Diagonal approximations for some fundamental crossed complexes: Torus, Klein Bottle, Projective Plane*

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Introduction

J.H.C. Whitehead developed in [Whi49]. following [Bla48], the tool of what he called the *homotopy system* of a CW-complex X and which we now call the *fundamental crossed complex* ΠX_* of the CW-complex X with its skeletal filtration X_*. Full details of this construction are covered in the book [BHS11].

A key innovation in Whitehead's paper was the notion of *free crossed module*, and the theorem that the group $\pi_2(X \cup \{e^2_\lambda\}, X, x)$, regarded as a crossed $\pi_1(X, x)$-module, can be presented as the *free crossed module* on the characteristic maps of the 2-cells. An exposition of Whitehead's proof is in [Bro80], and the proof of a much more general result using a 2-dimensional Seifert-van Kampen Theorem is in [BH78] and [BHS11].

Also covered in that book is the monoidal closed structure on crossed complexes, giving an exponential law

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)).$$

This is derived from a clear and intuitive description of a monoidal closed structure on the equivalent category of (strict) cubical ω-groupoids with connections. The complications of the description of the tensor product of crossed complexes relate to the complications of the cell structure of the product of two cells E^m, E^n where E^0 is a singleton, $E^1 = \{0, 1\} \cup e^1$ and for $n \geqslant 2$, $E^n = e^0 \cup e^{n-1} \cup e^n$. However for filtered spaces X_*, Y_* there is a natural morphism

$$\Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*) \qquad (1)$$

*This is a typed, expanded and revised version of some handwritten notes dated 6:00am 26/07/1992.

which is an isomorphism for CW-complexes with their skeletal filtrations.

Note that the tensor product $C \otimes D$ of two crossed complexes is equipped with two projections $p_1 : C \otimes D \to C, p_2 : C \otimes D \to D$. A *diagonal* for a crossed complex C is a morphiam $\phi : C \to C \otimes C$ such that $p_1\phi, p_2\phi$ are each homotopic to the identity $C \to C$. If X_* is a CW-filtration, then the cellular approximation theorem and the isomorphism (1) implies that ΠX_* admits a diagonal.

The book [BHS11] explains how the category of crossed complexes is convenient for many purposes of algebraic topology, and has advantages over the widely used category of chain complexes with a group of operators. As Whitehead wrote in the Introduction to [Whi49], but in our terminology: crossed complexes have better realisation properties than chain complexes with a group of operators. For example crossed complexes completely capture weak homotopy 2-types, which is not so for chain complexes with a group of operators. We note from [BHS11] that:

• there is a functor ∇ from crossed complexes to chain complexes with a groupoid of operators (§7.4.11);

• ∇ has a right adjoint, and so preserves colimits (Prop. 7.4.29);

• if X_* is the skeletal filtration of a CW-complex, then $\nabla\Pi(X_*)$ is isomorphic to the cellular chain complexes of the universal covers of X at its various vertices, with the fundamental groupoid $\pi_1(X, X_0)$ as groupoid of operators (Prop. 8.4.2); and

• ∇ preserves tensor products (Theorem 9.5.4).

One of the themes of the book [BHS11] is that the category of crossed complexes is convenient both for geometric intuition and for calculation. The aim of this note it to illustrate the latter by calculating diagonal approximations $C \to C \otimes C$ for the fundamental crossed complexes of the Klein Bottle and the Torus. The usual diagrams for these spaces are given in the next section, and we note that the 2-cell k of the Klein bottle can be regarded as the generator of a free crossed module with boundary

$$\delta(k) = b - a + b + a. \tag{2}$$

Thus this setup reflects the geometry more than the usual chain complex algebra, in which we have

$$\partial(k) = 2b,$$

unless we work with chain complexes of universal covers. In that context the equation (2) becomes the more complicated

$$\delta(k) = (\tilde{b})^{-a+b+a} - (\tilde{a})^{-a-b} + (\tilde{b})^{a} + \tilde{a} \tag{3}$$

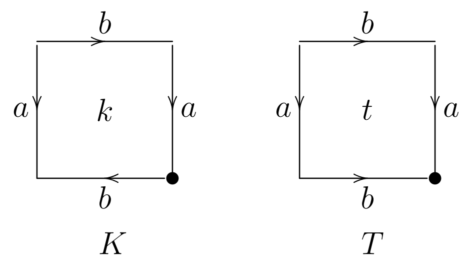
since the 1-dimensional chains of the universal cover are freely generated as a $\pi_1(K)$-module by elements $\tilde{a}, \tilde{b}$, and the map from $\pi_1 K^1$ to the 1-dimensional chains of the universal cover is a derivation. This fact is of course related to the Fox free differential calculus, which is foreseen in [Whi49].

A further advantage of the category of crossed complexes is that the functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ is defined homotopically; this contrasts with the usual somewhat circuitous route for the cellular chain complex of a CW-complex as determined using singular homology. However the major properties of Π are not established directly, but via a category of multiple groupoids hinted at above.

