thus groupoid theory satisfactorily models this geometry.

The groupoid I with its special arrow $\iota : 0 \to 1$ has also the following property: if g is an arrow of a groupoid G then there is a unique morphism $\hat{g} : \mathsf{I} \to G$ whose value on ι is g. Thus the groupoid I with ι plays for groupoids the same role as does for groups the infinite cyclic group $\mathbb{Z}$ with the element 1: they are each free on one generator in their respective category. However, we can draw a complete diagram of the elements of I as follows:



whereas we cannot draw a complete picture of the elements of $\mathbb{Z}$.

The fundamental group is a kind of anomaly in algebraic topology because of its *non-Abelian* nature. Topologists in the early part of the 20th century were aware that:

(1) The non-commutativity of the fundamental group was useful in applications; for path connected X there was an isomorphism

$$H_1(X) \cong \pi_1(X,x)^{\mathrm{ab}}.$$

(2) The Abelian homology groups existed in all dimensions.

Consequently there was a desire to generalize the non-Abelian fundamental group to all dimensions. We indicate some solutions to this in the next section.

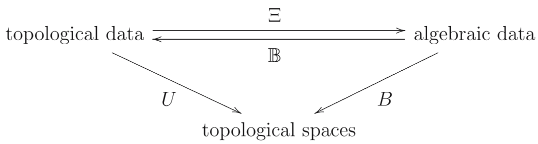
4.3 Wider Considerations

There is yet another approach to the Van Kampen Theorem which goes *via* the theory of *covering spaces*, and the *equivalence* between covering spaces of a reasonable space X and functors $\pi_1(X) \longrightarrow \mathsf{Set}$ (Brown 2004). See also an example (Douady and Douady 1979) that consists in an exposition of the relation of this approach with the Galois theory. Another paper, by Brown (1996), and Brown and Janelidze (1997, 2004), gives a general formulation of conditions for the theorem to hold in the case $X_0 = X$ in terms of the map $U \sqcup V \to X$ being an '*effective global* *descent morphism'* (the theorem is given in the generality of lextensive categories). The latter work has been developed for topoi (Bunge and Lack 2003). However, analogous interpretations of the topos work for higher dimensional Van Kampen theorems are not known so far.

The justification for changing from groups to groupoids is here threefold:

- the elegance and power of the results obtained with groupoids;
- the increased linking with other uses of groupoids (Brown 1987), and
- the opening out of new possibilities in higher dimensions, which allowed for new results, calculations in homotopy theory, and also suggested new algebraic constructions.

The notion of the fundamental groupoid of a space goes back at least to Reidemeister (1949), and an exposition of the main theorems of 1-dimensional homotopy theory in terms of the fundamental groupoid $\pi_1(X, A)$ on a set A of base points was given by the first author in 1968, 1988 (see Brown et al. 2008). This was inspired by work of Higgins in applying groupoids to group theory, (Higgins 1964). The success of the applications to 1-dimensional homotopy theory, as perceived by the writer, led to the idea of using groupoids in higher homotopy theory, as announced in Brown (1967). There was an idea of a proof in search of a theorem. The chief obstacle was constructing and applying *higher homotopy groupoids*. The overall aim became subsumed in the following diagram:



(4.3)

The aim is to find suitable categories of topological data, algebraic data and functors as above, where U is the forgetful functor and $B = U \circ \mathbb{B}$, with the following properties:

(1) the functor Ξ is defined homotopically and satisfies a higher homotopy van Kampen theorem (HHvKT), in that it preserves certain colimits;
(2) $\Xi \circ \mathbb{B}$ is naturally equivalent to 1;
(3) there is a natural transformation $1 \longrightarrow \mathbb{B} \circ \Xi$ preserving some homotopical information.

The purpose of (1) is to allow some calculation of Ξ. This condition also rules out at present some widely used algebraic data, such as simplicial groups or groupoids, since for those cases no such functor Ξ is known. (2) Shows that the algebraic data faithfully captures some of the topological data. The imprecise (3) gives further information on the algebraic modelling. The functor B should be called a *classifying space functor* because it often generalises the classifying space of a group or groupoid. Note here the essential involvement of certain *adjoint* functor pairs between categories of topological and correlated algebraic structures which preserve, respectively, colimits and limits.

We explain more about the HHvKT, in the case when the topological data is that of a filtered topological space

$$X_* : \quad X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X. \tag{4.4}$$

The advantage of this situation is to hope to obtain global information on X by climbing up the 'ladder' of the subspaces X_n, which again may be considered 'local'. But now we consider 'local' in another sense by supposing that there is given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\zeta \in \Lambda^n$ we set $U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}, U_i^\zeta = U^\zeta \cap X_i$. Then $U_0^\zeta \subset U_1^\zeta \subset \cdots$ is called the *induced filtration* U_*^ζ of U^ζ. Thus we can describe the filtered space X_* as a colimit in terms of the following diagram:

$$\bigsqcup_{\zeta \in \Lambda^2} U_*^\zeta \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} U_*^\lambda \xrightarrow{c} X_* \tag{4.5}$$

Here $\bigsqcup$ denotes disjoint union; a, b are determined by the inclusions $a_\zeta : U^\lambda \cap U^\mu \longrightarrow U^\lambda, b_\zeta : U^\lambda \cap U^\mu \longrightarrow U^\mu$ for each $\zeta = (\lambda, \mu) \in \Lambda^2$; and c is determined by the inclusions $c_\lambda : U^\lambda \longrightarrow X$. We would like this diagram to express that X_* is built from all the local filtered spaces U_*^λ by gluing them along the intersections $U_*^\zeta = U_*^\lambda \cup U_*^\mu$ whenever $\zeta = (\lambda, \mu)$. The useful categorical term for this is that diagram (4.5) is a *coequaliser diagram* in the category of filtered spaces.

We would like to turn this topological information into algebraic information. to enable us to understand and to calculate. So we apply the functor Ξ and if it preserves disjoint union we have the following diagram:

$$\bigsqcup_{\zeta \in \Lambda^2} \Xi(U_*^\zeta) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \Xi(U_*^\lambda) \xrightarrow{c} \Xi(X_*) \tag{4.6}$$

We would like this diagram (4.6) to be a coequaliser diagram in our category of algebraic data. This is not true in general but needs an extra condition, which we call *connected* for that topological data, not only on the U_*^λ but on all finite intersections of these. The conclusion of the HHvKT is then the important fact that X_* is also connected, and that diagram (4.6) is indeed a coequaliser diagram. This implies that the global algebraic invariant $\Xi\ X_*$ is *completely determined* by the local algebraic invariants ΞU_*^λ, and the way these are glued together using the information on the ΞU_*^ζ. Note that this is not a reductionist result: the whole is not just made up of its parts, but, as is only sensible, is made up of its parts and the way they are put together. In the case the open cover consists of two elements, then the above coequaliser reduces to a pushout, and so includes the cases of the van Kampen Theorem considered earlier.

A feature of this scheme is that the algebraic data that we use has structure in a range of dimensions. This is necessary for homotopy theory since change in a low dimension can considerably affect higher dimensional behaviour. We do not define the connectivity condition precisely here, but note that while it does considerably restrict the range of applications, it still allows for new proofs of classical theorems of homotopy theory, such as the relative Hurewicz theorem, and allows for totally new results, including non-Abelian results in dimension 2. The format of the above coequaliser (4.6) is similar to diagrams appearing in Grothendieck's descent theory, but which extend to the left indefinitely. That theory is a very sophisticated local-to-global theory. This is perhaps indicative for future work.

The examples of *topological data* for which these schemes are known to work are:

Topological data	Algebraic data
Spaces with one base point	Groups
Spaces with a set of base points	Groupoids
Filtered spaces	Crossed complexes
n-cubes of pointed spaces	Cat^n-groups
Hausdorff spaces	Double groupoids with connections

In fact crossed complexes are equivalent to a bewildering array of other structures, which are important for applications (Brown 1999). Cat^n-groups are also equivalent to *crossed n-cubes of groups*. The construction of the equivalences and of the functors Ξ in all these cases is difficult conceptually and technically. The general philosophy is that one type of category is sufficiently geometric to allow for the formulation and proof of theorems, in a higher dimensional fashion, while another is more 'linear' and suitable for calculation. The transformations between the two forms give a kind of synaethesia. The classifying space constructions are also significant, and allow for information on the homotopy classification of maps.

From the ontological point of view, these results indicate that it is by no means obvious what algebraic data will be useful to obtain precise local-to-global results, and indeed new forms of this data may have to be constructed for specific situations. These results do not give a TOE, but do give a new way of obtaining new information not obtainable by other means, particularly when this information is in a noncommutative form. The study of these types of results is not widespread, but will surely gain attention as their power becomes better known.

In Algebraic Topology crossed complexes have several advantages such as:

- They are good for *modelling CW-complexes*. Free crossed resolutions enable calculations with *small CW-models* of $K(G,1)$s *and their maps* (Brown and Razak Salleh 1999).

- Also, they have an interesting relation with the Moore complex of simplicial groups and of *simplicial groupoids*.
- They *generalise groupoids and crossed modules to all dimensions*. Moreover, the natural context for the second relative homotopy groups is crossed modules of groupoids, rather than groups.
- They are convenient for *calculation*, and the functor Π is classical, involving *relative homotopy groups*.
- They provide a kind of *'linear model'* for homotopy types which includes all 2-types. Thus, although they are not the most general model by any means (they do not contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. For example, this is how a general n-adic Hurewicz Theorem was found (Brown and Loday 1987).
- Crossed complexes have a *good homotopy theory*, with a *cylinder object, and homotopy colimits*. (A *homotopy classification* result generalises a classical theorem of Eilenberg–Mac Lane).
- They are close to chain complexes with a group(oid) of operators, and related to some classical homological algebra (e.g., *chains of syzygies*). In fact if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly *noncommutative version of the singular chains of a space*.

For more details on these points, we refer to Brown (2004).

4.4 Construction of the Homotopy Double Groupoid of a Hausdorff Space

In the previous section, we mentioned that higher homotopy groupoids have been constructed for filtered spaces and for n-cubes of spaces. It is also possible to construct a homotopy double groupoid for a Hausdorff space, and prove a higher homotopy van Kampen theorem for this functor. This illustrates the interest and difficulty of extending this construction to other situations, such as smooth manifolds, or for Quantum Axiomatics.

We shall begin by recalling the construction of *The Homotopy Double Groupoid* $\boldsymbol{\rho}^{\square}(X)$ as adapted from Brown et al. (2002), and the reader should refer to that source for complete details.

4.5 The Singular Cubical Set of a Topological Space

We shall be concerned with the low dimensional part (up to dimension 3) of the singular cubical set

$$R^{\square}(X) = (R_n^{\square}(X), \partial_i^-, \partial_i^+, \varepsilon_i)$$

of a topological space X. We recall the definition (cf. Brown and Hardie 1976). For $n \geq 0$ let

$$R_n^{\square}(X) = \mathsf{Top}(I^n, X)$$

denote the set of *singular n-cubes* in X, i.e., continuous maps $I^n \longrightarrow X$, where $I = [0,1]$ is the unit interval of real numbers. We shall identify $R_0^{\square}(X)$ with the set of points of X. For $n = 1,2,3$ a singular n-cube will be called a *path*, resp. *square*, resp. *cube*, in X. The *face maps*

$$\partial_i^-, \partial_i^+ : R_n^{\square}(X) \longrightarrow R_{n-1}^{\square}(X) \quad (i = 1, \ldots, n)$$

are given by inserting 0 resp. 1 at the ith coordinate whereas the *degeneracy maps*

$$\varepsilon_i : R_{n-1}^{\square}(x) \longrightarrow R_n^{\square}(X) \quad (i = 1, \ldots, n)$$

are given by omitting the ith coordinate. The face and degeneracy maps satisfy the usual cubical relations (cf. Brown and Higgins 1981). A path $a \in R_1^{\square}(X)$ has *initial point* $a(0)$ and *endpoint* $a(1)$. We will use the notation $a : a(0) \simeq a(1)$. If a,b are paths such that $a(1) = b(0)$, then we denote by $a + b : a(0) \simeq b(1)$ their *concatenation*, i.e.,

$$(a+b)(s) = \begin{cases} a(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ b(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

If x is a point of X, then $\varepsilon_1(x) \in R_1^{\square}(X)$, denoted e_x, is the *constant path* at x, i.e.,

$$e_x(s) = x \quad \text{for all } s \in I.$$

If $a : x \simeq y$ is a path in X, we denote by $-a : y \simeq x$ the *path reverse* to a, i.e., $(-a)(s) = a(1-s)$ for $s \in I$. In the set of squares $R_2^{\square}(X)$ we have two partial compositions $+_1$ (*vertical composition*) and $+_2$ (*horizontal composition*) given by concatenation in the first resp. second variable. Similarly, in the set of cubes $R_3^{\square}(X)$ we have three partial compositions $+_1$, $+_2$, $+_3$.

The standard properties of vertical and horizontal composition of squares are listed in Brown and Hardie (1976) Sect. 1. In particular we have the following *interchange law*. Let $u, u', w, w' \in R_2^{\square}(X)$ be squares, then

$$(u +_2 w) +_1 (u' +_2 w') = (u +_1 u') +_2 (w +_1 w')$$

whenever both sides are defined. More generally, we have an interchange law for rectangular decomposition of squares. In more detail, for positive integers m,n let $\varphi_{m,n} : I^2 \longrightarrow [0,m] \times [0,n]$ be the homeomorphism $(s,t) \longmapsto (ms, nt)$. An $m \times n$ *subdivision* of a square $u : I^2 \longrightarrow X$ is a factorization $u = u', \varphi_{m,n}$; its *parts* are the squares $u_{ij} : I^2 \longrightarrow X$ defined by

$$u_{ij}(s,t) = u'(s+i-1, t+j-1).$$

We then say that u is the *composite* of the array of squares (u_{ij}), and we use matrix notation $u = [u_{ij}]$. Note that as in Sect. 1 of Brown and Hardie (1976), $u +_1 u', u +_2 w$ and the two sides of the interchange law can be written respectively as

$$\begin{bmatrix} u \\ u' \end{bmatrix}, \quad [u \quad w], \quad [u \quad w\,u' \quad w']$$

Finally, *connections*:

$$\Gamma^-, \Gamma^+ : R_1^\square(X) \longrightarrow R_2^\square(X)$$

are defined as follows. If $a \in R_1^\square(X)$ is a path, $a : x \simeq y$, then let

$$\Gamma^-(a)(s,t) = a(max(s,t));\ \Gamma^+(a)(s,t) = a(min(s,t)).$$

The full structure of $R^\square(X)$ as a *cubical complex with connections and compositions* has been exhibited in (Al-Agl et al. 2002).

4.5.1 Thin Squares

In the setting of a geometrically defined double groupoid with connection there is an appropriate notion of *geometrically thin* square as shown in Brown et al. (2002). In the cases given there geometrically and algebraically thin squares coincide (Theorem 5.2 of that reference). In our context the explicit definition is as follows:

Definition 4.1 A square $u : I^2 \longrightarrow X$ in a topological space X is *thin* if there is a factorisation of u:

$$u : I^2 \xrightarrow{\Phi_u} J_u \xrightarrow{p_u} X,$$

where J_u is a tree and Φ_u is piecewise linear (PWL, see below) on the boundary ∂I^2 of I^2.

