CHAPTER 5

Johnstone's Topos

5.0 Introduction

In the previous chapter we studied the embedding of SEQ into the quasitopos SuSEQ . Johnstone [J2] embedded the category SEQ into a topos. In the next chapter we will study general topology in Johnstone's topos. So we devote this chapter to give Johnstones description of his topos. We will give the proofs in more detail trying to make them readable with a little background on topos theory.

Johnstone's topos can be described as follows:

Let Σ be the full subcategory of Top with two objects $1 = \{*\}$ and N^+ the one point compactification of the set N of the natural numbers. Impose on Σ the canonical Grothendieck topology J. Johnstone uses the anodyne notation $\mathcal{E}$ for the topos $sh(\Sigma,J)$ of sheaves on the site (Σ,J) , and this is the topos he is investigating. Note that an object X of $sh(\Sigma,J)$ gives rise to two sets $X(1)$ and $X(N^+)$; clearly $X(1)$ should be called the underlying set of X . Johnstone points out that each element a of $X(N^+)$ determines a map $\bar{a}: N^+ \to X(1)$; he suggests thinking of a as a proof that $\bar{a}$ is a convergent sequence.

Lawvere (in private conversation with R. Brown) has suggested that elements of $X(N^+)$ be called processes, thus dignifying them by a name, rather than making them subservient to the convergent sequences. We follow this sugestion, and so call X a process space and write Proc for the category $sh(\Sigma, J)$.

5.1 Grothendieck topologies

We begin this section by reminding the reader of the definition of Grothendieck topology. This material is adapted from [B-W] , [Fr] , [G-V] , [J1] , [K-M] , [K-W] , [M-R] and [Wr] . For a category C and object U of C , a sieve S on U is a subfunctor of the hom functor, $hom(-,U)$; that is if V is an object of C , then $S(V)$ is a subset of $hom(V,U)$ and these inclusions for all V form the components of a natural transformation.

Definition 5.1.1 Let C be a small category. A Grothendieck topology J on C is an assignment to each object U of C a set $J(U)$ of U-sieves, called covering sieves, such that:

(i) For each U in $ob(C)$, the maximal sieve, $hom(-,U)$ itself is in $J(U)$.

(ii) Stability with respect to change of basis:

If $R \in J(U)$ and $f: V \to U$ is a morphism of C , then the pullback $f^*(R) = \{\alpha: W \to V \mid f\alpha \in R\}$ of R along f is a V-covering sieve.

(iii) Local character:

If R is in $J(U)$ and S is another sieve on U such that for each $(f: V \to U)$ in R , $f^*(S)$ is in $J(V)$, then S is in $J(U)$.

A small category with a Grothendieck topology is called a <u>site</u> . (In the literature, this is often called a small site). The idea behind defining a topology on a category is to generalise the notion of a *classical* sheaf [T] to an arbitrary category, rather than the category of open sets of space. Now for a given family F of presheaves there is a Grothendieck topology J such that each presheaf in F is a sheaf, in fact a largest such topology exists. In particular the <u>canonical Grothendieck topology</u> is the largest for which each representable presheaf is a sheaf, more precisely J-sheaf.

We now discuss a simple way of describing the canonical Grothendieck topology on any small C with pullbacks.

Let (C,J) be a <u>site</u>. A presheaf X is a <u>J-sheaf</u> if and only if for each object U of C and for each covering sieve $R = \{\alpha_j : U_j \to U \mid j \in I\}$

$$i^*: \mathrm{Nat}\,[U,X] \to \mathrm{Nat}\,[R,X]$$

is bijective, where $i: R \to U$ is the inclusion of the subpresheaf R on the presheaf U .

This definition is equivalent to the following condition:

Given any compatible family $(s_j)_{j \in I}$, $s_j \in X(U_j)$ for each $U_j \xrightarrow{\alpha_j} U$ in R , there is a unique $s \in X(U)$ which restricts to s_j in $X(U_j)$ for each j .