The aim of this note is to illustrate the methods of crossed complexes by giving some simple examples of calculation with these tools, namely of diagonal approximations for the fundamental crossed complexes of the Torus and Klein Bottle.

1 Diagonals for the fundamental crossed complexes of the Klein Bottle, Torus and Projectivs Plane

Standard diagrams for the Klein Bottle K and Torus T are as follows:



Here $\bullet$ denotes the base point. Let $C = \Pi T_*,\ C' = \Pi K_*$. Then $C_1 = C'_1$ is the free group on generators a, b and C_2, C'_2 are the free crossed C_1-modules on the generator t for the torus, k for the Klein bottle with

$$\delta t = -b - a + b + a, \qquad \delta k = b - a + b + a.$$

In dimension 1, both of the tensor products $C \otimes C,\ C' \otimes C'$ are the free group on generators a_1, b_1, a_2, b_2; in dimension 2 they are respectively the free crossed $(C \otimes C)_1$- module on generators

$$a_1 \otimes a_2, a_1 \otimes b_2, b_1 \otimes a_2, b_1 \otimes b_2$$

and t_1, t_2 for the Torus, k_1, k_2 for the Klein bottle, where always if c, d are in dimension 1, then

$$\delta(c \otimes d) = -d - c + d + c,$$

while for $i = 1, 2$:

$$\delta t_i = -b_i - a_i + b_i + a_1,$$
$$\delta k_i = b_i - a_i + b_i + a_i.$$

A diagonal $\Delta : C \to C \otimes C, \Delta' : C' \to C' \otimes C'$ is given in dimension 1 by

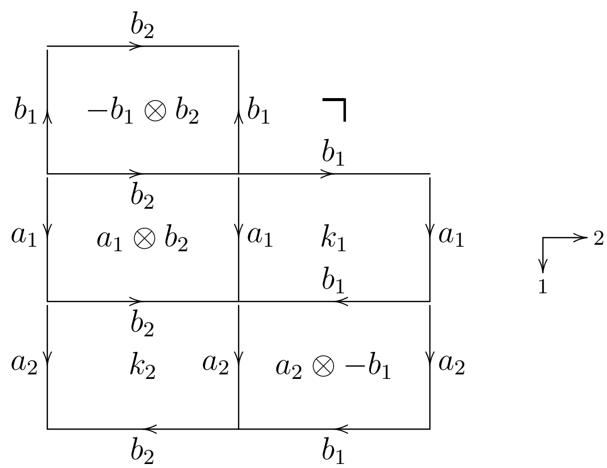
$$\Delta a = \Delta' a = a_1 + a_2, \quad \Delta b = \Delta' b = b_1 + b_2.$$

Hence

$$\begin{aligned}\Delta\delta t = \Delta(-b - a + b + a) &= -b_2 - b_1 - a_2 - a_1 + b_1 + b_2 + a_1 + a_2,\\ \Delta'\delta k = \Delta'(b - a + b + a) \ \ &= b_1 + b_2 - a_2 - a_1 + b_1 + b_2 + a_1 + a_2.\end{aligned}$$

We need to find elements of $(C \otimes C)_2, (C' \otimes C')_2$ with these elements as boundary.

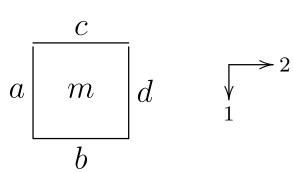
We first consider the Klein Bottle K and for this consider the following subdivided square diagram, whose boundary is $\Delta'\delta k$:



(4)

In order to determine the element of $(C' \otimes C')_2$ this represents, we use double groupoid techniques, which apply here since all the cells are of square shape.

Recall from [BS76, BHS11] that given a crossed module $\mathcal{M} = (\mu : M \to P)$ we obtain a double groupoid $\lambda(\mathcal{M})$ which is P in dimension 1 and whose squares are quintuples $(m : a\ {}^{c}_{b}\ d)$, such that $a, b, c, d \in P, m \in M$ and $\mu(m) = -b - a + c + d$. This is also written



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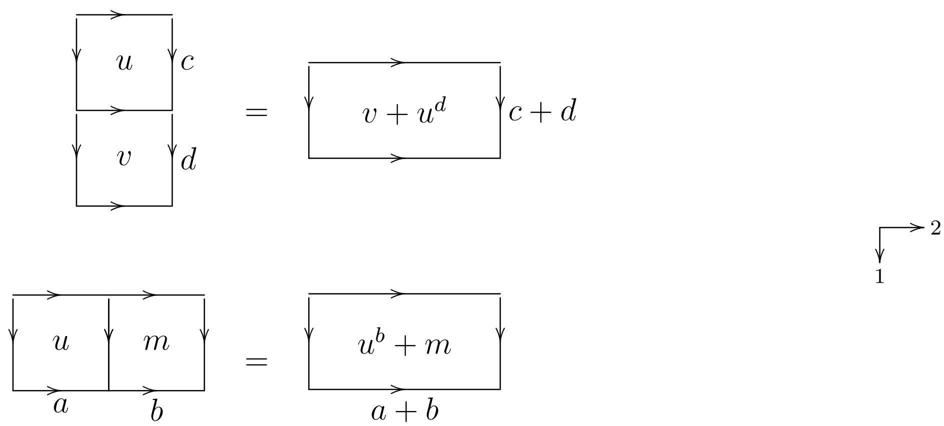
In particular the corner square denoted by ⌝ in diagram (4) corresponds to the quintuple $\big(0 : -b_1 \,{}^{0}_{b_1}\, 0\big)$.