Here, by a *tree*, we mean the underlying space $|K|$ of a finite 1-connected 1-dimensional simplicial complex K.

A map $\Phi : |K| \longrightarrow |L|$ where K and L are (finite) simplicial complexes is PWL (*piecewise linear*) if there exist subdivisions of K and L relative to which Φ is simplicial.

Let u be as above, then the homotopy class of u relative to the boundary ∂I^2 of I is called a *double track*. A double track is *thin* if it has a thin representative.

4.6 The Homotopy Double Groupoid of a Hausdorff Space

The full data for the homotopy double groupoid, $\boldsymbol{\rho}^\square(X)$, will be denoted by

$$(\boldsymbol{\rho}_2^\square(X), \boldsymbol{\rho}_1^\square(X), \partial_1^-, \partial_1^+, +_1, \varepsilon_1),\ (\boldsymbol{\rho}_2^\square(X), \boldsymbol{\rho}_1^\square(X), \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(\boldsymbol{\rho}_1^\square(X), X, \partial^-, \partial^+, +, \varepsilon).$$

Here $\boldsymbol{\rho}_1(X)$ denotes the *path groupoid* of X. We recall the definition. The objects of $\boldsymbol{\rho}_1(X)$ are the points of X. The morphisms of $\boldsymbol{\rho}_1^{\square}(X)$ are the equivalence classes of paths in X with respect to the following relation $\sim_T$.

Definition 4.2 Let $a, a' : x \simeq y$ be paths in X. Then a is *thinly equivalent* to a', denoted $a \sim_T a'$, if there is a thin relative homotopy between a and a'.

We note that $\sim_T$ is an equivalence relation, see Brown et al. (2002). We use $\langle a \rangle : x \simeq y$ to denote the $\sim_T$ class of a path $a : x \simeq y$ and call $\langle a \rangle$ the *semitrack* of a. The groupoid structure of $\boldsymbol{\rho}_1^{\square}(X)$ is induced by concatenation, +, of paths. Here one makes use of the fact that if $a : x \simeq x'$, $a' : x' \simeq x''$, $a'' : x'' \simeq x''$ are paths then there are canonical thin relative homotopies

$$\begin{array}{rcl} (a + a') & + & a'' \simeq a + (a' + a'') : x \simeq x''' \ (\mathit{rescale}) \\ a + e_{x'} \simeq a & : & x \simeq x';\ e_x + a \simeq a : x \simeq x' \ (\mathit{dilation}) \\ & & a + (-a) \simeq e_x : x \simeq x \ (\mathit{cancellation}). \end{array}$$

The source and target maps of $\boldsymbol{\rho}_1^{\square}(X)$ are given by

$$\partial_1^- \langle a \rangle = x, \quad \partial_1^+ \langle a \rangle = y,$$

if $\langle a \rangle : x \simeq y$ is a semitrack. Identities and inverses are given by

$$\varepsilon(x) = \langle e_x \rangle \quad \text{resp.} - \langle a \rangle = \langle -a \rangle.$$

In order to construct $\boldsymbol{\rho}_2^{\square}(X)$, we define a relation of cubically thin homotopy on the set $R_2^{\square}(X)$ of squares.

Let u,u' be squares in X with common vertices. (1) A *cubically thin homotopy* $U : u \equiv_T^{\square} u'$ between u and u' is a cube $U \in R_3^{\square}(X)$ such that

(i) U is a homotopy between u and u',

$$\text{i.e., } \partial_1^-(U) = u, \quad \partial_1^+(U) = u',$$

(ii) U is rel. vertices of I^2,

$$\text{i.e., } \partial_2^-\partial_2^-(U), \partial_2^-\partial_2^+(U), \partial_2^+\partial_2^-(U), \partial_2^+\partial_2^+(U) \ \text{ are constant,}$$

(iii) the faces $\partial_i^{\alpha}(U)$ are thin for $\alpha = \pm 1,\ i = 1, 2$.

(2) The square u is *cubically T-equivalent* to u', denoted $u \equiv_T^{\square} u'$ if there is a cubically thin homotopy between u and u'.

The relation $\equiv_T^{\square}$ can be seen to be an equivalence relation on $R_2^{\square}(X)$. For the proof of this result, the reader is referred to (Brown et al. 2002).

If $u \in R_2^{\square}(X)$ we write $\{u\}_T^{\square}$, or simply $\{u\}_T$, for the equivalence class of u with respect to $\equiv_T^{\square}$. We denote the set of equivalence classes $R_2^{\square}(X) \equiv_T^{\square}$ by $\boldsymbol{\rho}_2^{\square}(X)$. This inherits the operations and the geometrically defined connections from $R_2^{\square}(X)$ and so becomes a double groupoid with connections. A proof of the final, fine detail of the structure is given in (Brown et al. 2002).

An element of $\boldsymbol{\rho}_2^{\square}(X)$ is *thin* if it has a thin representative (in the sense of Definition in Brown (2004). From the remark at the beginning of this subsection we infer:

Lemma 4.1 *Let $f : \boldsymbol{\rho}^{\square}(X) \to \mathsf{D}$ be a morphism of double groupoids with connection. If $\alpha \in \boldsymbol{\rho}_2^{\square}(X)$ is thin, then f(α) is thin.*

Lemma 4.2 (The Homotopy Addition Lemma) *Let u: $I^3 \to X$ be a singular cube in a Hausdorff space X. Then by restricting u to the faces of I^3 and taking the corresponding elements in $\boldsymbol{\rho}_2^{\square}(X)$, we obtain a cube in $\boldsymbol{\rho}^{\square}(X)$ which is commutative by the homotopy addition lemma for $\boldsymbol{\rho}^{\square}(X)$ (Brown et al.* 2002*, Proposition 5.5). Consequently, if $f : \boldsymbol{\rho}^{\square}(X) \to \mathsf{D}$ is a morphism of double groupoids with connections, then any singular cube in X determines a commutative 3-shell in D.*

Now under the situation given earlier where the Hausdorff space X has an cover by sets $\{U_\lambda\}_{\lambda\in\Lambda}$ we get a diagram as follows:

$$\bigsqcup\nolimits_{\zeta\in\Lambda^2} \boldsymbol{\rho}^{\square}(U^{\zeta}) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup\nolimits_{\lambda\in\Lambda} \boldsymbol{\rho}^{\square}(U^{\lambda}) \xrightarrow{c} \boldsymbol{\rho}^{\square}(X) \tag{4.7}$$

The following is a statement of the Higher Homotopy van Kampen Theorem (HHvKT) expressed in terms of Double Groupoids with connections as developed and proven in Brown et al. (2002).

Theorem 4.3 (Brown et al. 2002) *The van Kampen theorem for Double Groupoids. If the interiors of the sets of $\mathcal{U}$ cover X, then in the above diagram (4.7), c is the coequaliser of a,b in the category of double groupoids with connections.*

The reader is referred to Brown et al. (2002), for the proof of this form of the Higher Homotopy van Kampen theorem. A special case of this result is when $\mathcal{U}$ has two elements. In this case the coequaliser reduces to a pushout. An important feature of the proof is the notion of commutative cube, the relation of these to thin cubes, and the fact that any multiple composition of commutative cubes is commutative. All these are facts whose analogues for squares are trivial. Thus the step from dimension to, i.e., for squares, to dimension 3, i.e., for cubes, is a large one technically and conceptually. Corresponding results in higher dimensions involve increasing difficulties, which are overcome for the groupoid case in Brown and Higgins (1981a), and in the category case in Higgins (2005).

4.7 Local-to-Global (LG) Construction Principles consistent with Quantum Axiomatics

A novel approach to QST construction in AQFT may involve the use of generalized fundamental theorems of algebraic topology from specialized, 'globally well-behaved' topological spaces, to arbitrary ones. In this category are the generalized, *Higher Homotopy van Kampen theorems (HHvKT)* of Algebraic Topology with novel and unique non-Abelian applications. Such theorems greatly aid the calculation of higher homotopy of topological spaces. In the case of the Hurewicz theorem, this was generalized to arbitrary topological spaces (Spanier 1966), and establishes that certain homology groups are isomorphic to 'corresponding' homotopy groups of an arbitrary topological space. Brown et al. (1999, 2004) went further and generalized the van Kampen theorem, at first to fundamental groupoids on a set of base points (Brown 1967), and then, to higher dimensional algebras involving, for example, homotopy double groupoids and 2-categories (Brown 2004). The more sensitive *algebraic invariant* of topological spaces seems to be, however, captured only by *cohomology* theory through an algebraic *ring* structure that is not accessible either in homology theory, or in the existing homotopy theory. Thus, two arbitrary topological spaces that have isomorphic homology groups may not have isomorphic cohomological ring structures, and may also not be homeomorphic, even if they are of the same homotopy type. The corollary of this statement may lead to an interesting cohomology-based classification in a category of certain Coh topological spaces that have isomorphic ring structures and are also homeomorphic. Furthermore, several *non-Abelian* results in algebraic topology could only be derived from the Higher Homotopy van Kampen Theorem (cf. Brown and Loday 1987; Brown 2004), so that there could be links of such results to the expected '*non-commutative* geometrical' structure of quantized spacetime (Connes 1994). In this context, the important algebraic-topological concept of a *Fundamental Higher Homotopy Groupoid* (replacing the traditional higher homotopy groups) might be applied to a Quantum Topological Space (QTS) as a "partial classifier" of the *invariant* topological properties of quantum spaces of *any* dimension; quantum topological spaces are then linked together in a *crossed complex over a quantum groupoid* (Baianu et al. 2006), thus suggesting the construction of global topological structures from local ones with well-defined quantum homotopy groupoids. The latter theme is then further pursued through defining locally topological groupoids to which the Globalization Theorem may be applied to give a *Holonomy Groupoid* encapsulating notions of change of phase.

We shall see and also next speculate how this concept of a Locally Lie Groupoid can be applied in the context of *Algebraic Quantum Field Theory* (AQFT) to provide a Local-to-Global Construction of Quantum Space Times in the presence of intense gravitational fields without generating singularities as in GR, even in the presence of black holes, 'with or without hair'. The result of this construction is a *Quantum Holonomy Groupoid* (QHG) which is unique up to an isomorphism.

4.7.1 Potential Applications of Novel Algebraic Topology Methods to the Fundamental Ontology Level and the Problems of Quantum Spacetimes

With the advent of Quantum Groupoids, Quantum Algebra and Quantum Algebraic Topology, several fundamental concepts and new theorems of Algebraic Topology may also acquire an enhanced importance through their potential applications to current problems in theoretical and mathematical physics, such as those described in an available preprint (Baianu et al. 2006), and also in Baianu et al. (2007a, b *in this issue*.) Such potential applications will be briefly outlined at the conclusion of this section as they are based upon the following ideas, algebraic topology concepts and constructions.

Traditional algebraic topology works by several methods, but all involve going from a space to some form of combinatorial or algebraic structure. The earliest of these methods was 'triangulation': a space was supposed represented as a simplicial complex, i.e., was subdivided into simplices of various dimensions glued together along faces, and an algebraic structure such as a chain complex was built out of this simplicial complex, once assigned an orientation, or, as found convenient later, a total order on the vertices. Then in the 1940s a convenient form of singular theory was found, which assigned to any space X a 'singular simplicial set' SX, using continuous mappings from $\Delta^n \to X$, where Δ^n is the standard n-simplex. From this simplicial set, the whole of the weak homotopy type could in principle be determined. An alternative approach was found by Čech, using open covers $\mathcal{U}$ of X to determine a simplicial set $N\mathcal{U}$, and then refining the covers to get better 'approximations' to X. It was this method which Grothendieck discovered could be extended, especially combined with new methods of homological algebra, and the theory of sheaves, to give new applications of algebraic topology to algebraic geometry, via his theory of schemes.

The 600-page manuscript, '*Pursuing Stacks'* by Grothendieck (2007) was aimed at a *non-Abelian homological algebra*; it did not achieve this goal but has been very influential in the development of weak n-categories and other *higher categorical structures*.

Now if quantum mechanics is to reject the notion of a continuum, then it must also reject the notion of the real line and the notion of a path. How then is one to construct a homotopy theory?

One possibility is to take the route signalled by Čech, and which later developed in the hands of Borsuk into 'Shape Theory' (see Cordier and Porter 1989). Thus a quite general space is studied by means of its approximation by open covers. Yet another possible approach is briefly pointed out in the next subsection.

4.7.2 *Potential Applications of the Van Kampen Theorem to Crossed Complexes. Possible Representations of Quantum SpaceTime in terms of Quantum Crossed Complexes over a Quantum Groupoid*

There are several possible applications of the higher homotopy van Kampen theorem in the development of physical representations of a quantized spacetime 'geometry'. For example, a possible application of the generalized van Kampen theorem is the construction of the initial, quantized spacetime as the *unique colimit* of *quantum causal sets (posets)* which was precisely described in Baianu et al. (2006) in terms of *the nerve of an open covering* $N\mathcal{U}$ of the topological space X that would be isomorphic to a k-simplex K underlying X. The corresponding, *non-commutative* algebra Ω associated with the finitary T_0-poset $P(S)$ is *the Rota algebra* Ω discussed in Raptis and Zapatrin (2000) and the *quantum topology* T_0 is defined by the partial ordering arrows for regions that can overlap, or superpose, coherently (in the quantum sense) with each other. When the poset $P(S)$ contains $2N$ points we write this as $P_{2N}(S)$. The *unique* (up to an isomorphism) $P(S)$ in the *colimit*, $\lim_{\leftarrow} P_N X$, recovers a space homeomorphic to X (Sorkin 1991). Other non-Abelian results derived from the generalized van Kampen theorem are discussed by Brown et al. (2002) and Brown (2004), respectively.

5 Categorical Ontology: Basic Structure and the Theory of Levels

An effective Categorical Ontology requires, or generates—in the constructive sense—a '*structure*' rather than a discrete set of items. The classification process itself generates collections of items, as well as a *hierarchy of higher-level* '*items'* of items, thus facing perhaps certain possible antimonies if such collections were to be just sets that are subject to the Axiom of Choice and problems arising from the set membership concept at different levels.

The categorical viewpoint as emphasized by Lawvere, etc., is that the key structure is that of *morphisms*, seen, for example, as abstract relations, mappings, functions, connections, interactions, transformations, etc. Therefore, in this section we shall consider both the Categorical viewpoint in the Ontology of Space and Time in complex/super-complex systems, as well as the fundamental structure of Categorical Ontology, as for example in the Ontological Theory of Levels (Poli 2001a, b; 2006a, b) which will be discussed briefly in the next section.

5.1 Towards a Formal Theory of Levels

The first subsection here will present the fundamentals of the ontological theory of levels together with its further development in terms of mathematical categories, functors and natural transformations, as well as the necessary non-commutative generalizations of Abelian categorical concepts to non-Abelian formal systems and theories.

