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Spaces of maps into classifying spaces for equivariant crossed complexes

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ABSTRACT

We give an equivariant version of the homotopy theory of crossed complexes. The applications generalize work on equivariant Eilenberg–Mac Lane spaces, including the non abelian case of dimension 1, and on local systems. It also generalizes the theory of equivariant 2-types, due to Moerdijk and Svensson. Further, we give results not just on the homotopy classification of maps but also on the homotopy types of certain equivariant function spaces.

INTRODUCTION

A *crossed complex* C is like a chain complex with a group of operators, but changed in two significant ways. The first change is that the part $C_2 \to C_1$ is a crossed module, and so may consist of non abelian groups. This amount of non abelian structure allows for the complete modelling of connected, pointed homotopy 2-types, as shown by Mac Lane and Whitehead [28], as well as for carrying detailed information on group presentations. In this form, crossed complexes were used by Blakers in [5], to relate homology and homotopy groups. In the free case, under the name 'homotopy systems', they were strongly used by Whitehead in his paper [38], and also were used to state realization results in the final section of his paper on simple homotopy types [41], as part of his general programme on 'Algebraic homotopy' [39]. They are used in the books by Baues [2, 3] under the name 'crossed chain complexes'.

The second change is that the groups involved are generalized to groupoids. This is necessary for dealing with the equivariant theory, since base points need not be preserved under a group action. It is also essential for dealing with function spaces, since we will be considering the internal hom $\mathcal{CRS}(B, C)$ of crossed complexes, which in dimension 0 consists of the morphisms $B \to C$, and in dimension 1 consists of the homotopies of morphisms. This use is analogous to the fact that the category of groupoids is cartesian closed, unlike that of groups. This general definition of crossed complex was introduced by Brown and Higgins in [9].

Crossed complexes were shown to be equivalent to ∞-groupoids in [11]. In particular, crossed modules over groupoids, i.e. crossed complexes trivial in dimensions > 2, are equivalent to 2-groupoids, and this explains why we are able to include the results of Moerdijk and Svensson in [31].

The notion of Eilenberg–Mac Lane space may be generalized via a 'nerve' functor

$$N : (\text{crossed complexes}) \to (\text{simplicial sets}),$$

which in fact goes back to Blakers [5]. This functor is right adjoint to the fundamental crossed complex functor

$$\pi : (\text{simplicial sets}) \to (\text{crossed complexes}),$$

a functor which is defined on all filtered spaces. Composition of N with geometric realization gives the classifying space functor

$$B : (\text{crossed complexes}) \to (\text{topological spaces}).$$

A first result of [14] is the 'homotopy adjunction' of B and π, that is, a bijection of sets of homotopy classes

(1) $$[\pi X_*, C]_{Crs} \cong [X, BC],$$

for any crossed complex C and CW-complex X with its cellular filtration X_*. This result implies classical results on the homotopy classification of maps into Eilenberg–Mac Lane spaces $K(A, n)$, including the case $n = 1$, and the case of local coefficients.

The category of crossed complexes Crs is shown in [12] to have a monoidal closed structure, so that for all crossed complexes A, B, C there is a natural bijection

(2) $$Crs(A \otimes B, C) \cong Crs(A, \mathcal{CRS}(B, C)).$$

The internal hom $\mathcal{CRS}(B, C)$ is used in [14] to generalize (1) to a natural weak equivalence

(3) $$B(\mathcal{CRS}(\pi X_*, C)) \to \mathcal{TOP}(X, BC)$$

where the right hand object is the space of maps. It is this result which we will generalize to the equivariant case (Theorem 4.1). This is to our knowledge the first equivariant version of results on function spaces of maps into Eilenberg–Mac Lane spaces.

The proof of Theorem 4.1 requires all the techniques used to prove (3), together with:

1. additional information on the Eilenberg–Zilber theorem for crossed complexes, proved by Tonks in [34];
2. results on homotopy coherence developed by Cordier and Porter in [16];
3. general techniques of equivariant theory, as given by say tom Dieck in [19], Lück in [25], Dwyer and Kan in [20].

Crossed complexes do not model all homotopy types, and an equivariant theory of n-types has recently been given by Garzón and Miranda in [23], using cat^n-groups. The advantages of the crossed complex theory are the more detailed results on function spaces, not at present available for more general n-types, and the close relation of crossed complexes will chain complexes with a group(oid) of operators [13]. This will be exploited elsewhere.

In this paper our results will be for the case when the group G of the equivariant theory is discrete (actually, we give a more general case which includes for example the case when G is totally disconnected). The more general case, say when G is a Lie group, will be dealt with elsewhere.

1. CROSSED COMPLEXES

We will refer the reader to the papers, [13], [14], [8], and the thesis [34] for a more detailed treatment of the theory of crossed complexes. We will need various elements of that theory here but will only give a summary.

1.1. Crossed complexes form a locally Kan simplicial category

A crossed complex C is a sequence

$$\longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\delta} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

where

1. C_1 is a groupoid over C_0;
2. $\delta : C_2 \to C_1$ is a crossed module over C_1;
3. C_n is a C_1-module for $n \geq 3$;
4. $\delta : C_n \to C_{n-1}$ is an operator morphism for $n \geq 3$;
5. $\delta\delta : C_n \to C_{n-2}$ is trivial if $n \geq 3$;
6. δC_2 acts trivially on C_n for $n \geq 3$.

Each C_n is thus a groupoid with object set C_0 and with no arrows between distinct objects if $n \geq 2$. With the obvious definition of morphism, we obtain a category *Crs* of crossed complexes.