Vertical and horizontal compositions of quintuples are defined by



Of course these formulae are determined by the convention for the quintuple and the need to get the correct boundaries for the compositions.

So we can evaluate in the associated crossed module the total composition in diagram (4). The first column evaluates to

$$\alpha = k_2 + (a_1 \otimes b_2)^{a_2} + (-b_1 \otimes b_2)^{a_1+a_2}$$

and the second column evaluates to

$$\beta = a_2 \otimes b_1 + (k_1)^{a_2}$$

so that the final result is

$$\gamma = \alpha^{-b_1} + \beta,$$

and we therefore may set

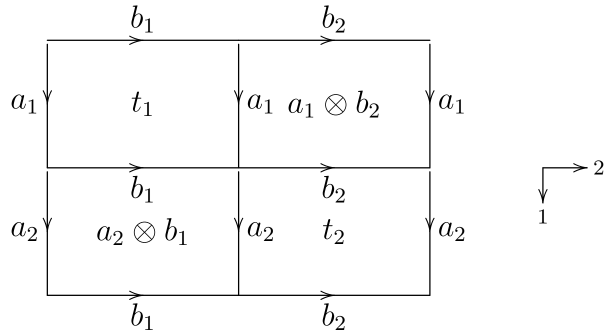
$$\Delta'(k) = \gamma.$$

Note that $p_1(\gamma) = k_1, p_2(\gamma) = k_2$, since $p_1(a_2) = 0, p_2(b_1) = 0$.

Maybe the conclusion is that the 2-dimensional diagram (4) is more revealing than the formula for γ. However the formula may be better for further computation.

Note also that equivalent ways of evaluation, e.g. rows first and then columns, are equal by the interchange law, and the resulting crossed module elements are equal by the rules for crossed modules, particularly the second rule[1], which is equivalent to the interchange law.

The work for the Torus is similar but simpler. We consider the diagram

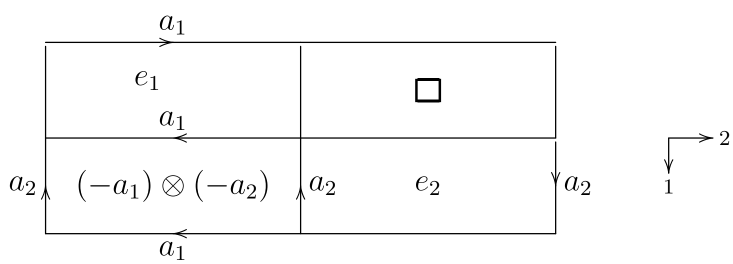


(5)

So we can set

$$\Delta(t) = (a_2 \otimes b_1)^{b_2} + (t_1)^{a_2+b_2} + a_2 \otimes b_2 + (t_2)^{a_2}.$$

Finally we consider the Projective Plane, P, with cell structure $e^0 \cup e^1 \cup e^2$, and with boundary of the 2-cell e given by $\delta e = 2a$, say. Thus a diagonal in dimension 1 is given by $\Delta = a_1 + a_2$. The following is a diagram whose boundary is $a_1 + a_2 + a_1 + a_2$:



(6)

This evaluates in the crossed module to

$$(-a_1) \otimes (-a_2) + e_1^{-a_2} + e_2$$

and this is the value we can take for Δe.

An alternative procedure is to use the methods of van Kampen diagrams as in Section 3.1.ii of [BHS11].

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[1]The reader of [ML98] should note that the final section omits the second rule for a crossed module.

References

[Bla48] Blakers, A. 'Some relations between homology and homotopy groups'. *Ann. of Math. (2)* **49** (1948) 428–461.

[Bro80] Brown, R. 'On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead'. *J. London Math. Soc. (2)* **22** (1) (1980) 146–152. 1

[BH78] Brown, R. and Higgins, P. J. 'On the connection between the second relative homotopy groups of some related spaces'. *Proc. London Math. Soc. (3)* **36** (2) (1978) 193–212. 1

1

[BHS11] Brown, R., Higgins, P.J. and Sivera, R., *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics Vol. 15, 703 pages. (August 2011). 1, 2, 4, 6

[BS76] Brown, R. and Spencer, C. B. 'Double groupoids and crossed modules'. *Cahiers Topologie Géom. Différentielle* **17** (4) (1976) 343–362. 4

[ML98] Mac Lane, S. *Categories for the working mathematician*, 2nd edition, *Graduate Texts in Mathematics*, Volume 5. Springer-Verlag, New York (1998). 6

[Whi49] Whitehead, J. H. C. 'Combinatorial homotopy. II'. *Bull. Amer. Math. Soc.* **55** (1949) 453–496. 1, 2, 3