5.1.1 Fundamentals of Poli's Theory of Levels

The ontological theory of levels (Poli 2001a, b, 2006a, b, 2008) considers a hierarchy of *items* structured on different levels of existence with the higher levels *emerging* from the lower, but usually *not* reducible to the latter, as claimed by widespread reductionism. This approach draws from previous work by Hartmann (1935, 1952) but also modifies and expands considerably both its vision and range of possibilities. Thus, Poli (1998, 2001a, 2006a, b, 2008) considers four realms or *levels* of reality: Material-inanimate/Physico-chemical, Material-living/Biological, Psychological and Social. We harmonize this theme by considering categorical models of complex systems in terms of an evolutionary dynamic viewpoint using the mathematical methods of category theory which afford describing the characteristics and binding of levels, besides the links with other theories which, *a priori*, are essential requirements. The categorical techniques which form an integral part of the discussion provide a means of describing a hierarchy of levels in both a linear and interwoven, or *entangled*, fashion, thus leading to the necessary bill of fare: emergence, higher complexity and open, non-equilibrium/irreversible systems. We further stress that the categorical methodology intended is *intrinsically 'higher dimensional'* and can thus account for 'processes between processes…' within, or between, the levels—and sub-levels—in question.

Whereas a strictly Boolean classification of levels allows only for the occurrence of *discrete* ontological levels, and also does not readily accommodate either *contingent* or *stochastic sub-levels*, the LM-logic algebra is readily extended to *continuous*, *contingent* or even *fuzzy* (Baianu and Marinescu 1968) sub-levels, or levels of reality (cf. Georgescu 2006; Baianu 1977, 1987a, b, 2007; Baianu et al. 2006). Clearly, a Non-Abelian Ontology of Levels would require the inclusion of either Q- or LM- logics algebraic categories because it begins at the fundamental quantum level—where Q-logic reigns—and 'rises' to the emergent ultra-complex level(s) with 'all' of its possible sub-levels represented by certain LM-logics.

Poli (2006a) has stressed a need for understanding *causal and spatiotemporal* phenomena formulated within a *descriptive categorical context* for theoretical levels of reality. There are three main points to be taken into account: differing spatiotemporal regions necessitate different (levels of) causation, for some regions of reality analytic reductionism may be inadequate, and there is the need to develop a *synthetic* methodology in order to compensate for the latter, although one notes (cf. Rosen 1999) that analysis and synthesis are not the exact inverse of each other. Following Poli (2001a, b), we consider a causal dependence on levels, somewhat apart from a categorical dependence. At the same time, we address the *internal dynamics*, the *temporal rhythm, or cycles*, and the subsequent unfolding of reality. The genera of corresponding concepts such as 'processes', 'groups', 'essence', 'stereotypes', and so on, can be simply referred to as '*items*' which allow for the existence of many forms of causal connection (Poli 2001a). The implicit meaning is that the *irreducible multiplicity* of such connections converges, or it is ontologically integrated within a *unified synthesis*. Rejecting reductionism thus necessitates accounting for an irreducible multiplicity of ontological levels, and possibly the ontological acceptance of many worlds also. In this regard, the Brentano hypothesis is that the class of physical phenomena and the class of psychological (or spiritual) phenomena are *complementary*; in other words, physical categories were said to be 'orthogonal' to psychological categories (Poli 2006a, b).

As befitting the situation, there are devised *universal* categories of reality in its entirety, and also subcategories which apply to the respective sub-domains of reality. Following Poli (2001a, b), the ontological procedures in question provide:

- coordination between categories (for instance, the interactions and parallels between biological and ecological reproduction as in Poli 2001a, b);
- modes of dependence between levels (for instance, how the co-evolution/interaction of social and mental realms depend and impinge upon the material);
- the categorical closure (or completeness) of levels.

Already we can underscore a significant component of this essay that relates the ontology to geometry/topology; specifically, if a level is defined via 'iterates of local procedures' (cf. 'items in iteration', Poli 2001a, b), then we have some handle on describing its intrinsic governing dynamics (with feedback) and, to quote Poli (2001a, b), to 'restrict the *multi-dynamic* frames to their linear fragments'.

On each level of this ontological hierarchy there is a significant amount of connectivity through inter-dependence, interactions or general relations often giving rise to complex patterns that are not readily analyzed by partitioning or through stochastic methods as they are neither simple, nor are they random connections. But we claim that such complex patterns and processes have their logico-categorical representations quite apart from classical, Boolean mechanisms. This ontological situation gives rise to a wide variety of networks, graphs, and/or mathematical categories, all with different connectivity rules, different types of activities, and also a hierarchy of super-networks of networks of subnetworks. Then, the important question arises what types of basic symmetry or patterns such super-networks of items can have, and how do the effects of their sub-networks percolate through the various levels. From the categorical viewpoint, these are of two basic types: they are either *commutative* or *non-commutative*, where, at least at the quantum level, the latter takes precedence over the former, as we shall further discuss and explain in the following sections.

5.2 Categorical Logics of Processes and Structures: Universal Concepts and Properties

The logic of classical events associated with either mechanical systems, mechanisms, universal Turing machines, automata, robots and digital computers is generally understood to be simple, *Boolean* logic. The same applies to Einstein's GR. It is only with the advent of quantum theories that quantum logics of events were introduced which are *non-commutative*, and therefore, also *non-Boolean*. Somewhat surprisingly, however, the connection between quantum logics (QL) and other *non-commutative* many-valued logics, such as the Łukasiewicz logic, has only been recently made (Dalla Chiara et al. 2004 and refs. cited therein; Baianu et al. 2004; Baianu et al. 2004, 2006). The universal properties of categories of LM-logic algebras are, in general, categorical constructions that can be, in particular cases, 'just universal objects'—which still involve categorical constructions; therefore such a danger of confusion does not arise at all in this context. Such considerations are of potential interest for a wide range of complex systems, as well as quantum ones, as it has been pointed out previously (Baianu 1977, 2004; Baianu et al. 2004, 2006). Furthermore, both the concept of 'Topos' and that of variable category, can be further generalized by the involvement of *many-valued* logics, as for example in the case of 'Łukasiewicz-Moisil, or LM Topos' (Baianu et al. 2006). This is especially relevant for the development of *non-Abelian dynamics* of complex and super-complex systems; it may also be essential for understanding human consciousness (as it will be discussed in the context of Sects. 6 and 7).

Whereas the hierarchical theory of levels provides a powerful, systematic approach to categorical ontology, the foundation of science involves *universal* models and theories pertaining to different levels of reality. Such theories are based on axioms, principles, postulates and laws operating on distinct levels of reality with a specific degree of complexity. Because of such distinctions, inter-level principles or laws are rare and over-simplified principles abound. As relevant examples, consider the Chemical/Biochemical Thermodynamics, Physical Biochemistry and Molecular Biology fields which have developed a rich structure of specific-level laws and principles, however, without 'breaking through' to the higher, emergent/integrative level of organismic biology. This does not detract of course from their usefulness, it simply renders them incomplete as theories of biological reality. With the possible exceptions of Evolution and Genetic Principles or Laws, Biology has until recently lacked other universal principles for highly complex dynamics in organisms, populations and species, as it will be shown in the following sections. One can therefore consider Biology to be at an almost 'pre-Newtonian' stage by comparison with either Physics or Chemistry.

It will be therefore worthwhile considering the structure of scientific theories and how it could be improved to enable the development of emergence principles for various complexity levels, including those of the *inter(active)-level* types.

5.3 The Object-based Approach versus Process-based, Dynamic Ontology

Part of the very essence of existence is its dynamic and impermanent character; the Universe itself is constantly changing as represented by the brilliant intuition and metaphor of Heraclitus: "*Panta rhei*". On the other hand, science and especially mathematics is seeking the most general, unchanged patterns that form the underlying essence of all reality; the latter may thus be viewed in a Platonic sense as the abstract unity of all items, large and small, either finite or infinite in their physical extent. Hence lies a potential, creative tension and challenge between Ontology and Science, especially Mathematics: whereas the former focuses on objective existence, classification and the essence of items, the latter focuses on the 'Platonic', unchangeable/abstract essence of an 'ideal, universal reality' assumed to be present 'behind' (e.g., underlying) all items, be they objects/things or processes. On the one hand, the *categorical* classification in ontology has a very broad philosophical meaning with increasingly practical connotations; on the other hand, the *categorical* approach provided by mathematics provides a powerful, precise tool for ontological studies whose full scope is yet to be explored.

This apparent 'clash' of the actual, existential essence and its abstract form, either discovered or created by the human mind, may be the origin of the dualistic, Cartesian philosophical viewpoint; it is in marked contrast to widespread Eastern philosophical standpoints—as for example, Buddhist or Taoist—that adopt a decidedly monistic, or holistic view of the world which chooses not to separate the mind's essence from that of its surrounding universe. Such apparently divergent, and indeed oftentimes considered as contradictory, philosophies may however *both* play their important roles for harmonizing technological development with environmental protection from those technologies that are crude, harmful, wasteful and inefficient. Philosophical biases notwithstanding, Ontology can no longer ignore the trends towards unity in Science, especially mathematics and physics, as well as the needs for unity between different societies, both West and East, or the needs for harmony between the global human society and its environment. On the other hand, it seems that ontological classification is becoming increasingly important in information-based and highly technological, modern societies. Thus, a balanced, rather than biased, philosophical approach to Ontology is here advocated.

In classifications, such as those developed over time in Biology for organisms, or in Chemistry for chemical elements, the *objects* are the basic items being classified even if the 'ultimate' goal may be, for example, either evolutionary or mechanistic studies. Rutherford's comment is pertinent in this context:

> "There are two major types of science: physics or stamp collecting."

An ontology based strictly on object classification may have little to offer from the point of view of its cognitive content. It is interesting that many psychologists, especially behavioural ones, emphasize the object-based approach rather than the process-based approach to the ultra-complex process of consciousness occurring 'in the mind'—with the latter thought as an 'object'. Nevertheless, as early as the work of William James in 1850, consciousness was considered as a *'continuous stream that never repeats itself'*—a Heraclitian concept that does also apply to super-complex systems and life, in general. Similarly, general systems theorists emphasize object-based representations of systems rather than process-based ones, akin to those abundantly present in engineering/technology.

On the other hand, it is often thought that the object-oriented approach can be readily converted from an ontological viewpoint into a process-based one. It would seem that the answer to this question depends critically on the ontological level selected. For example, at the quantum level, *object and process become inter-mingled*. Either comparing or moving between levels, requires ultimately a process-based approach, especially in Categorical Ontology where relations and inter-process connections are essential to developing any valid theory. At the fundamental level of 'elementary particle physics' however the answer to this question of process versus object becomes quite difficult as a result of the 'blurring' between the particle and the wave concepts. Thus, it is well-known that any 'elementary quantum object' is considered by all accepted versions of quantum theory not just as a 'particle' or just a 'wave' but both: the quantum 'object' is *both* wave and particle, *at the same-time*, a proposition accepted since the time when it was proposed by de Broglie. At the quantum microscopic level, the object and process are inter-mingled, they are no longer separate items. Therefore, in the quantum view the 'object-particle' and the dynamic process-'wave' are united into a single dynamic entity or item, called *the wave-particle quantum*, which strangely enough is *neither discrete nor continuous*, but both at the same time, thus 'refusing' intrinsically to be an item consistent with Boolean logic. Ontologically, the quantum level is a very important starting point which needs to be taken into account by any theory of levels that aims at completeness. Such completeness may not be attainable, however, simply because an 'extension' of Gödel's theorem may hold here also. The fundamental quantum level is generally accepted to be dynamically, or intrinsically *non-commutative*, in the sense of the *non-commutative quantum* *logic* and also in the sense of *non-commuting quantum operators* for the essential quantum observables such as position and momentum.

Therefore, any 'complete' theory of levels, in the sense of incorporating the quantum level, is thus—*mutatis mutandis*—*non-Abelian*. Therefore, at this point, there are two basic choices in Categorical Ontology: either to include the quantum level and thus generate a non-Abelian Ontology founded upon the non-commutative quantum logic, or to exclude the 'fundamental' level and remain strictly Abelian, that is accepting only strict determinism/linear causality and a commutative logic for its foundation such as Boolean or Brouwer-intuitionistic logic.

Furthermore, as the non-Abelian case is the more general one, from a strictly formal viewpoint, a non-Abelian Categorical Ontology is the preferred choice. Nevertheless, from the point of view of simplicity (see Occam's razor) or 'economy of thought', the *Abelian* form of Categorical Ontology may be often selected by reductionists, mathematicians or engineers, for example; the commutativity and/or symmetry present in the Abelian theory can be seen as quite attractive either from an esthetic viewpoint or from the standpoint of the rapid elaboration/development of Categorical Ontology. Regardless of the latter views, the paradigm-shift towards a *non-Abelian Categorical Ontology* has already started (Brown et al. 2008: '*Non-Abelian Algebraic Topology*'; Baianu et al. 2006: NA-QAT).

5.4 From Object and Structure to Organismic Functions and Relations: A Process-based Approach to Ontology

Wiener (1950) made the important remark that implementation of complex functionality in a (complicated) machine requires also the design and construction of a complex structure. A similar argument holds *mutatis mutandis*, or by induction, for variable machines, variable automata and variable dynamic systems (Baianu 1970 through 1986; Baianu and Marinescu 1974); therefore, if one represents organisms as variable dynamic systems, one *a fortiori* requires a *super-complex structure* to enable or entail *super-complex dynamics*, and indeed this is the case for organisms with their extremely intricate structures at both the molecular and *supra-molecular* levels. It is an open question how the first organism has emerged through *self-assembly*, or 'self-construction'. On the other hand, for simple automata, or machines, there is the famous mathematical result about the existence of an *unique, Universal Turing Automaton* (uUTA) that can build or construct any other automaton. Furthermore, the category of all automata, and also the category of **(M,R)**-systems have both limits and colimits (Baianu 1973; Baianu and Marinescu 1974; Baianu, 1987a, b, 2007). It would seem that the uUTA is isomorphic to the *colimit construction* in the category of all automata (Baianu 1973). One can also conjecture, and indeed, perhaps even prove formally, that a certain Variable Universal Automaton (VUA) also exists which can build *any* other variable automata; one may also hypothesize the metamorphosis of a certain selected variable automaton through an evolution-like process into variable automata of higher complexity and higher dimensionality, thus mimicking ontogeny, and possibly also phylogeny. Thus, an analogy is here suggested with the primordial organism as a specially selected variable universal automaton. Furthermore, the colimit of such an evolving, or developing, *direct system of variable automata* may be conjectured to exist as a VUA structure; such a VUA would then be a universal object in the supercategory of variable automata, and *a fortiori* would also be unique.

Although the essence of super- and ultra-complex systems is in the *interactions, relations and dynamic transformations* that are ubiquitous in such higher-level ontology, surprisingly many a psychology, cognitive and an ontology approach begins with a very strong emphasis on *objects* rather than relations. It would also seem that a basic 'trick' of human consciousness is to pin a subjective sensation, perception and/or feeling on an internalized *object*, or vice-versa to represent/internalize an object in the form of an internal symbol in the mind. The example often given is that of a human child's substituting a language symbol, or image for the *mother 'object'*, thus allowing 'her permanent presence' in the child's consciousness. Clearly, however, a complete approach to ontology must also include *relations and interconnections* between items, with a strong emphasis on *dynamic processes, complexity* and *functionality* of systems, which all require an emphasis on general relations, *morphisms* and the *categorical viewpoint* of ontology.