The basic example that will be used frequently in what follows is the *fundamental crossed complex* $C = \pi X_*$ of a filtered space $X_* = (X_n)_{n \in \mathbb{N}}$. Here $C_0 = \pi_0 X_* = X_0$, as a set; $C_1 = \pi_1 X_*$ is the fundamental groupoid, $\pi_1(X_1, X_0)$, of homotopy classes of paths in X_1 between points in X_0; and if $n \geq 2$, $C_n = (\pi_n(X_n, X_{n-1}, x))_{x \in X_0}$ is the family of relative homotopy groups based at the points of X_0. The boundary maps, δ, are the usual boundaries of the relative homotopy groups and the operation of $\pi_1(X_1, X_0)$ on each of the C_n corresponds to the usual 'change of base point'. We will use this fundamental crossed complex when X_* is the filtered space of a CW-complex with skeletal filtration. If K is a simplicial set, we write πK for $\pi|K|$, where $|K|$ is given the skeletal filtration. In particular, we write $\pi(n)$ for $\pi\Delta[n]$ where $\Delta[n]$ is here the standard simplicial n-simplex.

The *nerve* of a crossed complex C is the simplicial set NC, defined for $n \in \mathbb{N}$ by

$$(NC)_n = \mathcal{C}rs(\pi(n), C).$$

This construction is essentially the one used by Blakers in [5]. The nerve NC of a crossed complex is always a Kan complex, and in fact has the stronger property that any horn has a unique 'thin' filler, where for $n \geq 1$ the thin elements of $(NC)_n$ are the morphisms $f : \pi(n) \to C$ which are trivial on the top dimensional cell of Δ^n (cf. Ashley, [1], Brown and Higgins, [14]).

Proposition 1.1 ([14]). *The functor $\pi : \mathcal{S} \to \mathcal{C}rs$, $K \mapsto \pi K$, where $\mathcal{S}$ is the category of simplicial sets, is left adjoint to the nerve functor $N : \mathcal{C}rs \to \mathcal{S}$.*

The category $\mathcal{C}rs$ may be given the structure of a symmetric monoidal closed category, as shown by Brown and Higgins in [12]. So there is a tensor product $- \otimes -$ and 'internal hom' $\mathcal{CRS}(-,-)$ in $\mathcal{C}rs$ and an exponential law, given in (2) of the Introduction. There is always a natural morphism

$$\pi(X_*) \otimes \pi(Y_*) \to \pi(X_* \otimes Y_*)$$

which is an isomorphism for skeletally filtered CW-complexes X_* and Y_* cf. [12, 14], and even more generally [4]. We will also need a crossed complex form of the usual Eilenberg–Zilber theorem. A proof of this by the method of acyclic models is indicated in [14], but in fact there is an explicit formulation given by Tonks in [34, Theorem 2.3.1].

Proposition 1.2 ([34]). *For any simplicial sets K, L, the crossed complex $\pi K \otimes \pi L$ is a natural strong deformation retract of $\pi(K \times L)$. More precisely, there are natural maps*

$$a_{K,L} : \pi(K \times L) \to \pi(K) \otimes \pi(L)$$
$$b_{K,L} : \pi(K) \otimes \pi(L) \to \pi(K \times L)$$
$$h_{K,L} : \pi(K \times L) \otimes \pi(1) \to \pi(K \times L),$$

such that $a_{K,L}$ and $b_{K,L}$ are homotopy inverses, with $a_{K,L} \cdot b_{K,L} = Id$, $h_{K,L} : b_{K,L} \cdot a_{K,L} \simeq Id$, and $h_{K,L} \cdot (b_{K,L} \otimes Id) = b_{K,L} \otimes 0$.

The formulae for these maps are crossed complex analogues of those given by Eilenberg and Mac Lane in [21], and are given explicitly in [34, Proposition 2.2.3, 2.2.10].

We now show how this gives the category of crossed complexes a crucial simplicial enrichment. Our general convention is that if a category $\mathcal{C}$ is used in both an unenriched and a simplicially enriched form, then the enriched form will usually be underlined. Thus in this case, for objects C, C' of $\mathcal{C}$, the notation $\underline{\mathcal{C}}(C, C')$ denotes the simplicial set determined by this data.

The monoidal closed structure on *Crs* gives it a *Crs*-enriched structure with composition

$$c_{A,B,C} : \mathcal{CRS}(A,B) \otimes \mathcal{CRS}(B,C) \to \mathcal{CRS}(A,C).$$

Recall that

$$N(\mathcal{CRS}(A,B))_n = Crs(\pi(n), \mathcal{CRS}(A,B))$$

and write $\underline{Crs}(A,B)$ for the simplicial set $N(\mathcal{CRS}(A,B))$. The $f \in \underline{Crs}(A,B)_n$, $g \in \underline{Crs}(B,C)_n$ have an 'external composite'

$$(4) \qquad c_{A,B,C} \circ (f \otimes g) : \pi(n) \otimes \pi(n) \to \mathcal{CRS}(A,C).$$

Tensor products, unlike products, do not generally have a diagonal. However, πK has an analogue of the Alexander–Whitney map, namely the composite

$$(5) \qquad \pi K \longrightarrow \pi(K \times K) \xrightarrow{a_{K,K}} \pi K \otimes \pi K,$$

where $a_{K,L}$ is as in Proposition 1.2. Applying this to the simplicial n-simplex $K = \Delta[n]$, and composing with the external composite (4), gives us a composite or convolution product $f * g$ in $\underline{Crs}(A,C)_n$. Thus *Crs* has an $\mathcal{S}$-enriched structure with composition

$$\underline{Crs}(A,B) \times \underline{Crs}(B,C) \to \underline{Crs}(A,C)$$

given by $(f,g) \mapsto f * g$. This gives us the composition and so makes *Crs* into a complete and cocomplete simplicially enriched category in which all the 'hom-sets' are Kan complexes. Thus *Crs* is a locally Kan $\mathcal{S}$-category. It is this structure, rather than the related Quillen model category structure given by Brown and Golasiński in [8], that will be used later to obtain results not only on the homotopy classification of maps, but also on the description and homotopy type of the appropriate function spaces.