The above condition can be reformulated in terms of a compatible family of morphisms instead of a compatible family of elements s_j . This can be done since each element s_j in $X(U_j)$ can be identified with

$$s_j : \mathrm{hom}(-,U_j) \to X \quad .$$

Then the compatibility will mean the following:

For each i,j in I we have

$$s_i \circ \mathrm{hom}(-,\alpha_j') = s_j \circ \mathrm{hom}(-,\alpha_i')$$

where α_i' and α_j' are the pullback of α_j and α_i in C ,

$$\begin{tikzcd}
U_i \times_U J_j \arrow[r, "\alpha_j'"] \arrow[d, "\alpha_i'"'] & U_i \arrow[d, "U_i"] \\
U_j \arrow[r, "\alpha_j"'] & U
\end{tikzcd}$$

That is the following diagram

$$\begin{tikzcd}
& \mathrm{hom}(-,U_i) \arrow[dr, "\alpha_i^*"] \arrow[drr, "s_i"] & & \\
\mathrm{hom}(-,U_i \times_U U_j) \arrow[ur, "\mathrm{hom}(-,\alpha_j')"] \arrow[dr, "\mathrm{hom}(-,\alpha_j')"'] & & \mathrm{hom}(-,U) & X \\
& \mathrm{hom}(-,U_j) \arrow[ur, "\alpha_j^*"] \arrow[urr, "s_j"'] & &
\end{tikzcd}$$

commutes. Such a compatible family of morphisms is known as a compatible family of morphisms from the covering sieve R to the presheaf X. The sheaf condition will be equivalent to the existence of a unique morphism $\hom(-,U) \xrightarrow{\alpha} X$ such that

$$s_i = \alpha \circ \alpha_i^* \quad \text{and} \quad s_j = \alpha \circ \alpha_j^*$$

$$\begin{tikzcd}
\hom(-,U_i) \arrow[dr, "\alpha_i^*"'] \arrow[drr, "s_i"] & & \\
& \hom(-,U) \arrow[r, "\alpha"] & X \\
\hom(-,U_j) \arrow[ur, "\alpha_j^*"] \arrow[urr, "s_j"'] & &
\end{tikzcd}$$

Let $R = \{\alpha_j : U_j \to U \mid j \in I\}$ be a family of morphism in C.

<u>Definition 5.1.2</u> R is said to be an <u>epimorphic family</u> if for any pair $U \overset{\beta}{\underset{\gamma}{\rightrightarrows}} V$ of morphisms with $\beta\alpha_j = \gamma\alpha_j$ for all j, then $\beta = \gamma$.

R is called an <u>effective epimorphic family</u> if any compatible family $\{\beta_j : U_j \to V \mid j \in I\}$ of morphism factors uniquely through R.

<u>Remark 5.1.3</u> Clearly any effective epimorphic family is an epimorphic family. And it is easy to see that a covering U-sieve R is effective epimorphic if and only if $\hom(-,U)$ satisfies the sheaf condition for R.

<u>Definition 5.1.4</u> R is said to be <u>universally</u>, or stable, <u>effective epimorphic</u> if its pullback along any morphism $U' \xrightarrow{\alpha} U$ of C is effective epimorphic. That is the family $\{\alpha_j'\}$ is effective epimorphic

$$\begin{tikzcd} U_j \times_U U' \arrow[r, "\alpha_j'"] \arrow[d] & U' \arrow[d, "\alpha"] \\ U_j \arrow[r, "\alpha_j"'] & U \end{tikzcd}$$

<u>Lemma 5.1.5</u> A topology J on C is the canonical Grothendieck topology if and only if each covering sieve is universally effective epimorphic.

The following lemma is needed in the next section.

<u>Lemma 5.1.6</u> If R is the empty sieve on U, then R is effective epimorphic if and only if U is an initial object.

2. Johnstone's topos

Let Σ be the full subcategory of Top whose objects are $1 = \{*\}$, the terminal object in Top, and N^+, the one point compactification of the set N of natural numbers.

We now follow [J2] in describing a topology J on Σ in terms of its family of sieves and showing that J is the canonical Grothendieck topology on Σ (Prop. 5.2.6). We will however give the proofs in more detail than in [J2] in order to make them readable for non-experts in topos theory.

<u>Lemma 5.2.1</u> [J2] If R is a universally effective sieve on N^+ in Σ, then $n: 1 \to N^+$ is in R for each $n \in N^+$.

<u>Proof</u>. Assume that $n: 1 \to N^+$ is not in R for some $n \in N^+$. Then it is clear that the pullback $n^*(R)$ of R along n is the empty sieve on 1. But 1 is not an initial object of Σ, so $n^*(R)$ is not epimorphic (Lemma 5.1.4). That is R is not a universally effective sieve. □

Let U be an infinite subset of N. Let $f_U: N^+ \to N^+$ denote the unique strictly increasing function with image $U \cup \{\infty\}$. Then f_U is continuous.

<u>Lemma 5.2.2</u> [J2] Let R be an effective epimorphic sieve on N^+. Then R contains f_U for some infinite subset U of N.