The *process-based approach* to universal ontology is therefore essential to an understanding of the ontology of levels, hierarchy, complexity, anticipatory systems, Life, Consciousness and Universe(s). On the other hand, the opposite approach, based on objects, is perhaps useful only at the initial cognitive stages in experimental science, as the reductionist approach of 'cutting off' functional connectivities and relations, retaining the object pieces, and then attempting 'to put back together the pieces' does *not* work for complex, super-complex or ultra-complex systems. Psychologists would be horrified at the proposition of 'taking a mind to pieces and attempting to put it back together afterwards'; not only it would not work, but it would also be *highly unethical*. One could also argue that if chimpanzees are very close to humans genetically (and maybe also to some extent functionally, even though separated from a 'common', hypothetical ancestor by 5–8 million years of evolution), their use in reductionist-inspired neurophysiological 'experiments' involving cutting and poking with electrodes, thus presumably, altering their chimpanzee 'consciousness' is also unethical

5.5 Categorical Representations in the Ontological Theory of Levels: From Abelian Categories to Non-Abelian Theories in Ontology

General system analysis seems to require formulating ontology by means of categorical concepts (Poli 2008; Baianu and Poli 2008). Furthermore, category theory appears as a natural framework for any general theory of transformations or dynamic processes, just as group theory provides the appropriate framework for classical dynamics and quantum systems with a finite number of degrees of freedom. Therefore, we shall adopt here a categorical approach as the starting point, meaning that we are looking for *"what is universal"* (in some domain, or in general), and that for simple systems this involves *commutative* modelling diagrams and structures (as, for example, in Fig. 1 of Rosen 1987). Note that this ontological use of the word '*universal'* is quite distinct from the mathematical use of '*universal property'*, which means that a property of a construction on particular objects is defined by its relation to *all* other objects (i.e., it is a *global* attribute), usually through constructing a morphism, since this is the only way, in an *abstract* category, for objects to be related. With the first (ontological) meaning, the most universal feature of reality is that it is *temporal*, i.e., it changes, it is subject to countless transformations, movements and alterations. In this select case of *universal temporality*, it seems that the two different meanings can be brought into superposition through appropriate formalization. Furthermore, *concrete* categories may also allow for the representation of ontological 'universal items' as in certain previous applications to *cat-neurons*, or categories of neural networks (Baianu 1972; Ehresmann and Vanbremeersch 1987, 2006; Healy and Caudell 2004,2006).

As we shall be considering here only a few special cases of modelling diagrams that include simple, reductionist systems in order to compare them with super-complex biological systems, the following discussion in Sects. 5 through 7 will require just the use of such 'concrete' categories of 'sets with structure' (e.g., groups, groupoids, crossed complexes, etc.) For general categories, however, each object is a kind of a Skinnerian black box, whose only exposure is through input and output, i.e., the object is given by its *connectivity* through various morphisms, to other objects. For example, the opposite of the category of sets has objects but these have *no structure* from the categorical viewpoint. Other types of category are important as expressing useful relationships on structures, for example *lextensive* categories, which have been used to express a general van Kampen theorem by Brown (1996), and Brown and Janelidze (1997, 2004).

This concrete categorical approach seems also to provide an elegant formalization that matches the ontological theory of levels briefly described above. The major restriction—as well as for some, attraction—of the 3-level categorical construction outlined above seems to be its built-in *commutativity* (see also Sects. 2 and 3 for further details). Note also how 2-arrows become '3-objects' in the meta-category, or '3-category', of functors and natural transformations. This construction has already been considered to be suitable for representing dynamic processes in a generalized Quantum Field Theory. The presence of mathematical structures is just as important for highly complex systems, such as organisms, whose organizational structure—in this mathematical and biological function/physiological sense—may be superficially apparent but difficult to relate unequivocally to anatomical, biochemical or molecular 'structures'. Thus, abstract mathematical structures are developed to define *relationships*, to deduce and calculate, to exploit and define analogies, since *analogies are between relations* between things rather than between things themselves.

As *structures* and *relations* are present at the very core of mathematical developments (Ehresmann 1965, 1966), the theories of categories and toposes distinguish at least two fundamental types of items: *objects* and *arrows* (also called suggestively '*morphisms'*).

The object structure is categorically investigated through its relations with other objects, or otherwise objects are categorically studied by ‘transforming’, or representing them by morphisms, for example as in the case of natural transformations.

Thus, first-level arrows may represent mappings, relations, interactions, dynamic transformations, and so on, whereas categorical objects are usually endowed with a selected type of structure only in ‘concrete’ categories of ‘sets with structure’. Note, however, that simple sets have only the ‘discrete topology structure’, consisting of just discrete elements, or points.

A description of a new structure is thus in some sense a development of part of a new language. The notion of *structure* is also related to the notion of *analogy*. It is one of the triumphs of the mathematical theory of categories in the 20th century to make progress in *unifying* mathematics through the finding of *analogies* between various behaviour of structures across different areas of mathematics. This theme is further elaborated in the article by Brown and Porter (2002) who argue that many analogies in mathematics, and in many other areas, are *not* between objects themselves but *between the relations* between objects. Here, we mention as an example, only the categorical notion of a *pushout*, which was used in Sect. 4 in discussing the higher homotopy, generalized van Kampen theorems. A pushout has the same definition in different categories even though the construction of pushouts in these categories may be widely different. Thus, focusing on the *constructions* rather than on the *universal properties* may lead to a failure to see the analogies. Super-pushouts, on the other hand, were reported to be involved in multi-stability and metamorphoses of living organisms (Baianu 1970). Ehresmann developed new concepts and new language which have been very influential in mathematics; we mention here only those of holonomy groupoid, Lie groupoid, fibre bundles, foliations, germs and jets. There are other concepts whose time perhaps is just coming or has yet to come: included here might be ordered groupoids, *variable groupoids* and *multiple categories*. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures. For example, we are also seeking *non-Abelian* methods for higher dimensional local-to-global problems in homotopy theory.

One must note in the latter case above the use of a very different meaning of the word ‘structure’, which is quite distinct from that of the organizational/physiological and mathematical structure introduced at the beginning of this section. Even though concrete, molecular or anatomical ‘structures’ could also be defined with the help of ‘concrete sets with structure’, the physical structures representing ‘anatomy’ are very different from those representing physiological-functional/organizational structures. Further aspects of this representation problem for systems with highly complex dynamics, together with their structure–functionality relationships, will be discussed in the following sections.

In reference to the above discussion, one of the major goals of category theory is to see how the properties of a particular mathematical structure, say S, are reflected in the properties of the category $\mathsf{Cat}(S)$ of all such structures and of morphisms between them. Thus the first step in category theory is that a definition of a structure should come with a definition of a morphism of such structures. Usually, but not always, such a definition is obvious. The next step is to compare structures. This might be obtained by means of a *functor* $A : \mathsf{Cat}(S) \longrightarrow \mathsf{Cat}(T)$. Finally, we want to compare such functors $A, B : \mathsf{Cat}(S) \longrightarrow \mathsf{Cat}(T)$. This is done by means of a natural transformation η: $A \Rightarrow B$. Here η assigns to each object X of $\mathsf{Cat}(S)$ a morphism $\eta(X) : A(X) \longrightarrow B(X)$ satisfying a commutativity condition for any morphism $a : X \longrightarrow Y$. In fact we can say that η assigns to each morphism a of $\mathsf{Cat}(S)$ a commutative square of morphisms in $\mathsf{Cat}(T)$ (such as (2.7)). This notion of *natural transformation* is at the heart of category theory. As Eilenberg-Mac Lane write: "*to define natural transformations one needs a definition of functor, and to define the latter one needs a definition of category*".

As explained in Sect. 2, the second level arrows, or 2-arrows ('*functors*') representing relations, or comparisons, between the first level 'concrete' categories of 'sets with structure' do not 'look inside' the 1-objects, which may appear as necessarily limiting the mathematical construction; however, the important ability to 'look inside' 1-objects at their structure, for example, is recovered by the third level arrows, or 3-arrows, in terms of natural transformations. For example, if A is an object in a mathematical category C, E is a certain 'corresponding' object in a category D and F is a covariant functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, such that $F(A) = E$, then one notes that F carries the whole object A into the category D without 'looking' inside the object A at its components; in the case when A is a set the functor F does not 'look' at the elements of A when it 'transforms' the whole set A into the object E (which does not even have to be a set; a functor F, therefore, does not act like a 'mapping' on elements). On the other hand, natural transformations in the case of *concrete* categories do define mappings of objects with structure by acting first on functors, and then by imposing the condition of naturality on diagrams that also include comparisons between *functorial mappings of morphisms*.

From the point of view of mathematical modelling, the mathematical theory of categories models the dynamical nature of reality by representing temporal changes through either *variable* categories or through *toposes*. According to Mac Lane and Moerdijk (1992) certain variable categories can also be generated as a topos. For example, the category of sets can be considered as a topos whose only generator is just a single point. A variable category of varying sets might thus have just a generator set.

The claim advanced by several recent textbooks and reports is that standard topos theory may also suit to a significant degree the needs of complex systems. Such claims, however, do not seem to draw any significant, qualitative ontological distinction between 'simple' and 'complex' systems, and furthermore, they do not satisfy also the second condition (naturality of modelling diagrams, as pointed out in Rosen 1987). As it will be shown in Sect. 6, a qualitative distinction *does exist*, however, between organisms—considered as complex systems—and 'simple', inanimate dynamical systems, in terms of the modelling process and the type of predictive mathematical models or representations that they can have (Rosen 1987, and also, previously, Baianu 1968 through 1987).

5.5.1 *A Hierarchical, Formal Theory of Levels. Commutative and Non-Commutative Structures: Abelian Category Theory versus Non-Abelian Theories*

One could formalize-for example as outlined in Baianu and Poli (2008, TAO-1)—the hierarchy of multiple-level relations and structures that are present in biological, environmental and social systems in terms of the mathematical Theory of Categories, Functors and Natural Transformations (TC-FNT, Mac Lane 2000). On the first level of such a hierarchy are the links between the system components represented as '*morphisms'* of a structured category which are subject to several axioms/restrictions of Category Theory, such as *commutativity* and associativity conditions for morphisms, functors and natural transformations. Among such mathematical structures, *Abelian* categories have particularly interesting applications to rings and modules (Popescu 1973; Gabriel 1962) in which commutative diagrams are essential. Commutative diagrams are also being widely used in Algebraic Topology (Brown 2004; May 1999). Their applications in computer science also abound.

Then, on the second level of the hierarchy one considers '*functors*', or links, between such first level categories, that compare categories without 'looking inside' their objects/ system components.

On the third level, one compares, or links, functors using *'natural transformations'* in a 3-category (meta-category) of functors and natural transformations. At this level, natural transformations not only compare functors but also look inside the first level objects (system components) thus 'closing' the structure and establishing 'the universal links' between items as an integration of both first and second level links between items. The advantages of this constructive approach in the mathematical theory of categories, functors and natural transformations have been recognized since the beginnings of this mathematical theory in the seminal paper of Eilenberg and Mac Lane (1945). Note, however, that in general categories the objects have no 'inside', though they may do so for example in the case of 'concrete' categories.

A relevant example of applications to the natural sciences, e.g., neurosciences, would be the higher-dimensional algebra representation of processes of cognitive processes of still more, linked sub-processes (Brown 2004). Additional examples of the usefulness of such a categorical constructive approach to generating higher-level mathematical structures would be that of groups of groups of items, 2-groupoids, or double groupoids of items. The hierarchy constructed above, up to level 3, can be further extended to higher, n-levels, always in a consistent, natural manner, that is using commutative diagrams. Let us see therefore a few simple examples or specific instances of commutative properties. The type of global, natural hierarchy of items inspired by the mathematical TC-FNT has a kind of *internal symmetry* because at all levels, the link compositions are *natural*, that is, all link compositions that exist are subject to the logical restrictions: *transitive*, i.e., $x < y$ and $y < z \Longrightarrow x < z$, or $f : x \longrightarrow y$ and $g : y \longrightarrow z \Longrightarrow h : x \longrightarrow z$, yielding a composition $h = g \circ f$. This general property of such link composition chains or diagrams involving any number of sequential links is called *commutativity*, and is often expressed as a *naturality* *condition for diagrams*. This key mathematical property also includes the mirror-like symmetry $x \bigstar y = y \bigstar x$ when x and y are operators and the symbol '★' represents the operator multiplication. Then, the equality of $x \bigstar y$ with $y \bigstar x$ defines the statement that "the x and y operators *commute*"; in physical terms, this translates into a sharing of the same set of eigenvalues by the two commuting operators, thus leading to 'equivalent' numerical results i.e., up to a multiplication constant); furthermore, the observations X and Y corresponding, respectively, to these two operators would yield the same result if X is performed before Y in time, or if Y is performed first followed by X. This property, when present, is very convenient for both mathematical and physical applications (such as those encountered in quantum mechanics). Unfortunately, not all operators 'commute', and not all categorical diagrams or mathematical structures are, or need be, commutative. *Non-commutativity* may therefore appear as a result of 'breaking' the 'internal symmetry' represented by commutativity. As a physical analogy, this might be considered a kind of *'symmetry breaking'* which is thought to be responsible for our expanding Universe and CPT violation, as well as other physical phenomena such as phase transitions and superconductivity (Weinberg 1995, 2000).

5.6 Ontological Organization of Systems in Space and Time: Items Categorical Classification in SpaceTime Reference Frames

Ontological classification based on items involves the organization of concepts, and indeed theories of knowledge, into a *hierarchy of categories of items at different levels of 'objective reality'*, as reconstructed by scientific minds through either a *bottom-up* (induction, synthesis, or abstraction) process, or through a *top-down* (deduction) process (Poli 2008), which proceeds from abstract concepts to their realizations in specific contexts of the 'real' world. A more formal approach to this problem will be considered in the following Sect. 6, with several ontological examples being also provided in subsequent sections and three related articles (Baianu and Poli 2008; Baianu et al. 2007a, b; in this issue). The conceptual foundation for such effective formulations in terms of different level categories and their higher-order relations has been already outlined in the preceding subsections.

6 Modelling and Classification of Systems in Relation to the Categorical Theory of Levels: Simple, Complex and Super-Complex Systems. Logical Models of Higher Complexity Levels

The mathematician John von Neumann regarded 'complexity' as a measurable property of natural systems below the threshold of which systems behave 'simply', but above which they evolve, reproduce, self-organize, etc. Rosen (1987) proposed a refinement of these ideas by a more exact classification between 'simple' and 'complex'. Simple systems can be characterized through representations which admit maximal models, and can be therefore re-assimilated via a hierarchy of informational levels. Besides, the duality between dynamical systems and states is also a characteristic of such simple dynamical systems. It was claimed that any 'natural' system fits this profile. But the classical assumption that natural systems are simple, or 'mechanistic', is too restrictive since 'simple' is applicable only to machines, closed physicochemical systems, computers, or any system that is recursively computable. On the other hand, an *ultra-complex* system as applied to psychological–sociological structures is describable in terms of *variable categories* or structures, and cannot be reasonably represented by a fixed state space for its entire lifespan. Replacements by limiting dynamical approximations lead to increasing system 'errors' and through such approximations a complex system can be viewed in its acting as a single entity, but not conversely.