1.2. Classifying spaces

Many of the applications of crossed complexes require a classifying space construction and our first main application in this paper will be to extend this to the equivariant setting.

A cubical version of the nerve and classifying space of a crossed complex was introduced in [10], and some properties, such as its homotopy groups, were obtained. The main homotopy classification results were obtained simplicially in [14] (a previous preprint gave cubical versions of these results). We now summarize some of the main facts.

The *classifying space BC* of a crossed complex C is defined to be $|NC|$, the geometric realization of the nerve of C.

• If X is a CW-complex and C is a crossed complex, then there is a weak homotopy equivalence,

$$B(\mathcal{CRS}(\pi X_*, C)) \to \mathcal{TOP}(X, BC)$$

and thus a bijection of sets of homotopy classes,

$$[\pi X_*, C]_{Crs} \cong [X, BC],$$

which is natural with respect to morphisms of C and cellular maps of X [14].

• If Y is a CW-complex with skeletal filtration Y_*, then the map $Y \to B\pi(Y_*)$ determined up to homotopy by the above adjunction has at the base point $y \in Y_0$ a homotopy fibre F_y, say, and the homotopy exact sequence at the point y of the homotopy fibration $F_y \to Y \to B\pi Y_*$ is isomorphic to Whitehead's 'certain exact sequence' [40],

$$\cdots \to \Gamma_n(Y, y) \to \pi_n(Y, y) \to H_n(\tilde{Y}_y) \to \cdots$$

where $\tilde{Y}_y$ denotes the universal cover of Y based at y. Further, if $\pi_i(Y, y) = 0$ for $1 < i < n$, then the fibre F_y is n-connected [10, 1].

• If Y is a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$, and X is a CW-complex with $\dim X \leq n$, then the homotopy fibration $Y \to B\pi Y_*$ induces a bijection

$$[X, Y] \to [X, B\pi(Y_*)]$$

and hence $[X, Y] \cong [\pi(X_*), \pi(Y_*)]_{Crs}$ [10, 38].

In these results we have written $[C, D]_{Crs}$ for $\pi_0 \underline{Crs}(C, D)$, the set of homotopy classes of maps in Crs from C to D.

2. ELEMENTS OF HOMOTOPY COHERENCE

A common method in equivariant theory is that of Quillen model categories. However, this theory is not a strong enough abstract homotopy theory for describing the equivariant analogues of the above results on the homotopy types of spaces of maps from a space to a classifying space. The basis for our theory is instead that of simplicially enriched categories and homotopy coherence (cf. [17]).

The classifying space of an equivariant crossed complex will be defined in Section 4 using the nerve functor and the 'coalescence functor' c of (6) below. Comparison of the formulae we will give for c (see (8) and (9)) with well-known formulae for the double bar construction as studied by May, [29], Elmendorf, [22], Seymour, [32], Meyer, [30], and others, shows that the two concepts of cobar and of coalescence are essentially the same. The terminology we use is designed to emphasize certain universal properties of the constructions, rather than their origins in the classical bar construction.

If G is a Hausdorff topological group, the orbit category, OrG, can be given a natural simplicial enrichment, cf. Dwyer and Kan, [20]. The objects of OrG are the coset spaces G/H with H a closed subgroup of G. The morphism sets of

G-maps from G/H to G/K have a natural topology and some authors (e.g. Elmendorf, [22]) have used this. The category G-$\mathcal{T}op$ of G spaces is naturally $\mathcal{S}$-enriched by

$$G\text{-}\underline{\mathcal{T}op}(X, Y)_n = G\text{-}\mathcal{T}op(X \times \Delta^n, Y),$$

where G acts trivially on Δ^n, $n \in \mathbb{N}$. The simplicial set G-$\underline{\mathcal{T}op}(X, Y)$ also corresponds (in a convenient category of topological spaces) to the singular complex of the space of G-maps from X to Y. The simplicial sets $\underline{Or}G(G/H, G/K)$ are given by G-$\underline{\mathcal{T}op}(G/H, G/K)$. Having said this our *standing assumption* from now on will be that these simplicial sets are discrete (i.e. of the form $K(S, 0)$ for some set S, and thus having S in all dimensions and identity mappings for all simplicial operators). This means that for us OrG is a 'discretely simplicial' category, i.e. is *just* a category. Examples of topological groups G for which this holds are not only discrete groups, but also totally disconnected topological groups. The results in tom Dieck [19] show how in this case G-equivariant homotopy theory can be reduced to considerations of the category of diagrams indexed by OrG (cf. also Cordier and Porter, [17], and the references therein, as well as several papers by Dwyer and Kan in this area).

In [17] a proof of an enriched version of a result of Elmendorf is given. This enriched result states that there are functors

$$(6) \qquad R : G\text{-}\mathcal{T}op \to \mathcal{S}^{OrG^{\mathrm{op}}}, \qquad c : \mathcal{S}^{OrG^{\mathrm{op}}} \to G\text{-}\mathcal{T}op$$

such that for any G-CW-complex X and diagram $T : OrG^{\mathrm{op}} \to \mathcal{S}$, with each $T(G/H)$ Kan, there is a homotopy equivalence,

$$(7) \qquad G\text{-}\underline{\mathcal{T}op}(X, c(T)) \simeq Coh\,\underline{\mathcal{S}}(R(X), T).$$

Here $Coh\,\underline{\mathcal{S}}(R(X), T)$ denotes the simplicial set of *homotopy coherent transformations* from the diagram $R(X)$ to T (see Proposition 2.1 and (11)), and R is defined by

$$R(X)(G/H) = Sing(X^H).$$

The 'coalescence functor' $c : T \mapsto c(T)$ is given by a cobar construction (9).