<u>Proof</u>. Assume not, that is R is an effective epimorphic sieve on N^+ such that f_U is not in R for all infinite subsets U of N. We then claim that for each $f \in R$ the image of f is infinite. Suppose there is an f in R with U = image f infinite; then it is easy to see that there is $g: N^+ \to N^+$ such that $f \circ g = f_U$. By the sieve axioms, $f_U \in R$, and this contradicts our assumption.

Now consider the function $f_\infty: N^+ \to N^+$ defined as $f_\infty(n) = n$ for all $n \in N$ and $f_\infty(\infty) = 0$. Then f_∞ is continuous on every finite subset, and it follows that

$$\mathrm{id}: R(1) \to N^+(1) = \hom(1, N^+)$$
$$f_\infty^*: R(N^+) \to N^+(N^+) = \hom(N^+, N^+)$$

form the components of a morphism $R \to N^+ = \hom(-,N^+)$ which has no extension to a morphism $N^+ = \hom(-,N^+) \to N^+ = \hom(-,N^+)$. That is $N^+ = \hom(-,N^+)$ fails to satisfy the sheaf axioms for R , so R is not an effective epimorphic (Remark 1.2). □

Proposition 5.2.3 [J2] If R is a universally effective epimorphic sieve on N^+ , then for each infinite subset S of N there is an infinite $U \subseteq S$ with $f_U \in R$.

Proof. Assume that $f_U \notin R$ for all infinite subsets U of S . Then the pullback of R along f_S is not an effective epimorphic, fails to satisfy lemma 5.2.2. Otherwise, let $f_U \in f_S^*(R)$ for some infinite U of N , so the composite

$$N^+ \xrightarrow{f_U} N^+ \xrightarrow{f_S} N^+$$

is in R . But $f_S \circ f_U = f_{f_S[U]} \in R$, which is impossible since $f_U \notin R$ for all infinite U of N . □

Proposition 5.2.4 [J2] Let $J(1)$ be the set consisting of the maximal sieve on 1 . Let $J(N^+)$ be the set of sieves on N^+ such that a sieve R is in $J(N^+)$ if and only if

(i) $n: 1 \to N^+$ is in R for each $n \in N^+$, and

(ii) if T is an infinite subset of N , then f_U is in R for some infinite subset U of T .

Then J is a Grothendieck topology on Σ .

Proof. The maximal sieves on N^+ and 1 are in $J(N^+)$ and $J(1)$.

For the proof of stability with respect to change of base, we consider two cases.

Case 1: Consider a sieve R on $J(1)$. Then R is the maximal sieve on 1, so the pullback of R along any morphism $f: V \to U$ of Σ, is the maximal sieve on V.

Case 2: Let $R \in J(N^+)$. Then $n: 1 \to N^+$ is in R for each $n \in N^+$, so $n^*(R)$ is in $J(1)$ for all n.

If $g: N^+ \to N^+$ is continuous, then $gn \in R$ for each $n: 1 \to N^+$. So n is in $g^*(R)$ for each n. To show that $g^*(R)$ satisfies (ii), let T be an infinite subset of N. if $g[T]$ is infinite then there is an infinite $S \subseteq g[T]$ such that $f_S \in R$, since R is in $J(N^+)$. Then $W = T \cap g^{-1}(S)$ is an infinite subset of T and $g \circ f_W = f_S \in R$, so f_W is in $g^*(R)$.

Now if the image of g is finite, then $Z = T \cap g^{-1}(\{n\})$ is infinite for some n in $g[T]$. So $g \circ f_Z = C_n$ the constant sequence with value n. But C_n is in R, since n is in R. So f_Z is in $g^*(R)$.

For the proof of the local character;

Let R and S be sieves on N^+ such that R is a covering sieve and for each $(f: V \to U)$ in R, $f^*(S)$ is in $J(V)$. Let $n \in N^+$, then $n^*(S)$ is in $J(1)$ so $n^*(S) = \{\text{maximal sieve on } 1\}$. Thus $\mathrm{id}: 1 \to 1$ is in $n^*(S)$, that is $n \circ \mathrm{id}$ is in S. So S satisfies (i).

Let T be an infinite subset of N , then there is an infinite subset U of T such that $f_U \in R$. Since $f_U^*(S) \in J(N^+)$, so there is an infinite set W such that $f_W \in f_U^*(S)$. So $f_{f_U[W]} = f_U \circ f_W \in S$, and $f_U[W]$ is an infinite subset of T. Hence S satisfies (ii) , so S is in $J(N^+)$. □

Proposition 5.2.5 [J2] Let X be an object of SuSEQ , the category of subsequential spaces. Then

$$\hom_{SuSEQ}(1,X) : \Sigma \to \mathrm{Set}$$

is a J-sheaf.