Just as for simple systems, both *super-complex* and *ultra-complex* systems admit their own orders of causation, but the latter two types are different from the first—by inclusion rather than exclusion—of the mechanisms that control simple dynamical systems.

6.1 Formal Representations of Dynamic Systems as Stable Spacetime Structures

As defined in Baianu and Poli (2008), a system is a dynamical whole able to maintain its working conditions; the system definition is here spelt out in detail by the following, general definition, **D1**.

D1 A simple system is in general a bounded, but not necessarily closed, entity—here represented as a category of stable, interacting components with inputs and outputs from the system's environment, or as a supercategory for a complex system consisting of subsystems, or components, with internal boundaries among such subsystems.

As proposed by Baianu and Poli (2008) in order to define a system one therefore needs specify: (1) components or subsystems, (2) mutual interactions or links; (3) a separation of the selected system by some boundary which distinguishes the system from its environment; (4) the specification of the system's environment; (5) the specification of the system's categorical structure and dynamics; (6) a supercategory will be required when either components or subsystems need be themselves considered as represented by a category, i.e., the system is in fact a super-system of (sub) systems, as it is the case of emergent super-complex systems or organisms.

Point (5) claims that a system should occupy a macroscopic spacetime region: a system that comes into birth and dies off extremely rapidly may be considered either a short-lived process, or rather, a 'resonance'—an instability rather than a system, although it may have significant effects as in the case of 'virtual particles', 'virtual photons', etc., as in quantum electrodynamics and chromodynamics. Note also that there are many other, different mathematical definitions of 'systems' ranging from (systems of) coupled differential equations to operator formulations, semigroups, monoids, topological groupoids and categories. Clearly, the more useful system definitions include algebraic and/or topological structures rather than simple, structureless sets, classes or their categories (cf. Baianu 1970, and Baianu et al.

2006). The main intuition behind this first understanding of system is well expressed by the following passage:

> The most general and fundamental property of a system is the inter-dependence of parts/components/sub-systems or variables.

As discussed by Baianu and Poli (2008), *inter-dependence* thus consists in the existence of determinate relationships among the parts or variables as contrasted with randomness or extreme variability. In other words, inter-dependence is the presence or existence of a certain organizational order in the relationship among the components or subsystems which make up the system. It can be shown that such organizational order must either result in a stable attractor or else it should occupy a stable spacetime domain, which is generally expressed in *closed* systems by the concept of equilibrium. On the other hand, in non-equilibrium, open systems, one cannot have a static but only a *dynamic self-maintenance* in a 'state-space region' of the open system—which cannot degenerate to either an equilibrium state or a single attractor spacetime region. Thus, non-equilibrium, open systems capable of self-maintenance (seen as a form of autopoiesis) are also generic/structurally-stable: their perturbation from a homeostatic maintenance regime does not result either in completely chaotic dynamics with a single attractor or the loss of their stability. It may however involve an ordered process of change—a process that follows a determinate pattern rather than random variation relative to the starting point.

6.1.1 Selective Boundaries and Homeostasis. Varying Boundaries versus Horizons

Boundaries are especially relevant to *closed* systems. According to Poli (2008): "they serve to distinguish what is internal to the system from what is external to it", thus defining the fixed, overall structural topology of a closed system. By virtue of possessing boundaries, "a whole (entity) is something on the basis of which there is an interior and an exterior. The initial datum, therefore, is that of a difference, of something/a key attribute which enables a difference to be established between the whole closed system and environment." (cf. Baianu and Poli 2008). One notes however that a boundary, or boundaries, may change or be quite selective/directional—in the sense of dynamic fluxes crossing such boundaries—if the system is *open* and grows/develops as in the case of an organism, which will be thus characterized by a *variable* topology that may also depend on the environment, and is thus *context-dependent* as well. Perhaps the simplest example of a system that changes from *closed to open*, and thus has a *variable topology*, is that of a pipe equipped with a functional valve that allows flow in only one direction. On the other hand, a semi-permeable membrane such as a cellophane, thin-walled 'closed' tube—that allows water and small molecule fluxes to go through but blocks the transport of large molecules such as polymers through its pores—is *selective* and may be considered as a primitive/'simple' example of an open, selective system. Organisms, in general, are *open systems with variable topology* that incorporate both the valve and the selectively permeable membrane boundaries—albeit much more sophisticated and dynamic than the simple/fixed topology cellophane membrane—in order to maintain their stability and also control their internal structural order, or low microscopic entropy.

The formal definition of this important concept of 'variable topology' will be introduced in this essay for the first time in the context of the spacetime evolution of organisms, populations and species in Sect. 7.

As shown by Baianu and Poli (2008), an essential feature of boundaries in open systems is that they can be crossed by matter; however, all boundaries may be crossed by either fields or by quantum wave-particles if the boundaries are sufficiently thin, even in 'closed' systems. Thus, there are more open boundaries and less open ones, but they can all be crossed in the above sense. The boundaries of closed systems, however, cannot be crossed by molecules or larger particles. On the contrary, a horizon is something that one cannot reach or cross. In other words, a horizon is not a boundary. This difference between horizon and boundary might be useful in distinguishing between systems and their environment. "Since the environment is delimited by open horizons, not by boundaries capable of being crossed, it is not a system." (cf. Luhmann 1984, as cited in Poli 2008). We note here, however, that one can define both open horizons and varying boundaries in terms of variable topologies, but with different organization or structure. As far as open systems are concerned, the difference between inside and outside loses its common sense, or 'spatial' understanding. As a matter of fact, 'inside' doesn't anymore mean 'being placed within', but it means 'being part of' the system. One of the earlier forerunners of system theory clarified the situation in the following way: "Bacteria in the organism… represent complexes which are, in the organizational sense, not 'internal', but external to it, because they do not belong to the system of its organizational connections. And those parts of the system which go out of its organizational connections, though spatially located inside it, should also be considered as being… external." (Bogdanov 1981–84; as cited in Poli 2008). In essence, the attributes internal and external are first and foremost relative to the system, not to its actual location in physical space. The situation is, however, much less clear-cut in the case of viruses that insert themselves into the host genome and are expressed by the latter as if the viral genes 'belonged' to the host genome. Even though the host may not always recognize the viral genes as 'foreign', or 'external' to the host, their actions may become incompatible with the host organization as in the case of certain oncogenic viruses that cause the death of their host. These key attributes—internal and external—might also be taken as features describing the difference between the world of 'inanimate' things/machines and the world of organisms. In the mechanistic, 'linear' order of things or processes, the world is regarded as being made, or constituted, of entities which are outside of each other, in the sense that they exist independently in different regions of spacetime and interact through forces. By contrast, in a living organism, *each part grows in the context of the whole*, so that it does not exist independently, nor can it be said that it merely 'interacts' with the others, without itself being essentially affected in this relationship. The parts of an organism grow and develop together as a result of cell division, migration, and other related processes.

Boundaries may be fixed, clear-cut, precise, rigid, or they may be vague, blurred, mobile, varying/variable in time, or again they may be intermediate between these two typical cases, according to how the differentiation is structured. In the beginning there may be only a slightly asymmetric distribution in perhaps just one direction, but usually still maintaining certain symmetries along other directions or planes. Interestingly, for many multi-cellular organisms, including man, the overall symmetry retained from the beginning of development is bilateral—just one plane of mirror symmetry—from Planaria to humans. The presence of the head-to-tail asymmetry introduces increasingly marked differences among the various areas of the head, middle, or tail regions as the organism develops.

The formation of additional borderline phenomena occurs later as cells divide and differentiate thus causing the organism to grow and develop. Generally speaking, a closed boundary generates an internal situation characterized by limited differentiation. Open boundaries allow instead, and indeed stimulate, greater internal differentiation, and therefore, a greater degree of development of the system than would occur in the presence of just closed boundaries. In its turn, a population with marked internal differentiation, that is, with a higher degree of development, in addition to having numerous internal boundaries is also surrounded by a nebula of functional and non-coincident boundaries. This non-coincidence is precisely one of the principal reasons for the dynamics of the system. Efforts to harmonize, coordinate or integrate boundaries, whether political, administrative, social, etc, generate a dynamic which constantly maintains the boundary situation at a steady-state.

Note, however, that in certain ‘chaotic’ systems, organized patterns of spatial boundaries do indeed occur, albeit established as a direct consequence of their ‘chaotic’ dynamics. The multiplicity of boundaries, and the dynamics that derive from it, generate interesting phenomena. As pointed out by Campbell, boundaries tend to reinforce one another (Campbell 1958, as cited in Poli 2008). One may also quote Platt’s view on this phenomenon: “The boundary-surface for one property… will tend to coincide with the boundary surfaces for many other properties… because the surfaces are mutually-reinforcing.”

According to Poli (2008) this somewhat astonishing regularity of nature has not been sufficiently emphasized in perception philosophy. It is this that makes it useful and possible for us to identify sharply-defined regions of space as ‘objects’. “This is what makes a collection of properties a ‘thing’ rather than a smear of overlapping images”.

On the other hand, the underlying quantum-theoretical reason for the macroscopically sharp-definition of objects is the *decoherence* of the wave-function in many-particle systems in the presence of overwhelming thermal motions. The surfaces thus appear to be ‘mutually-reinforcing’ because their quantum phases are sharply different and vary from location to location.

6.2 Topological Transformations and Discontinuities in Biological Development. Organisms Represented as Variable Dynamic Systems. Generic States and Dynamic System Genericity

In actual fact, the super-complexity of the organism itself emerged through the generation of dynamic, variable structures which then entail variable/flexible functions, homeostasis, autopoiesis, anticipation, and so on. In this context, it is interesting that Wiener (1950) proposed the simulation of living organisms by variable machines/automata that did not exist in his time. The latter were subsequently formalized independently in two related reports (Baianu 1971a, b).

In D'Arcy Thompson's extensive book "On Growth and Form" (ca. 1900) there are many graphic examples of coordinate, continuous transformations (in fact *homotopies*) of anatomical structure from one species to another, rates of growth in organisms and populations, as well as a vast array of dynamic data serving as a source of inspiration in a valiant attempt to understand morphogenesis in terms of physical forces and chemical reactions. It is a remarkable, very early attempt to depart from Darwin's historical approach to evolution, and to understand organismic forms in terms of their varied and complex dynamic growth; it is often criticized for disagreeing with Darwin's theory of evolution, and also for being a physicalist attempt. Yet, some of the issues raised by D'Arcy W. Thompson are of interest even today, as he explicitly pointed out in his book that the 'morphogenetic dynamics' he is considering does not exhaust the real, *very complex dynamics* of biological development.

Separated in time by almost a century is René Thom's work on Catastrophe Theory (1980) that attempts to explain 'topologically' the presence of discontinuities and 'chaotic' behaviour, such as bifurcations, 'catastrophes', etc. in organismic development and evolution. Often criticized, his book does have the insight of *structural stability* in biodynamics *via* 'generic' states that when perturbed lead to other similarly stable states. The use of the term 'catastrophe' was 'gauche' as it reminds one of Cuvier's catastrophic theory for the formation of species, even though Thom's theory had no connection to the former. When analyzed from a categorical standpoint, organismic dynamics has been suggested to be characterized not only by homeostatic processes and steady state, but also by *multi-stability* (Baianu 1970). The latter concept is clearly equivalent from a dynamic/topological standpoint to super-complex system genericity, and the presence of *multiple dynamic attractors* (Baianu 1971) which were categorically represented as *commutative super-pushouts* (Baianu 1970). The presence of generic states and regions in super-complex system dynamics is thus linked to the emergence of complexity through both structural stability and the *open* system attribute of any living organism that enable its persistence in time, in an accommodating niche, suitable for its competitive survival.

6.3 Simple versus Complex Dynamics—Closed versus Open Systems

In an early report (Baianu and Marinescu 1968), the possibility of formulating a (Super-) Categorical Unitary Theory of Systems (i.e., both simple and complex, etc.) was pointed out both in terms of organizational structure and dynamics. Furthermore, it was proposed that the formulation of any model or 'simulation' of a complex system—such as living organism or a society—involves generating a first-stage *logical model* (not-necessarily Boolean!), followed by a *mathematical* one, *complete with structure* (Baianu 1970). Then, it was pointed out that such a modelling process involves a diagram containing the complex system, (**CS**) and its dynamics, a corresponding, initial logical model, **L**, '*encoding'* the essential dynamic and/or structural properties of **CS**, and a detailed, structured mathematical model (**M**); this initial modelling diagram may or may not be commutative, and the modelling can be iterated through modifications of ***L***, and/or **M**, until an acceptable agreement is achieved between the behaviour of the model and that of the natural, complex system (Baianu and Marinescu 1968; Comoroshan and Baianu 1969). Such an *iterative modelling* process may ultimately 'converge' to appropriate models of the complex system, and perhaps a best possible model could be attained as the categorical colimit of the directed family of diagrams generated through such a modelling process. The possible models **L**, or especially **M**, were not considered to be necessarily either numerical or recursively computable (e.g., with an algorithm or software program) by a digital computer (Baianu 1971b, 1986).

6.3.1 Commutative versus Non-commutative Modelling Diagrams

Interestingly, Rosen (1987) also showed that complex dynamical systems, such as biological organisms, cannot be adequately modelled through a *commutative* modelling diagram—in the sense of digital computer simulation—whereas the simple ('physical'/engineering) dynamical systems can be thus numerically simulated. Furthermore, his modelling commutative diagram for a *simple dynamical system* included both the 'encoding' of the 'real' system **N** in [**M**] as well as the 'decoding' of [**M**] back into **N**:

$$\begin{array}{ccc} [N] & \xrightarrow{\textbf{(Encoding)}} & L \oplus M \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \aleph_M} \\ N & \xleftarrow[\textbf{(Decoding)}]{} & [M] \end{array}$$

where δ is the real system dynamics and $\aleph$ is an algorithm implementing the numerical computation of the mathematical model (**M**) on a digital computer. First, one notes the ominous absence of the *Logical Model*, ***L***, from Rosen's diagram published in 1987. Second, one also notes the obvious presence of logical arguments and indeed (*non-Boolean*) 'schemes' related to the entailment of organismic models, such as MR-systems, in the more recent books that were published last by Rosen (1994, 1999, 2004). This will be further discussed in the next sections with the full mathematical details.

Furthermore, Elsasser (1981) pointed out a fundamental, logical difference between physical systems and biosystems or organisms: whereas the former are readily represented by *homogeneous* logic classes, living organisms exhibit considerable variability and can only be represented by *heterogeneous* logic classes. One can readily represent homogeneous logic classes or endow them with 'uniform' mathematical structures, but heterogeneous ones are far more elusive and may admit a multiplicity of mathematical representations or possess variable structure. This logical criterion may thus be useful for further distinguishing simple systems from highly complex systems.