To define these functors and the equivalence (7), we will work with a general indexing category $\mathcal{A}$ instead of OrG^{op}, and a simplicially enriched receiving category $\mathcal{C}$, which could be simplicial sets, topological spaces, or crossed complexes. (To handle the case of a Lie group G, we would need the domain category $\mathcal{A}$ to be $\mathcal{S}$-enriched, but that extra complication will not be dealt with here.)

We start by introducing coherent coends. These take the place of homotopy colimits in this context. There is a dual theory of coherent ends, which we explain later.

As we want to form indexed colimits, we require the receiving category $\mathcal{C}$ to be cocomplete. Then the $\mathcal{S}$-category $\mathcal{C}$ will be *tensored*, which means that if K is a simplicial set and C an object of $\mathcal{C}$, there is an object $K \overline{\otimes} C$ of $\mathcal{C}$, with a natural isomorphism of simplicial sets

$$\underline{C}(K \bar{\otimes} C, C') \cong \underline{S}(K, \underline{C}(C, C')).$$

If A, A' are objects of $\mathcal{A}$, let $X(A, A')$ be the simplicial set $Ner(A \downarrow \mathcal{A} \downarrow A')$, the nerve of the category of objects under A and over A'. Suppose given a functor

$$Q : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{C}.$$

Then the *homotopy coherent coend* of Q is defined by

$$\oint^{A} Q(A, A) = \int^{A, A'} X(A, A') \bar{\otimes} Q(A', A).$$

As one might expect, this coend has a description as a 'diagonal' of a simplicial object. Define a simplicial object $Z(Q)$ in $\mathcal{C}$ by

$$Z(Q)_n = \coprod \{Q(A_n, A_0) : u \in (Ner\,\mathcal{A})_n, u = (A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} A_n)\}$$

cf. Bousfield and Kan [7]. Then we have

$$\text{(8)} \qquad \oint^{A} Q(A, A) \cong \int^{[n]} \Delta[n] \bar{\otimes} Z(Q)_n.$$

Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A}^{\text{op}} \to \mathcal{S}$ be functors with $\mathcal{C}$ a cocomplete $\mathcal{S}$-category. We define the *coherent mean tensor* to be

$$G \bar{\otimes} F = \oint^{A} GA \bar{\otimes} FA.$$

In particular we will write $\underline{F} : \mathcal{A} \to \mathcal{C}$ for the $\mathcal{S}$-functor given by $\underline{F}(A) = \mathcal{A}(-, A) \bar{\otimes} F$. The natural transformation $\epsilon^F : \underline{F} \to F$ is a levelwise homotopy equivalence. Analysis of the construction of $\underline{F}$ for simple examples of the $\mathcal{S}$-category $\mathcal{A}$, reveals $F \mapsto \underline{F}$ to be a generalization of the construction of a cofibration from an ordinary map. Thus although we are not seeking a Quillen model category structure in this setting, we can think of $\underline{F}$ as 'F made cofibrant'. We will later, (11), give a definition of homotopy coherent transformations in terms of homotopy coherent ends, closely related to the usual representation of natural transformations in terms of ends. However, the following proposition gives a convenient definition using the above notions.

Proposition 2.1 ([18]). *If $\mathcal{C}$ is cocomplete, and $F, G : \mathcal{A} \to \mathcal{C}$, there is a natural isomorphism*

$$Coh(\mathcal{A}, \underline{C})(F, G) \cong \underline{C}^{\mathcal{A}}(\underline{F}, G).$$

We can now define the *coalescence functor* $c : \mathcal{S}^{OrG^{\text{op}}} \to G\text{-}\mathcal{T}op$ as

$$\text{(9)} \qquad c(T) = \oint^{G/H} |T(G/H)| \times G/H,$$

where $|T(G/H)|$ denotes geometric realization of the simplicial set $T(G/H)$. That is, $c(T)$ is the homotopy coherent coend of $Q_T : (G/H, G/K) \mapsto |T(G/H)| \times G/K$.

We are now able to state our enriched version of the result of Elmendorf, [22]. Other authors have obtained variants of this, notably Seymour, [32], and Dwyer and Kan, [20]. The following theorem is one half of [17, Theorem 3.11(i)]. In this theorem, by G-complex we mean G-CW-complex.

Theorem 2.2 ([17]). *The above pair of $\mathcal{S}$-functors*

$$R : G\text{-}\mathcal{T}op \to \mathcal{S}^{OrG^{\mathrm{op}}}, \qquad c : \mathcal{S}^{OrG^{\mathrm{op}}} \to G\text{-}\mathcal{T}op$$

has the properties that if Y is a G-complex, and T is a OrG^{op}-diagram taking Kan values, then there is a homotopy equivalence of Kan simplicial sets

$$G\text{-}\underline{\mathcal{T}op}(Y, c(T)) \simeq Coh\,\underline{\mathcal{S}}(R(Y), T).$$

In order to prove that certain key maps are homotopy equivalences, we need the alternative construction of the simplicial set of homotopy coherent transformations using ends rather than coends. Since we will form indexed limits, we require the receiving $\mathcal{S}$-category $\mathcal{C}$ to be complete. Then $\mathcal{C}$ will be *cotensored*, which means that if K is a simplicial set and C an object of $\mathcal{C}$, there is an object which we will denote by $\bar{\mathcal{C}}(K, C)$ such that there is a natural isomorphism

$$\underline{\mathcal{S}}(K, \underline{\mathcal{C}}(C', C)) \cong \underline{\mathcal{C}}(C', \bar{\mathcal{C}}(K, C)).$$

Suppose

$$Q : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}.$$

We can now define the *homotopy coherent end* of Q by

$$\oint_A Q(A, A) = \int_{A, A'} \bar{\mathcal{C}}(X(A, A'), Q(A, A'))$$

(cf. Cordier and Porter, [18]).

Example 2.3. Suppose $F, G : \mathcal{A} \to \mathcal{C}$ are two $\mathcal{S}$-functors, and set $Q(A, A') = \underline{\mathcal{C}}(FA, GA')$. Then $\oint_A Q(A, A)$ can be interpreted as the simplicial set of homotopy coherent transformations from F to G. This will be denoted by $Coh(\mathcal{A}, \underline{\mathcal{C}})(F, G)$ or more simply by $Coh\,\underline{\mathcal{C}}(F, G)$ if the codomain is the important information to remember whilst the domain is fixed, or just by $Coh(F, G)$ if there is no danger of confusion.

Given an $\mathcal{S}$-functor $Q : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$, we can construct a cosimplicial object in $\mathcal{C}$, denoted $Y(Q) : \Delta \to \mathcal{C}$, by

$$(10) \qquad Y(Q)^n = \prod \{Q(A_0, A_n) : u \in (Ner\,\mathcal{A})_n, u = (A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} A_n)\}.$$

The coface and codegeneracy maps are given by formulae analogous to those of the 'cosimplicial replacement' construction of Bousfield and Kan, [7], and are given in detail in [18]. As is now standard, the Bousfield–Kan homotopy limit of a diagram of simplicial sets can be given as a 'total complex' of a cosimplicial simplicial set constructed from the given data. If Y is a cosimplicial simplicial set,

$$Tot(Y) = \int_{[n]} \underline{S}(\Delta[n], Y^n_\bullet)$$

and so is the simplicial set of natural transformations with domain the Yoneda embedding $\Delta : \boldsymbol{\Delta} \to \mathcal{S}$, considered as a cosimplicial simplicial set, and with codomain Y. Analogously, for $Y : \boldsymbol{\Delta} \to \mathcal{C}$ a cosimplicial object in a cotensored complete $\mathcal{S}$-category $\mathcal{C}$, one can define the 'total object' by

$$Tot(Y) = \int_{[n]} \bar{\mathcal{C}}(\Delta[n], Y^n).$$

Applying this to (10) gives another formulation of coherent ends.

Lemma 2.4 ([18]). *Given an $\mathcal{S}$-functor $Q : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{C}$ as above, there is a natural isomorphism*

$$\oint_A Q(A, A) \cong Tot(Y(Q)).$$

Suppose as before that $F, G : \mathcal{A} \to \mathcal{C}$ are two functors. Then the simplicial set $Coh(F, G)$ is given by

$$(11) \qquad Coh(F, G) = \int_{A, A'} \underline{S}(X(A, A'), \underline{\mathcal{C}}(FA, GA')),$$

or as above by $Tot(Y(F, G))$ where

$$Y(F, G)^n = \prod \{\underline{\mathcal{C}}(FA_0, GA_n) : u \in (Ner\,\mathcal{A})_n, u = (A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} A_n)\}.$$

We will need later the following results from [16]:

- If $\mathcal{A}$ is an $\mathcal{S}$-category, and $Q : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{S}$ is an $\mathcal{S}$-functor such that each $Q(A, A')$ is a Kan complex, then $\oint_A Q(A, A)$ is a Kan complex. This follows from Axiom SM7 on page 277 of [7], since in this case $Y(Q)$ is a fibrant cosimplicial simplicial set.
- If $P, Q : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{S}$ are $\mathcal{S}$-functors such that each $P(A, A')$ and $Q(A, A')$ is a Kan complex, and

$$\theta(A, A') : P(A, A') \to Q(A, A')$$

is a natural transformation of $\mathcal{S}$-bifunctors such that each $\theta(A, A')$ is a homotopy equivalence, then θ induces a homotopy equivalence

$$\oint_A \theta(A, A) : \oint_A P(A, A) \to \oint_A Q(A, A).$$

3. DIAGRAMS OF CROSSED COMPLEXES AND THEIR NERVES

In this section we enrich and generalize Proposition 1.1, i.e. Corollary 3.5 of Brown and Higgins, [14].

Proposition 3.1. *Let K be a simplicial set and C a crossed complex. Then there is a natural set of strong deformation retraction data*:

$$\underline{S}(K,NC) \underset{a^*}{\overset{b^*}{\rightleftarrows}} \underline{Crs}(\pi K, C),$$

with b^*a^* the identity, and a natural homotopy

$$h^* : \underline{S}(K,NC) \to \underline{S}(\Delta[1], \underline{S}(K,NC))$$

between a^*b^* and the identity.

Note that N and π are not adjoint at the enriched level even though they are adjoint in the ordinary sense. The obvious 'enrichment' of π, while it does not strictly respect the enriched composition structures, is a *homotopy coherent enrichment*, as made precise in [34]; similarly a^* and b^* are $\mathcal{S}$-natural in C but only up to coherent homotopy in K. All this results from the fact that the Eilenberg–Zilber maps are natural homotopy equivalences, but are not isomorphisms, and because the Alexander–Whitney diagonal is cocommutative only up to homotopy.