The importance of *Logic Algebras*, and indeed of *Categories of Logic Algebras*, is rarely discussed in modern Ontology even though categorical formulations of specific Ontology domains such as Biological Ontology and Neural Network Ontology are being extensively developed. For a recent review of such categories of logic algebras the reader is referred to the concise presentation by Georgescu (2006); their relevance to network biodynamics was also recently assessed (Baianu 2004a, b; Baianu and Prisecaru 2004; Baianu et al. 2006).

6.4 Dynamic Emergence of the Higher Complexity Levels: Organisms, the Human Mind and Society

We shall be considering the question of how biological, psychological and social functions are entailed through *emergent* processes of increasing complexity in higher-dimensional spacetime structures that are essential to Life, Evolution of Species and Human Consciousness. Such emergent processes in the upper three levels of reality considered by Poli (2006b) have corresponding, defining levels of increasing dynamic complexity from biological to psychological and, finally, to the social level. It is therefore important to distinguish between the *emergent* processes of higher complexity and the underlying, component physicochemical processes, especially when the latter are said to be *'complex'* by physicists only because they occur either as a result of 'sensitivity to initial conditions, small perturbations, etc., or because they give rise to unpredictable behaviour that cannot be completely simulated on any digital computer; the latter systems with (deterministic) chaotic dynamics are *not*, however, *emergent* systems because their existence does not belong to a higher level of reality than the simple dynamic systems that are completely predictable. We are here defending the claim that all 'true' dynamic complexity of higher order is *irreducible* to the dynamics of sub-processes—usually corresponding to a lower level of reality—and it is therefore a truly *emergent*, real phenomenon. In other words, **no emergence** $\Rightarrow$ **no complexity** higher than that of physicochemical systems with chaos, whereas reductionists now attempt to reduce everything, from life to societies and ecology, to systems with just chaotic behaviour.

The detailed nature of the higher level emergence will be further treated in a more precise manner in Sect. 7 after introducing the novel, pre-requisite concepts that allow an improved understanding of dynamic emergent processes in higher dimensions of spacetime structures.

There is an ongoing ambiguity in the current use of the term 'complex', as in 'complex dynamics and dynamical systems'—which is employed by chaotic physics reports and textbooks with a very different meaning from the one customarily employed in Relational Biology (Rosen 1987; and also earlier, more general definitions proposed by Baianu (19680–1987). We propose to retain the term 'complexity'—in accord with the use adopted for the field of physicochemical chaotic dynamics established by modern physicists and chemists. Then, in order to avoid the recurring confusion that would occur between inanimate, chaotic or robotic, systems that are 'complex' and living organisms which are at a distinctly higher level of dynamic complexity, we propose to define the latter, higher complexity level of *biosystems* as '*supercomplex*'. Thus, we suggest that the *biological* complex systems—whose dynamics is quite distinct from that of *physical* 'complex systems'—should be called '*supercomplex*' (Baianu and Poli 2008). (Elsasser also claimed that living organisms are 'extremely complex', as discussed in a recent report (Baianu 2006)). From a reductionist's viewpoint, such a distinction may appear totally unnecessary because a reductionist does believe (*without any possibility of proof*) that all systems—complex or otherwise—ultimately obey only known physical laws, as the complex systems can be 'reduced' (by unspecified, and/or unspecifiable, procedures!) to a finite collection of the simple component systems contained in any selected complex system. For example, such a collection of parts could be assembled through a categorical *colimit*, as it will be shown in a subsequent section (8). Note also that a categorical colimit is defined not just by its parts but also by the morphisms between the objects, which conforms with the naive view that an engine, say, is not just a collection of parts, but depends crucially on how they are put together, if it is to work! Any suggestion of alternative possibilities is regarded by the reductionist approach as an attempt to introduce either 'ghosts' or undefinable entities/relations that 'could not physically exist', according to (simple) physical principles that govern the dynamics of (simple) physical systems. Although this line of reasoning seems to satisfy Occam's razor principle—taken as an 'economy' of thought—it does exclude both life and human consciousness from having any independent, or even *emergent*, ontological existence. Taken to its ultimate extreme, this 'simple' reductionist approach would seem to demand the reduction of even human societies not only to collections of individual people but also to the 'elementary' particles and quantum-molecular fields of which humans are made of.

Interestingly, the term 'super-complex' is already in use in the computer industry for high performance digital computer systems designed with a high-degree of parallel processing, whose level of complexity is, however, much lower than that of physicochemical chaotic systems that are called 'complex' by physicists. On the other hand, in the fields of structural and molecular biology, the term 'super-complex' recently designates certain very large super-aggregates of biopolymers that are functional within a cell. Thus, our proposed use of the term ⟨*super-complex*⟩ is for the higher level of organization—that of the *whole, functional organism*, not for the first (physicochemical) level of reality—no matter how complicated, 'chaotic' or intricate it is at the molecular/atomic/quantum level. Therefore, in our proposed terminology, *the level of super-complex dynamics is the first emergent level*—which does correspond to the first emergent level of reality in the ontological theory of levels recently proposed by Poli (2006a, b). A more precise formulation and, indeed, resolution of such emergent complexity issues will be presented in Sect. 7.

Our approach from the perspectives of spacetime ontology and dynamic complexity thus requires a reconsideration of the question how new levels of dynamic complexity arise at both the biological and psychological levels. Furthermore, the close interdependence/two-way relations of the psychological and social levels of reality (Poli 2006a) do require a consideration of the correlations between the dynamic complexities of human consciousness and human society. The *emergence* of one is ultimately determined by the other, in what might be expressed as *iterated feedback and/or feedforward loops*, though not restricted to the engineering meaning which is usually implied by these terms. Thus, *feedforward* loops should be understood here in the sense of *anticipatory* processes, that can, for example, lead in the future to the improvement of social interactions through deliberate, conscious human planning—or even more—to the prevention of the human, and other species, extinction. Further *inter-relations* among the different ontological levels of system complexity are discussed in Baianu and Poli (2008).

7 The Logics of Life and Consciousness

7.1 Emergence of Organisms, Essential Organismic Functions and Life

Whereas it would be desirable to have a well-defined definition of living organisms, the list of attributes needed for such a definition can be quite lengthy. In addition to super-complex, recursively non-computable and open, the attributes: auto-catalytic, self-organizing, structurally stable/generic, self-repair, self-reproducing, highly interconnected internally, multi-level, and also possessing multi-valued logic and anticipatory capabilities would be recognized as important. One needs to add to this list at least the following: diffusion processes, inter-cellular flows, essential thermodynamically-linked, irreversible processes coupled to bioenergetic processes and (bio)chemical concentration gradients, and fluxes selectively mediated by semi-permeable biomembranes. This list is far from being complete. Some of these important attributes of organisms are inter-dependent and serve to define life categorically as a super-complex dynamic process that can have several alternate, or complementary descriptions/representations; these can be formulated, for example, in terms of variable categories, variable groupoids, generalized Metabolic-Repair systems, organismic sets, hypergraphs, memory evolutive systems (MES), organismic toposes, interactomes, organismic super-categories and higher dimensional algebra.

Perhaps the most important attributes of Life are those related to the logics ‘immanent’ in those processes that are essential to Life. As an example, the logics and logic-algebras associated with functioning neuronal networks in the human brain—which are different from the multi-valued (Łukasiewicz-Moisil) logics (Georgescu 2006) associated with functional genetic networks (Baianu 1977, 1987a, b, 2007; Baianu et al. 2006) and self-reproduction (Lofgren 1968; Baianu 1970; 1987a, b)—were shown to be different from the simple Boolean-crysippian logic upon which machines and computers are built by humans. The former *n*-valued (LM) logics of functional neuronal or genetic networks are *non-commutative* ones, leading to *non-linear, super-complex* dynamics, whereas the simple logics of ‘physical’ dynamic systems and machines/automata are *commutative* (in the sense of involving a commutative lattice structure). Here, we find a fundamental, logical reason why living organisms are *non-commutative*, super-complex systems, whereas simple dynamical systems have *commutative modelling diagrams* that are based on *commutative Boolean* logic. We also have here the reason why a *commutative* Categorical Ontology of Neural networks leads to advanced robotics and AI, but has indeed little to do with the *'immanent logics'* and functioning of the living brain, contrary to the proposition made by McCulloch and Pitts (1943).

Rashevsky (1969) attempted to define life in terms of the essential functional relations arising between organismic sets of various orders, i.e., ontological levels, beginning with genetic sets, their activities and products as the lowest possible order, zero, of on 'organismic set' (OS). Then he pursued the idea in terms of logical Boolean predicates (1969), attempting to provide the simplest model possible he proposed the organismic set, or OS, as a basic representation of living systems, but he did not attempt himself to endow his OS with either a topological or categorical structure, in spite of the fact that he previously reported on the fundamental connection between Topology and Life (Rashevsky 1954; 1959). He did attempt, however, a logical analysis in terms of formal symbolic logics and Hilbert's predicates. Robert Rosen did take up the challenge of representing organisms in terms of simple categorical models—his Metabolic-Repair, (**M,R**)-systems, or (**MR**)s (Rosen 1958a b). Further extensions and generalizations of MR's were subsequently explored by considering abstract categories with both algebraic and topological structures (Baianu 1973; Baianu and Marinescu 1974, 1980a, 1984, 1987a, b).

7.2 Łukasiewicz and LM-Logic Algebra of Genome Network Biodynamics. Quantum Genetics and Q-Logics

The representation of categories of genetic network biodynamics **GNET**s as subcategories of LM-Logic Algebras (**LMA**s) was recently reported (Baianu et al. 2006) and several theorems were discussed in the context of morphogenetic development of organisms. The **GNET** section of the cited report was a review and extension of an earlier article on the 'immanent' logic of genetic networks and their complex dynamics and non-linear properties (Baianu 1977). Comparison of GNET universal properties relevant to *Genetic Ontology* can be thus carried out by colimit- and/or limit-preserving functors of GNETs that belong to adjoint functor pairs (Baianu and Scripcariu 1974; Baianu 1987a, b, 2007; Baianu et al. 2006). Furthermore, evolutionary changes present in functional genomes can be monitored by natural transformations of such universal—property preserving functors, thus pointing towards evolutionary patterns that are of importance to the emergence of increasing complexity through evolution, and also to the emergence of man and ultra-complexity in the human mind. Missing from this approach is a consideration of the important effects of social, human interactions in the formation of language, symbolism, rational thinking, cultural patterns, creativity, and so on… to full human consciousness. The space, and especially time, ontology of such societal interaction effects on the development of human consciousness will also be briefly considered in the following sections.

7.3 The Organismic LM-Topos

As reported previously (Baianu et al. 2006) it is possible to represent directly the actions of LM, many-valued logics of genetic network biodynamics in a categorical structure generated by selected LM-logics. The combined logico-mathematical structure thus obtained may have several operational and consistency advantages over the GNET-categorical approach of 'sets with structure'. Such a structure was called an 'LM-Topos' and represents a significant, non-commutative logic extension of the standard Topos theory which is founded upon a commutative, intuitionist (Heyting-Brouwer) logic. Whereas the latter topos may be more suitable for representing general dynamics of simple systems, machines, computers, robots and AI structures, the non-commutative logic LM-topos offers a more appropriate foundation for structures, relations and organismic or societal functions that are respectively super-complex or ultra-complex. This new concept of an LM-topos thus paves the way towards a Non-Abelian Ontology of SpaceTime in Organisms and Societies regarded and treated precisely as super- or ultra-complex dynamic systems.

7.4 Simple vs. Complex Dynamic State Space Structures. Dynamic Groups, Groupoids and Variable Topology

Whereas simple dynamic systems, or general automata, have *canonically decomposable semigroup* state spaces (the Krone-Rhodes Decomposition Theorem), super-complex systems do not have state spaces that are known to be canonically decomposable, or partitioned into functionally independent subcomponent spaces, that is within a living organism all organs are inter-dependent and integrated; one cannot generally find a subsystem or organ which retains organismic life—the full functionality of the whole organism. However, in some of the simpler organisms, for example in *Planaria*, regeneration of the whole organism is possible from several of its major parts. Pictorially, and typically, living organisms are not chimeras that can be arbitrarily and functionally subdivided into independent smaller subsystems (even though cells form the key developmental and ontological levels of any multi-cellular organism, they cannot survive independently unless transformed.) By contrast, automata do have in general such *canonical sub-automata/machine decompositions* of their state-space. It is in this sense also that recursively computable systems are 'simple', whereas organisms are not. We note here that an interesting, incomplete but computable, model of multi-cellular organisms was formulated in terms of 'cellular' or 'tessellation' automata simulating cellular growth in planar arrays with such ideas leading and contributing towards the 'mirror neuron system hypothesis'. This incomplete model of 'tessellation automata' is often borrowed in one form or another by seekers of computer-generated/algorithmic, artificial 'life'.

On the one hand, simple dynamical (physical) systems are often represented through groups of dynamic transformations. In GR, for example, these would be Lorentz–Poincare' groups of spacetime transformations/reference frames. On the other hand, super-complex systems, or biosystems, emerging through self-organization and complex aggregation of simple dynamical ones, are therefore expected to be represented mathematically—at least on the next level of complexity—through an extension, or generalization of mathematical groups, such as, for example, *groupoids*. Whereas simple physical systems with linear causality have high symmetry, a single energy minimum, and thus they possess only *degenerate* dynamics, the super-complex (living) systems emerge with lower symmetries but higher dynamic and functional/relational complexity. As symmetries get 'broken' the complexity degree increases sharply. From groups that can be considered as very simple categories that have just one object and reversible/invertible endomorphisms, one moves through 'symmetry breaking' to the structurally more complex groupoids, that are categories with many objects but still with all morphisms invertible. Dynamically, this reflects the transition from degenerate dynamics with one, or a few stable, isolated states ('degenerate' ones) to dynamic state regions of many generic states that are metastable; this multi-stability of biodynamics is nicely captured by the many objects of the groupoid and is the key to the 'flow of life' occurring as multiple transitions between the multiple metastable states of the homeostatic, living system. More details of how the latter emerge through biomolecular reactions, such as catabolic/anabolic reactions, will be presented in the next subsections, and also in the next section, especially under natural transformations of functors of biomolecular categories.

Various groupoids were considered in the first paper of this series as some of the 'simplest' illustrations of the mathematical structures present in super-complex biological systems and classes thereof, such as *biogroupoids* (the groupoids featuring in biosystems) and variable biogroupoids to represent evolving biological species. Relevant are here also *crossed complexes* of variable groupoids and/or *multi-groupoids* as more complex representations of biosystems that follow the emergence of ultra-complex systems (the mind and human societies, for example) from super-complex dynamic systems (organisms).