Proof of 3.1. [34] This will make extensive use of the Eilenberg–Zilber theorem for crossed complexes (Proposition 1.2). For ease of typing we will abbreviate the maps of that theorem as $a_{K,L}$ to $a_{K,n}$, etc. when $L = \Delta[n]$. We note that $a_{K,0}$ is an isomorphism with inverse $b_{K,0}$, and $h_{K,0}$ is a trivial homotopy.

Following the argument in Brown and Higgins, [14], we obtain a map from $\underline{S}(K,NC)$ to $\underline{Crs}(\pi K, C)$ as the following composite: in dimension n

$$\begin{aligned}\underline{S}(K,NC)_n &\cong \mathcal{S}(\Delta[n] \times K, NC) \\ &\cong Crs(\pi(\Delta[n] \times K), C) \xrightarrow{b^*_{K,n}} Crs(\pi(n) \otimes \pi(K), C) \\ &\cong \underline{Crs}(\pi(K), C)_n.\end{aligned}$$

As $b_{K,n}$ is naturally cosimplicial in n, this does define a simplicial map in the right direction. The reverse map is obtained by reversing the isomorphisms and replacing $b^*_{K,n}$ by $a^*_{K,n}$ in the opposite direction. The result thus reduces to finding homotopies from $b^*_{K,n} \cdot a^*_{K,n}$ and $a^*_{K,n} \cdot b^*_{K,n}$ to the corresponding identities. However, by using $h_{K,n}$ and $a_{K,n} \cdot b_{K,n} = Id$ and the Eilenberg–Zilver maps once more, this is quite easy. □

The naturality of a^*, b^* and h^* enables us to pass quickly to the case when K and C vary functorially, $K : \mathcal{A} \to \mathcal{S}$, $C : \mathcal{A} \to Crs$ with $\mathcal{A}$ a small category.

Proposition 3.2. *Let $K : \mathcal{A} \to \mathcal{S}$, $C : \mathcal{A} \to Crs$ be functors. Then there is a homotopy equivalence*

$$\underline{S}^{\mathcal{A}}(K,NC) \simeq \underline{Crs}^{\mathcal{A}}(\pi K, C).$$

In fact $\underline{Crs}^{\mathcal{A}}(\pi K, C)$ is a strong deformation retract of $\underline{S}^{\mathcal{A}}(K,NC)$.

Proof. The natural transformations a^*, b^* and h^* induce:

$$\int_{\mathcal{A}} \underline{S}(K,NC) \underset{\int a^*}{\overset{\int b^*}{\rightleftarrows}} \int_{\mathcal{A}} \underline{Crs}(\pi K, C),$$

and

$$\int h^* : \int_{\mathcal{A}} \underline{S}(K,NC) \to \underline{S}(\Delta[1], \int_{\mathcal{A}} \underline{S}(K,NC))$$

and these give the result. □

A similar proof yields

Proposition 3.3. *Let $K : \mathcal{A} \to \mathcal{S}$, $C : \mathcal{A} \to Crs$ be functors. Then there is a homotopy equivalence*

$$Coh\,\underline{S}(K,NC) \simeq Coh\,\underline{Crs}(\pi K, C).$$

An elegant proof of this uses the formulation in terms of two bifunctors

$$P(A,A') = \underline{S}(KA, NCA') \qquad \text{and} \qquad Q(A,A') = \underline{Crs}(\pi KA, CA'),$$

both of which take Kan values, and then it uses the fact that $b^* : P \to Q$ is natural and levelwise a homotopy equivalence to conclude that $\oint_{\mathcal{A}} b^* : \oint_{\mathcal{A}} P \to \oint_{\mathcal{A}} Q$ is a homotopy equivalence.

4. THE EQUIVARIANT CLASSIFYING SPACE OF A CROSSED COMPLEX

The usual constructions of classifying spaces involve the use of nerve and geometric realization functors. In the equivariant set-up, R behaves like a singular complex constructor, while c is a 'realization functor'. (This viewpoint has already been noted and used by Dwyer and Kan in [20] in a similar context.) It should therefore come as no surprise that the construction of $B^G C$, the equivariant classifying space of the OrG^{op}-crossed complex C, is directly given as cNC. The proof that this works uses much of the machinery we have developed earlier.

Recall that G is a topological group such that OrG is a discretely simplicial category.

Theorem 4.1. *If C is an OrG^{op}-diagram of crossed complexes, then there is a functorially defined G-space $B^G C$ such that for any G-CW-complex X, there is a natural homotopy equivalence*

$$G\text{-}\underline{Top}(X, B^G C) \simeq Coh\,\underline{Crs}(\pi R(X), C)$$

of Kan complexes. Consequently there is a bijection

$$[X, B^G C]_G \cong \pi_0\, Coh\,\underline{Crs}(\pi R(X), C).$$

Corollary 4.2. *There is a modified fundamental crossed complex functor*

$$\underline{\pi R(X)} : OrG^{\text{op}} \to Crs$$

such that

$$G\text{-}\underline{Top}(X, B^G C) \simeq \underline{Crs}^{OrG^{\text{op}}}(\pi R(X), C).$$