7.5 Variable Topologies

Let us recall the basic principle that a *topological space* consists of a set X and a 'topology' on X where the latter gives a precise but general sense to the intuitive ideas of 'nearness' and 'continuity'. Thus the initial task is to axiomatize the notion of 'neighbourhood' and then consider a topology in terms of open or of closed sets, a compact-open topology, and so on (see Brown 2006). In any case, a topological space consists of a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on the set X. For instance, suppose an *open set topology* is given by the set $\mathcal{U}$ of prescribed open sets of X satisfying the usual axioms (Brown 2006 Chapter 2). Now, to speak of a variable open-set topology one might conveniently take in this case a family of sets $\mathcal{U}_\lambda$ of *a system of prescribed open sets*, where λ belongs to some indexing set Λ. The system of open sets may of course be based on a system of contained neighbourhoods of points where one system may have a different geometric property compared say to another system (a system of disc-like neighbourhoods compared with those of cylindrical-type). In general, we may speak of a topological space with a varying topology as a pair $(X, \mathcal{T}_\lambda)$ where $\lambda \in \Lambda$. The idea of a varying topology has been introduced to describe possible topological distinctions in bio-molecular organisms through stages of development, evolution, neo-plasticity, etc. This is indicated schematically in the diagram below where we have an n-stage dynamic evolution (through complexity) of categories D_i where the vertical arrows denote the assignment of topologies $\mathcal{T}_i$ to the class of objects of the D_i along with functors $\mathcal{F}_i : \mathsf{D}_i \longrightarrow \mathsf{D}_{i+1}$, for $1 \leq i \leq n-1$:

$$\begin{array}{ccccccccc} \mathcal{T}_1 & & \mathcal{T}_2 & & \cdots & & \mathcal{T}_{n-1} & & \mathcal{T}_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \mathsf{D}_1 & \xrightarrow{\mathcal{F}_1} & \mathsf{D}_2 & \xrightarrow{\mathcal{F}_2} & & \cdots & \mathsf{D}_{n-1} & \xrightarrow{\mathcal{F}_{n-1}} & \mathsf{D}_n \end{array}$$

In this way a variable topology can be realized through such n-levels of complexity of the development of an organism. Another instance is when cell/network topologies are prescribed and in particular when one considers a categorical approach involving concepts such as *the free groupoid over a graph* (Brown 2006). Thus a varying graph system clearly induces an accompanying system of variable groupoids.

8 Biological Evolution as a Local-to-Global Problem. Generalized (M,R)-Systems as Variable Groupoids developed from A Primordial System. Variable Bionetworks and Quantum-Enzymatic Realizations of (M,R)-Systems

We have the important example of MR-systems with *metabolic groupoid* structures (that is, *reversible enzyme reactions/metabolic functions—repair replication* groupoid structures), for the purpose of studying RNA, DNA, epigenomic and genomic functions. For instance, the relationship of

$$\text{METABOLISM} = \text{ANABOLISM} \Longrightarrow \Longleftarrow \text{CATABOLISM}$$

can be represented by a metabolic groupoid of '*reversible', anabolic/catabolic processes*. In this respect the simplest MR-system can be represented as a *topological groupoid* with the open neighbourhood topology defined for the entire dynamical state space of the MR-system, that is an open/generic—and thus, a structurally stable—system, as defined by the dynamic realizations of MR-systems. This necessitates a descriptive formalism in terms of *variable groupoids* following which the human MR-system would then arise as the *colimit* of its complete biological family tree expressible in terms of a family of many linked/connected groupoids; this variable biogroupoid formalism is briefly outlined in the next section.

8.1 Biological Species

From an ontology viewpoint, the biological species can be defined as a class of equivalent organisms from the point of view of sexual reproduction and or/

functional genome, or as a *biogroupoid* (Baianu et al. 2006). Whereas satisfactory as taxonomic tools these two definitions are not directly useful for understanding evolution. The biogroupoid concept, however, can be readily extended to a more flexible concept, the *variable groupoid*, which can be then utilized in theoretical evolutionary studies, and through predictions, impact on empirical evolutionary studies, as well as possibly organismic taxonomy.

8.2 Evolving Species as Variable Biogroupoids

For a collection of *variable groupoids* we can firstly envisage a parametrized family of groupoids $\{\mathsf{G}_\lambda\}$ with parameter λ (which may be a time parameter, although in general we do not insist on this). This is one basic and obvious way of seeing a variable groupoid structure. If λ belongs to a set M, then we may consider simply a projection $\mathsf{G} \times M \longrightarrow M$, which is an example of a trivial fibration. More generally, we could consider a *fibration of groupoids* $\mathsf{G} \hookrightarrow Z \longrightarrow M$ (Higgins and Mackenzie 1990). However, we expect in several of the situations discussed in this paper (such as, for example, the metabolic groupoid introduced in the previous subsection) that the systems represented by the groupoid are interacting. Thus, besides systems modelled in terms of a *fibration of groupoids*, we may consider a multiple groupoid as defined as a set with a number of groupoid structures any distinct pair of which satisfy an *interchange law* which can be expressed as: each is a morphism for the other, or alternatively: there is a unique expression of the following composition:

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \begin{array}{l} \ulcorner\!\longrightarrow j \\ \downarrow \\ i \end{array} \tag{8.1}$$

where i and j must be distinct for this concept to be well defined. This uniqueness can also be represented by the equation

$$(x \circ_j y) \circ_i (z \circ_j w) = (x \circ_i z) \circ_j (y \circ_i w). \tag{8.2}$$

This illustrates the principle that a 2-dimensional formula may be more comprehensible than a linear one!

Brown and Higgins (1981) showed that certain multiple groupoids equipped with an extra structure called *connections* were equivalent to another structure called a *crossed complex* which had already occurred in homotopy theory.

In general, we are interested in the investigation of the applications of the inclusions

$$(\text{groups}) \subset (\text{groupoids}) \subset (\text{multiple groupoids}).$$

The applications of groups, and Lie groups, in mathematics and physics are well known. Groupoids and Lie groupoids are beginning to be applied in such areas as quantization (see for example Landsman 1998). Indeed it is well known that groupoids allow for a more flexible approach to symmetry than do groups alone. There is probably a vast field open to further exploitation at the doorstep.

One of the difficulties, however, is that multiple groupoids can be very complex algebraic objects. It is known for example that they model weak homotopy n-types. This allows the possibility of a revolution in algebraic topology.

Another important notion is the *classifying space* BC of a crossed complex C. This, and the monoidal closed structure on crossed complexes, have been applied by Porter and Turaev to questions on Homotopy Quantum Field Theories (these are TQFT's with a 'background space' which can be helpfully taken to be of the form BC as above), and by Martins and Porter (2004), as *invariants* of interest in physics.

The *patching mechanism of a groupoid atlas* connects the iterates of local procedures (Bak et al. 2006). One might also consider in general a *stack in groupoids* (Borceux 1994), and indeed there are other options for constructing relational structures of higher complexity, such as *double, or multiple* groupoids (Brown 2004). As far as we can see, these are different ways of dealing with gluing or patching procedures, a method which goes back to Mercator! As we have pointed out in the previous paper (Baianu et al. 2007b) the *atlas* of structures should, in principle, apply to a lot of interesting, topological and/or algebraic, structures: groupoids, multiple groupoids, Heyting algebras, n-valued logic algebras and C^*-convolution -algebras. On the other hand, for 'simple' physical systems it is quite reasonable to suppose that structures associated with symmetry and transitions could well be represented by 1-groupoids, whereas transitions between *quantum* transitions, could be then represented by a special type of quantum symmetry double groupoid that we shall call here simply a *quantum double groupoid* (QDG; Baianu et al. 2007c), as it refers to *fundamental quantum* dynamic processes (cf. Heisenberg, as cited by Brown 2002).

Developmental processes, and in general, ontogeny considered from a structural or anatomical viewpoint involves not only geometrical or topological transformations but more general/complex transformations of even more flexible structures such as variable groupoids. The natural generalizations of variable groupoids lead to 'variable topology' concepts that are considered in the next subsection.

8.3 Super-Complex Network Biodynamics in Variable Biogroupoid Categories. Variable Bionetworks and their Super-Categories

This section is an extension of the previous one in which we introduced variable biogroupoids in relation to speciation and the evolution of species. The variable category concept generalizes the concept of variable groupoid which can be thought as a variable category whose morphisms are invertible. The latter is thus a more 'symmetric' structure than the general variable category.

We have seen that variable biogroupoid representations of biological species, as well as their categorical limits and colimits, may provide powerful tools for tracking evolution at the level of species. On the other hand, the representation of organisms, with the exception of unicellular ones, is likely to require more general structures, and super-structures of structures (Baianu 1970). In other words, this leads towards higher-dimensional algebras (HDA) representing the super-complex hierarchies present in a complex-functional, multi-cellular organism, or in a highly-evolved functional organ such as the human brain. The latter (HDA) approach will be discussed in a later section in relation to neurosciences and consciousness, whereas we shall address here the question of representing biosystems in terms of variable categories that are lower in complexity than the ultra-complex human mind. A variable category and/or variable topology approach is, on the other hand, a simpler alternative to the organismic LM-topos that was employed in Sect. 7.3 to represent the emergence and evolution of genetic network biodynamics, comparative genomics and phylogeny. In terms of representation capabilities, the range of applications for variable categories may also extend to the neurosciences, neurodynamics and brain development, in addition to the evolution of the simpler genomes and/or interactomes. Last-but-not-least, it does lead directly to the more powerful 'hierarchical' structures of higher dimensional algebra.

8.4 Evolution as a Local-to- Global Problem: The Metaphor of Chains of Local Procedures. Alternate Representations of Evolution by MES and Colimits of Transforming Species. Bifurcations, Phylogeny and the 'Tree of Life'

Darwin's theory of natural selection, sometimes considered as a reductionist attempt in spite of its consideration of both specific and general biological functions such as adaptation, reproduction, heredity and survival, has been substantially enriched over the last century; this was achieved through more precise mathematical approaches to population genetics and molecular evolution which developed new solutions to the key problem of speciation (Sober 1984). Modified evolutionary theories include neo-Darwinism, the 'punctuated evolution' (Gould 1977) and the 'neutral theory of molecular evolution' of Kimura (1983). The latter is particularly interesting as it reveals that evolutionary changes do occur much more frequently in unexpressed/silent regions of the genome, thus being 'invisible' phenotypically. Therefore, such frequent changes ('silent mutations') are uncorrelated with, or unaffected by, natural selection. For further progress in completing a logically valid and experimentally-based evolutionary theory, an improved understanding of speciation and species is required, as well as substantially more extensive, experimental/genomic data related to speciation than currently available. Furthermore, the ascent of man, as often proposed by evolutionary theories of *H. sapiens* beginning with that of Huxley, is apparently not the result of only natural selection but also that of co-evolution through society interactions; thus, simply put: the emergence of human speech and consciousness occurred both through selection and co-evolution, with the former not being all that 'natural' as society played a protective, as well as selective role from the very beginnings of hominin and hominid societies more than 2.2 million years ago. Somewhat surprisingly, the subject of *social selection* in human societies is rarely studied even though it may have played a crucial role in the emergence of *H. sapiens*, and occurs in every society that we know without exception. To the extent that social selection is not driven—at least not directly— by the natural environment it might be classified also as 'artificial' even though it does not involve any artificial breeding procedures, and it cannot be therefore assimilated in any way with the artificial selection of plants or animals.

Furthermore, there is a theory of levels, ontological question that has not yet been adequately addressed, although it has been identified: *at what level does evolution operate: species, organism or molecular (genetic)?* According to Darwin the answer seems to be the species; however, not everybody agrees because in Darwin's time a valid theory of inherited characters was neither widely known nor accepted. Moreover, molecular evolution and concerted mutations are quite recent concepts whose full impact has not yet been realized. As Goodwin (1994) puts it succinctly:

> "Where has the organism disappeared in Darwin's evolutionary theory?"

The answer in both Goodwin's opinion, and also in ours, lies in the presence of key functional/relational patterns that emerged and were preserved in organisms throughout various stages over four billion years or so of evolution. The fundamental relations between organism, species and the speciation process itself do need to be directly addressed by any theory that now claims to explain the Evolution of species and organisms. Furthermore, an adequate consideration of the biomolecular levels and sub-levels involvement in Speciation and Evolution must also be present in any modern evolutionary theory. These fundamental questions will be addressed for the first time from the categorical ontology standpoint in this and the next section.

To date there is no complete, direct observation of the formation of even one live, new multi-cellular species through *natural* selection, in spite of the rich paleontological, indirect evidence of evolution towards organisms of increasingly higher complexity with evolutionary time. However, man has generated many new species through selective breeding/artificial selection based on a fairly detailed understanding of hereditary principles, both Mendelian and non-Mendelian. Still more species of the simpler organisms are being engineered by man through molecular genetic manipulations, often raising grave concerns to the uninitiated layman leading to very restrictive legislation, especially in Europe. There are several differences between natural and artificial selection, with the main difference being seen in the pseudo-randomness of natural selection as opposed to the sharply directed artificial selection exerted by human breeders. This is however a matter of degree rather than absolute distinction: natural selection is not a truly random process either and artificial selection does involve some trial and error as it is not a totally controllable exercise. Furthermore, natural selection operates through several mechanisms on different levels whereas artificial selection involves strictly controlled reproduction and may involve just the single organism level to start with, followed by deliberate inbreeding, as an example. Therefore, one can reasonably argue that natural selection mechanisms differ from those of artificial selective breeding, with *adaptive* 'mechanisms' being largely eliminated in the latter, even though the laws of heredity are of course respected by both, but with fertilization and embryonic/organismal development being often under the breeder's control.

In this section, we shall endeavour to address the question of super-complex systems' evolution as a *local-to-global* problem and we shall seek solutions in terms of the novel categorical concepts that we introduced in the previous subsections. Thus, we shall consider biological evolution by introducing the unifying metaphor of *'local procedures'* which may represent the formation of new species that branch out to generate still more evolving species.

In his widely read book, D-Arcy W. Thompson (1994, re-printed edition) gives a large number of biological examples of organismic growth and forms analyzed at first in terms of physical forces. Then, he is successful in carrying out analytical geometry coordinate transformations that allow the continuous, homotopic mapping of series of species that are thought to belong to the same branch—phylogenetic line—of the tree of life. However, he finds it very difficult or almost impossible to carry out such transformations for fossil species, skeleton remains of species belonging to different evolutionary branches. Thus, he arrives at the conclusion that the overall evolutionary process is not a continuous sequence of organismic forms or phenotypes (see p. 1094 of his book).

Because genetic mutations that lead to new species are discrete changes as discussed above in Sect. 3, we are therefore not considering evolution as a series of continuous changes—such as a continuous curve drawn analytically through points representing species—but heuristically as a *tree of 'chains of local procedures'* (Brown 2006). Evolution may be alternatively thought of and analyzed as a *composition of local procedures*. Composition is a kind of combination and so it might be confused with a colimit, but they are substantially different concepts.

Therefore, one may attempt to represent biological evolution as an evolutionary tree, or tree of life, with its branches completed through chains of local procedures (pictured in Fig. 1 as overlapping circles) involving certain groupoids, which informally we call *variable topological biogroupoids*, and with the overlaps corresponding to 'intermediate' species or classes/populations of organisms which are rapidly evolving under strong evolutionary pressure from their environment (including competing species, predators, etc., in their niche).

A more specific formalization follows. The notion of 'local procedure' is an interpretation of Ehresmann's formal definition of a *local admissible section* s for a groupoid G in which $X = \mathrm{Ob}(G)$ is a topological space. Then s is a section of the source map $\alpha: G \to X$ such that the domain of s is open in X. If s, f are two such sections, their composition st is defined by $\mathsf{st}(x) = \mathsf{s}(\beta t(x)) \circ t(x)$ where $\circ$ is the composition in G. Thus the domain of st may be empty. One may also put the additional condition that s is 'admissible', namely $\beta\mathsf{s}$ maps the open domain of **s**

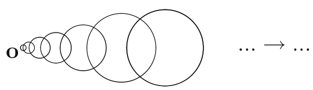


Fig. 1 Pictorial representation of Biological Evolution as a composition of local procedures involving variable biogroupoids that represent biological speciation phenomena. COLPs may form the branches of the evolutionary tree, oriented in this diagram with the time arrow pointing to the right

homeomorphically to the image of βs, which itself is open in X. Then an admissible local section is invertible with respect to the above composition.

The categorical colimits of MES, that may also be heuristically thought as ‘chains of local procedures’ (COLP), have their vertex object at the branching point on the evolutionary tree. The entire evolutionary tree—tracked to present day—is then intuitively represented through such connected chains of local procedures beginning with the primordial(s) and ending with *Homo*, thus generating an intuitive *global colimit* in the *2-category* of all variable topological biogroupoids (VTBs) that correspond to all classes of evolving organisms (either dead or alive). Such VTBs have a generic-dynamic, pictorial illustration which is shown as circles in the diagram of this global (albeit intuitive) evolutionary colimit (“$\lim \hookrightarrow$ ”). The primordial can be selected in this context as represented by the special PMR which is (was) realized by ribozymes as described in Sect. 8.5.

Note also that organisms were previously represented in terms of categories of dynamic state-spaces (Baianu 1970, 1980a, b, 1983, 1987a, b; Baianu et al. 2006) which are defined in terms of the various stages of ontogenetic development with increasing numbers of cells and functions as specialization and morphogenesis proceed in real time. This representation leads to the concept of a *direct limit* of organisms or equivalence classes of organisms of increasing complexity during evolution. We start with the definition of a *direct system* of objects and homomorphisms or homeomorphisms, (‘transformations’, functors, super-functors, natural transformations, etc). Let $(\mathsf{I}, \leq)$ be a *directed poset* whose elements $\underline{i}$ are the *complexity indices* of evolving organisms; an index of complexity is defined for example in terms of the genome complexity, with genetic network dynamics represented in terms of an **LM**-logic algebra and **LM**-algebra morphisms (Baianu 1977, 1987a, b, 2007; Baianu et al. 2006). Let $O_i | i \in \mathbf{I}$ be a family of objects (organisms or organismic supercategories (Baianu 1970, 1971)) indexed by **I** and suppose we have a family of homomorphisms (or homeomorphisms, or transformations, functors/ super-functors, etc.) $f_{ij} : O_i \to O_j$ for all $i \leq j$ with the following properties:

1. f_{ii} is the identity in O_i,
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq \; j \leq \; k$.

Then the pair (O_i, f_{ij}) is called a *direct system* over **I**. The *direct limit* **O**, of the direct system (O_i, f_{ij}) is defined as the coproduct of the O_i’s *modulo* a certain equivalence relation defined by evolutionary complexity:

$$\varinjlim O_i = (\amalg_i)/[x_i \sim x_j| \text{ there exists } k \in \mathbf{I} \text{ such that } f_{ik}(x_i) = f_{jk}(x_j).$$

Two elements in the disjoint union can be regarded as ‘equivalent’ if and only if they “eventually become equal” in the direct system. Thus, one naturally obtains from this equivalence definition the corresponding *canonical* morphisms $\varphi_i : O_i \to \mathsf{O}$ sending each ‘element’ (organism) to its equivalence (complexity) class. The algebraic operations on O are defined via these maps in an obvious manner. A general definition is also possible (Mac Lane 2000). The *direct limit* can be defined abstractly in an arbitrary category by means of a *universal property*. Let (X_i, f_{ij}) be a direct system of objects and morphisms in a category C (same definition as above). The abstract *direct limit* of this system of evolving organisms is an object X in C together with morphisms $\varphi_i : X_i \to X$ satisfying $\varphi_i = \varphi_j \circ f_{ij}$. The pair (X, φ_i) must be *universal* with the meaning that for any other such pair (Y, ψ_i) there exists a unique morphism $u : X \to Y$ making all the "obvious" identities hold, i.e., the cocone diagram (such as 10.2) must commute for all i, j. The direct limit is often denoted as:

$$X = \varinjlim X_i,$$

with the direct system (X_i, f_{ij}) being tacitly assumed to exist and also to be completely specified. Unlike the case of algebraic objects, the direct limit may not exist in an arbitrary category. If it does, however, it is unique in a strong sense: given any other direct limit X there exists is a unique isomorphism $\mathsf{X}' \longrightarrow \mathsf{X}$ commuting with the canonical morphisms. One notes also that a direct system in the category C admits an alternative description in terms of functors. Any directed poset I can be regarded as a small category where the morphisms consist of arrows $i \longrightarrow j$ if and only if $i \leq j$. The direct system is then just a covariant functor $L : \mathsf{I} \longrightarrow \mathsf{C}$. Similarly, a *colimit* can be thus defined by the family of ontogenetic development stages/ dynamic state-spaces, indexed by their corresponding complexity indices at specified instants of (ontogenetic) developmental time ($\Delta t \, \varepsilon \, R$), as fully specified in previous papers (Baianu 1970; Baianu and Scripcariu 1974; Baianu 1980a, b, 1983, 1984).

Such constructions of ontogenetic development colimits in terms of *cocone* diagrams of objects and morphisms (see Fig. 1) can be viewed as specific examples of 'local procedures'. Nevertheless, in a certain specific sense, these organismic (ontogenetic) development (OOD) colimits play the role of 'local procedures' in the 2-category of evolving organisms. Thus, the global colimit of the evolutionary 2-category of organisms may be regarded as a super-colimit, or an evolutionary colimit of the OOD colimits briefly mentioned above from previous reports. A tree-graph that contains only single-species biogroupoids at the 'core' of each 'local procedure' does define precisely an evolutionary branch without the need for subdivision because a species is an 'indivisible' entity from a breeding or reproductive viewpoint. Interestingly, in this dynamic sense, biological evolution 'admits' super-colimits (Baianu and Marinescu 1968; Comoroshan and Baianu 1969; Baianu 1970, 1980a, b, 1983, 1987a, b; Baianu et al. 2006), with a higher-dimensional structure which is less restrictive than either MES (Ehresmann and Vanbremeersch 1987), or simple MR's represented as categories of sets (in which case direct and inverse limits can *both* be constructed in a canonical manner, cf. Baianu 1973).

We note that several different concepts introduced by distinct ontological approaches to organismal dynamics, stability and variability *converge* here on the metaphor of (chains of) 'local procedures' for evolving organisms and species. Such distinct representations are: the dynamic genericity of organismic states which lead to structural stability—as introduced by Rosen (1987) and Thom (1980), the logical class heterogeneity of living organisms introduced by Elsasser (1981), the inherent 'bio-fuzziness' of organisms (Baianu and Marinescu 1968; also discussed by Comoroshan and Baianu 1969) in both their structure and function, or as ranges of autopoietic 'structural variability' exhibited by living systems (Maturana and Varela 1980), imposed to the organism through its coupling with a specific environmental niche.

This dynamic intuition of evolution—unlike Darwin's historical concept—may be hard to grasp at first as it involves several construction stages on different ontological levels: it begins with organisms (or even with biomolecular categories!), emerges to the level of populations/subspecies/ species that evolve into classes of species, that are then further evolving,… and so on, towards the point in time where the emergence of man's, *Homo* family of species began to separate from other hominin/hominide families of species some 5–8 million years ago. Therefore, it is not at all surprising that most students of evolutionary biology have had, or still have, difficulties in understanding the real intricacies of evolutionary processes that operate on several different levels/sublevels of reality, different time scales, and also aided by geographical barriers or geological accidents. In this case, Occam's razor may seem to patently fail as the simplest 'explanations', or the longest-lasting myths, ultimately cannot win when confronted by the reality of emerging higher levels of complexity.

Furthermore, we note also that the organisms within the species represented by VTBs have an ontogenetic development represented in the dynamic state space of the organism as a categorical colimit. Therefore, the evolutionary, global colimit is in fact a *super-colimit* of all organismic developmental colimits up to the present stage of evolution. This works to a good approximation insofar as the evolutionary changes occur on a much longer timescale than the lifespan of the 'simulation' model. Thus, the degree of complexity increases above the level of super-complexity characteristic of individual organisms, or even species (biog-roupoids), to a next, evolutionary meta-level, that we shall call *evolutionary meta-complexity*. Whenever there are uncertainties concerning taxonomy one could compare the alternate evolutionary possibilities by means of pairs of functors that preserve limits or colimits, called respectively, right- and left-adjoint functors. Moreover, such adjoint functor pairs also arise in comparing different developmental stages of the same organism from the viewpoint of preserving their developmental potential (Baianu and Scripcariu 1974), *dynamic* colimits preserved by the right-adjoint functor, G, and/or the *functional*, projective limits preserved by a left-adjoint functor of G (cf. Rashevsky's Principle of Biological Epimorphism, or the more general Postulate of Relational Invariance (cf. Baianu et al. 2006); see also Baianu and Scripcariu (1974) for both the relevant definitions and theorems.)

8.5 An Example of an Emerging Super-Complex System as A Quantum-Enzymatic Realization of the Simplest (**M,R**)-System

Note that in the case of either uni-molecular or multi-molecular, *reversible* reactions one obtains a *quantum-molecular groupoid*, QG, defined as above in terms of the mcv-observables. In the case of an enzyme, E, with an activated complex, $(ES)^*$, a *quantum biomolecular groupoid* can be uniquely defined in terms of mcv—observables for the enzyme, its activated complex $(ES)^*$ and the substrate S. Quantum tunnelling in $(ES)^*$ then leads to the separation of the reaction product and the enzyme E which enters then a new reaction cycle with another substrate molecule S', indistinguishable—or equivalent to—S. By considering a sequence of two such reactions coupled together,

$$QG_1 \leftrightarrows QG_2,$$

corresponding to an enzyme f coupled to a ribozyme ϕ, one obtains a *quantum-molecular realization of the simplest* **M,R**-*system* (f, ϕ) (see also Baianu et al. 2007a for further details about the MR/PMR).

The non-reductionist caveat here is that the relational systems considered above are *open* ones, exchanging both energy and *mass* with the system's environment in a manner which is dependent on time, for example in cycles, as the system 'divides'—reproducing itself; therefore, even though generalized quantum-molecular observables can be defined as specified above, neither a stationary nor a dynamic Schrödinger equation holds for such examples of 'super-complex' systems. Furthermore, instead of just energetic constraints—such as the standard quantum Hamiltonian—one has the constraints imposed by the diagram commutativity related to the mcv—observables, canonical functors and natural transformations, as well as to the concentration gradients, diffusion processes, chemical potentials/activities (molecular Gibbs free energies), enzyme kinetics, and so on. Both the canonical functors and the natural transformations defined above for uni- or multi-molecular reactions represent the relational increase in complexity of the emerging, super-complex dynamic system, such as, for example, the simplest (**M,R**)-system, (f, ϕ).

9 Conclusions and Discussion

The conceptual development of a logical and categorical framework for the SpaceTime Ontology of Complex, Super-Complex and Ultra-Complex Systems was here proposed that may be suitable for representing a very wide range of highly complex systems, such as the human brain and neural network systems that are supporting processes such as perception, consciousness and logical/abstract thought.

Mathematical generalizations such as higher dimensional algebra are concluded to be logical requirements of the unification between complex system and consciousness theories (Brown and Porter 2003, 2006) that would be leading towards a deeper understanding of man's own spacetime ontology, which is claimed here to be both *unique* and *universal*.

To what extent the concepts of Categorical Ontology and Higher Dimensional Algebra are suitable for the latter three items remains thus an open question. Furthermore, the possible extensions of our approach to investigating globally the *biosphere* and also the interactions with the environment:

Biosphere $\Longleftrightarrow$ Environment interactions

remain as a further object of study in need of developing a formal definition of the horizon concept, only briefly touched upon in previous papers.

New areas of Categorical Ontology are likely to develop as a result of the recent paradigm shift towards non-Abelian theories. Such new areas would be related to recent developments in: non-Abelian Algebraic Topology, non-Abelian gauge theories of Quantum Gravity, non-Abelian Quantum Algebraic Topology and Noncommutative Geometry, that were briefly mentioned here in relation to spacetime ontology.

Although the thread of the current essay strongly entails the elements of 'non-linear' and 'non-commutative' science, we adjourn contesting the above strictures. One can always adopt the Popperian viewpoint that theoretical models, at best, are approximations to the truth, and the better models (or the hardest to de-bunk *myths*, according to Goodwin (1994)) are simply those that can play out longer than the rest, such as Darwin's theory on the origin of species. As Chalmers (1996) and others suggest, re-conceptualizing the origins of the universe(s) may provide an escape route towards getting closer to a definitive explanation of consciousness. Whether such new explanations will dispel the traditional metaphysical problems of the phenomenal world, that remains to be seen.

On the one hand, Wittgenstein (1967) claimed that we cannot expect language to help us realize the effects of language. On the other hand, Mathematics—the democratic Queen of sciences (cf. Gauss)—is, or consists to a large extent of, precise, formal type(s) of language(s), (cf. Hilbert, or more recently, the Bourbaki school) which do allow one to have 'clear, sharp and verifiable representations of items'; these, in turn, enable one to make powerful deductions and statements through Logics, intuition and abstract thoughts, even about the undecidability of certain types of its own theorems (Gödel). Another misconception promoted by some mathematicians, as well as Wittgenstein, is that mathematics is merely a 'tautological exercise', presumably this label being reserved for 'pure' mathematics which is just an editorial convenience mode of operation. Perhaps, if all of mathematics could be reduced to, or based upon, only Boolean logic this might be a possibility; however, recent trends in mathematics are towards greater emphasis on the use of intuitionistic logic such as Brouwer-Heyting logic, and also of many-valued logics (Georgescu 2006) in defining universal mathematical concepts.

Two formal claims that were defended in our three papers appearing in this issue are summarized here as follows:

- The *non-commutative*, fundamentally 'asymmetric' character of Categorical Spacetime Ontology *relations and structure*, both at the top and bottom levels of reality; the origins of a paradigm shift towards non-Abelian theories in science and the need for developing a *non-Abelian Categorical Ontology*, especially a complete, non-commutative theory of levels founded in LM- and Q-logics.
- The potential now exists for exact, symbolic calculation of the non-commutative invariants of spacetime through logical or mathematical, precise language tools

(categories of LM-logic algebras, generalized LM-toposes, HHvKT, higher Dimensional Algebra, ETAS, and so on).

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