Proof of Corollary. Use the functor

$$\underline{\pi R(X)}(G/H) = OrG(G/H, \_) \bar{\otimes} \pi R(X)(G/H),$$

i.e. make $\pi R(X)$ a cofibrant diagram. □

Proof of Theorem. As suggested above, let $B^G C = cNC$. We have

$$\begin{aligned} G\text{-}\underline{Top}(X, B^G C) &\simeq Coh\,\underline{S}(R(X), NC) \\ &\simeq Coh\,\underline{Crs}(\pi R(X), C), \end{aligned}$$

as required. Taking π_0 of both sides gives the last statement. □

This diagram $\pi R(X)$ of crossed complexes is closely linked to a construction used by Lück [25, p. 144] and many others, namely the *fundamental category* $\Pi(G, X)$ of a G-space, X. This category has as objects, G-maps $x : G/H \to X$ (abbreviated to $x(H)$), and a morphism $(\sigma, [\omega]) : x(H) \to y(K)$ consists of a map $\sigma : G/H \to G/K$ in OrG and a homotopy class $[\omega]$ relative to $G/H \times \partial I$ of G-maps $\omega : G/H \times I \to X$ with $\omega_1 = x$, and $\omega_0 = y \circ \sigma$. The composition is given by a formula of the semidirect product type. Lück remarks [25, p. 145, Remark 8.17] that $\Pi(G, X)$ is the homotopy colimit of the functor

$$OrG^{\text{op}} \to Gpd$$

given by $G/H \mapsto \Pi X^H$, where Π is the fundamental groupoid functor.

Our results can be applied to equivariant groupoids, because a groupoid C may also be considered as a crossed complex of rank ≤ 1, i.e. with no non-trivial structure above level 1.

Proposition 4.3. *Let C be a groupoid, considered as a crossed complex of rank ≤ 1. Denote also by C the OrG^{op}-diagram of crossed complexes which is constant with value C. Then for any G-CW-complex X, there is a natural homotopy equivalence*

$$Coh\,\underline{Crs}(\pi R(X), C) \simeq \underline{Gpd}(\Pi(G, X), C).$$

Proof. The bottom level of $\pi R(X)(G/H)$ has the fundamental groupoid of X^H as quotient. Then the coherent end passes inside the 'internal-hom' giving a homotopy colimit. Lück's remark completes the proof. □

A classifying space $B^G C$ in the case when C is an OrG^{op}-groupoid and G is a compact Lie group is considered by Lück in [24]. It is claimed there that the existence of an equivalence $\mu : \Pi R(B^G C) \to C$ (applied in [26] and [27]) follows from [22], but this is not the case. This equivalence was proved by Jan-Alve

Svensson in [33] for the case when G is a discrete group, and in this case follows also from our results.

Proposition 4.4. *Let C be an OrG^{op}-diagram of crossed complexes. Then, in the diagram category $Crs^{OrG^{op}}$, there exists a natural map*

$$\mu : \pi R(B^G C) \to C$$

which is a level weak homotopy 1*-equivalence, that is induces a level homotopy equivalence*

$$\mu : \Pi R(B^G C) \to \Pi C$$

in the diagram category $Gpd^{OrG^{op}}$.

Proof. By [17, Proposition 3.5 (ii)], there is a natural level homotopy equivalence $\eta' : Rc(T) \to T$ for any $T \in S^{OrG^{op}}$, and in particular for $T = NC$. So we have a natural level homotopy equivalence $\pi(\eta') : \pi Rc(NC) \to \pi NC$. But it is standard for crossed complexes that there is a natural map $\pi NC \to C$ which is a level weak homotopy 1-equivalence. □

We now show briefly how these results specialize to other results in the literature.

For a group Γ, Γ-module A, and $n \geq 2$, let $\mathbb{C}(\Gamma, A, n)$ denote the crossed complex which is Γ in dimension 1, A in dimension n, with the given action of Γ, and all boundaries are 0. Then $\mathbb{C}(\Gamma, A, n)$ is the crossed complex obtained by applying the functor Θ of [13] to the chain complex with Γ as group of operators and in which only A in dimension n is non zero. It is shown in [14] how for a CW-complex X the set of homotopy classes $[X, B\mathbb{C}(\Gamma, A, n)]$ corresponds to the union of certain cohomology groups $H^n_\chi(X, A)$ with local coefficients determined by the system χ. It is in this way that our results specialize to those of [24].

There are various categories equivalent to the category of crossed complexes, for example the category of ∞-groupoids [11], the category of ω-groupoids [9], and the category of simplicial T-complexes [1]. We have chosen to give the exposition here in the category of crossed complexes because it is nearer to the traditional category of chain complexes with a group, or groupoid, of operators, and also because the functor π on filtered spaces is generally familiar in this case. Indeed, the direct constructions of equivalent functors with values in the other categories are considerably more complicated: the proof for ω-groupoids is given in [10], and for simplicial T-complexes in [1]. A further point is that an explicit construction of the monoidal closed structure is, at present, published only for the categories of crossed complexes and of ω-groupoids [12].

As explained earlier, homotopy 2-types are modelled by the category of crossed modules over groupoids, which can be regarded as crossed complexes which are trivial in dimensions > 2. So we are able to recover results of Moerdijk and Svensson [31] on equivariant 2-types, but with results on function spaces and not just homotopy classes of maps.

It is also possible to formulate a result for equivariant homotopy types with non-trivial homotopy groups only in dimensions 1 and n. Such a homotopy type is modelled by a crossed complex C which has precisely these groups as its homology.

REFERENCES

1. Ashley, N.D. – Simplicial T-complexes and crossed complexes. University of Wales PhD thesis, 1978, published as Diss. Math. **165** (1988).
2. Baues, H.J. – Algebraic homotopy. Cambridge Studies in Advanced Mathematics **15**, Cambridge University Press (1988).
3. Baues, H.J. – Combinatorial homotopy and 4-dimensional CW-complexes. De Gruyter Expositions in Mathematics **2**, De Gruyter, Berlin, New York (1991).
4. Baues, H.J. and R. Brown – On relative homotopy group of the product filtration, the James construction and a formula of Hopf. J. Pure Appl. Algebra **89**, 49–61 (1993).
5. Blakers, A.L. – Some relations between homology and homotopy groups. Annals of Math. **49**, 428–461 (1948).
6. Bourn, D. and J.-M. Cordier – A general formulation of homotopy limits. J. Pure Appl. Algebra **29**, 129–141 (1983).
7. Bousfield, A.K. and D.M. Kan – Homotopy limits, completions and localizations. Lecture Notes in Math. **304**, Springer, Berlin (1972).
8. Brown, R. and M. Golasiński – The closed model category of crossed complexes. Cahiers Topologie et Géom. Différentielle Catégoriques **30**, 61–82 (1989).
9. Brown, R. and P.J. Higgins – On the algebra of cubes. J. Pure Applied Algebra **21**, 233–260 (1981).
10. Brown, R. and P.J. Higgins – Colimit theorems for relative homotopy groups. J. Pure Appl. Algebra **22**, 11–41 (1981).
11. Brown, R. and P.J. Higgins – The equivalence of ∞-groupoids and crossed complexes. Cahiers Topologie et Géom. Différentielle Catégoriques **22**, 371–386 (1981).
12. Brown, R. and P.J. Higgins – Homotopies and tensor products for ω-groupoids and crossed complexes. J. Pure Appl. Algebra **47**, 1–33 (1987).
13. Brown, R. and P.J. Higgins – Crossed complexes and chain complexes with operators. Math. Proc. Camb. Phil. Soc. **107**, 33–57 (1990).
14. Brown, R. and P.J. Higgins – The classifying space of a crossed complex. Math. Proc. Camb. Phil. Soc. **110**, 95–120 (1991).
15. Cordier, J.-M. – Sur la notion de diagramme homotopiquement cohérent. Cahiers Topologie et Géom. Différentielle Catégoriques **23**, 93–112 (1982).
16. Cordier, J.-M. and T. Porter – Vogt's theorem on categories of homotopy coherent diagrams. Math. Proc. Cambridge Philos. Soc. **100**, 65–90 (1986).
17. Cordier, J.-M. and T. Porter – Categorical aspects of equivariant homotopy. Applied Categorical Structures, to appear.
18. Cordier, J.-M. and T. Porter – Homotopy coherent category theory. Trans. Amer. Math. Soc., to appear.
19. tom Dieck, T. – Transformation groups. De Gruyter Studies in Mathematics **8**, De Gruyter, Berlin, New York (1987).
20. Dwyer, W.G. and D.M. Kan – Singular functors and realization functors. Proc. Kon. Akad. van Wetensch. A **87** = Indag. Math. **46**, 147–153 (1984).
21. Eilenberg, S. and S. Mac Lane – On the groups $H(\pi, n)$, I. Ann. Math. **59**, 55–106 (1953).
22. Elmendorf, A. – Systems of fixed point sets. Trans. Amer. Math. Soc. **277**, 275–284 (1983).
23. Garzón, A.R. and J.G. Miranda – Models for homotopy n-types in diagram categories. Preprint, Granada, March 1994, 13 pages.

24. Lück, W. – Equivariant Eilenberg–Mac Lane spaces $K(G, \mu, 1)$ for possibly non-connected or empty fixed point sets. Manuscripta Math. **58**, 67–75 (1987).
25. Lück, W. – Transformation groups and algebraic K-theory. Lecture Notes in Mathematics, No. **1408**, Springer-Verlag (1989).
26. Lück, W. and I. Madsen – Equivariant L-theory, I. Math. Z. **203** (3), 503–526 (1990).
27. Lück, W. and I. Madsen – Equivariant L-theory, II. Math. Z. **204** (2), 253–268 (1990).
28. Mac Lane, S. and J.H.C. Whitehead – On the 3-type of a complex. Proc. Natl. Acad. Sci., 41–48 (1950).
29. May, J.P. – Classifying spaces and fibrations. Mem. Amer. Math. Soc. **155** (1975).
30. Meyer, J.-P. – Bar and cobar constructions, I. J. Pure Appl. Algebra **33**, 163–207 (1984).
31. Moerdijk, I. and J.-A. Svensson – Algebraic classification of equivariant homotopy 2-types, I. J. Pure Appl. Algebra **89**, 187–216 (1993).
32. Seymour, R.M. – Some functorial constructions on G-spaces. Bull. London Math. Soc. **15**, 353–359 (1983).
33. Svensson, J.-A. – Equivariant Eilenberg–Mac Lane spaces of type 1. Math. Scand. **68** (1), 19–26 (1991).
34. Tonks, A.P. – Theory and applications of crossed complexes. Ph.D. Thesis, University of Wales, Bangor (1993).
35. Vogt, R.M. – Homotopy limits and colimits. Math. Z. **134**, 11–52 (1973).
36. Whitehead, J.H.C. – On adding relations to homotopy groups. Ann. Math. **42**, 409–428 (1941).
37. Whitehead, J.H.C. – Note on a previous paper entitled 'On adding relations to homotopy groups'. Ann. Math. **47**, 806–810 (1946).
38. Whitehead, J.H.C. – Combinatorial homotopy, II. Bull. Amer. Math. Soc. **55**, 453–496 (1949).
39. Whitehead, J.H.C. – Algebraic Homotopy Theory. In: Proc. ICM Harvard Vol. II, 354–357; also in: Coll. Works, Vol. III (I.J. James, ed.), Pergamon, Oxford, 257–260 (1962).
40. Whitehead, J.H.C. – A certain exact sequence. Ann. Math. **52**, 51–110 (1950).
41. Whitehead, J.H.C. – Simple homotopy types. Amer. J. Math. **72**, 1–57 (1950).

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