Proof. For $R \in J(1)$, $R \cong \hom(-,1)$. So $i^*: \mathrm{Nat}\,[\hom(-,1)\,,\,\hom(-,X)] \to \mathrm{Nat}\,[R,\hom(1,X)]$ is bijective.

Let $R \in J(N^+)$ and $f: R \to \hom_{SuSEQ}(-,X)$ be a morphism in $\mathrm{Set}^{\Sigma^{op}}$.

Since $R(1) = \{n: 1 \to N^+ \mid n \in N^+\}$, so f determines a sequence (x_n) of points of $X = \hom_{SuSEQ}(1,X)$ together with a point x_∞ . Then the naturality of f implies that each subsequence of (x_n) contains a sequence that converges to x_∞ . By definition of subsequential space, the sequence (x_n) must itself converge to x_∞ . So there exists a unique extension of

$$f: \; R \longrightarrow \hom_{SuSEQ}(-,X)$$

to

$$N^+ \longrightarrow \hom_{SuSEQ}(-,X)$$

namely

$$((x_n), x_\infty) \in \hom_{SuSEQ}(N^+, X)$$

$$\cong \mathrm{Nat}\,[\hom_{Top}(-, N^+), \hom_{SuSEQ}(-, X)] \ . \quad \square$$

In particular for a topological space X the presheaf $\hom_{Top}(-,X)$, which is isomorphic to $\hom_{SuSEQ}(-,\hat{X})$ where $\hat{X}$ is the coreflection of X in SEQ , Chapter 4 , is a J-sheaf.

Proposition 5.2.6 [J2] The topology J coincides with the canonical Grothendieck topology on Σ .

Proof. By Lemma 5.2.1 and Proposition 5.2.3 , every universally effective epimorphic sieve is in J . So J contains the canonical Grothendieck topology. Also the above remark shows that each representable presheaf is a J-sheaf , so J is contained in the canonical Grothendieck topology. $\square$

Let Proc denote the category of sheaves on Σ , that is $\mathrm{Proc} = \mathrm{Sh}(\Sigma, J)$.

Proposition 5.2.7 [J2] Let X be an object of Top . Then

$$H: \ \mathrm{Top} \longrightarrow \mathrm{Proc}$$
$$X \longrightarrow \hom(-,X)$$

is a functor and the restriction of H to SEQ is fully faithful.

Proof. Clearly H is a functor.

To show that H is fully faithful, let $\eta \in \mathrm{Proc}(HX,HY)$, for X and Y objects of SEQ . Then $\eta\colon HX \to HY$ has components

$$\eta_1 : \; HX(1) \longrightarrow HY(1)$$
$$\eta_{N^+}: \; HX(N^+) \longrightarrow HY(N^+)$$

so η is induced by a unique function η_1 . We have to show that η_1 is continuous. By naturality of η , η_1 maps convergent sequences to convergent sequences, so η_1 is continuous. Hence H is fully faithful. □

Let X be an object of Proc . Then X is a sheaf on Σ . We mentioned in the introduction to this chapter that X determines two sets $X(N^+)$ and $X(1)$. We follow a suggestion of W. Lawvere and call the elements of $X(N^+)$ processes on the underlying set $X(1)$.

Remark: Let $p \in X(N^+)$. Then p determines in $X(1)$ a sequence (p_n) and a point p_∞ as follows:

each $n\colon 1 \to N^+$ induces a map

$$X_n\colon X(N^+) \to X(1) \quad ,$$

so each process p in $X(N^+)$ determines a sequence $(X_n(p))_{n \in N}$ together with a point $X_\infty(p)$ thought of as a limit of the sequence. Then the definition of subsequential can be considered as rules of deduction for these processes. For $p \in X(N^+)$ we will also write $\bar{p} = ((x_n),x_\infty)$ for the underlying sequence determined by p .

We will describe the subobject classifier using the processes, but first we need the following definition.

<u>Definition 5.2.8</u> [J2] A <u>closed ideal</u> of subsets of N is a pair (E,I), with E a subset of N and I is a set of infinite subsets of E satisfying the following conditions:

1. If M is in I, then every infinite subset of M is in I.
2. If M is an infinite subset of E such that every infinite subset of M contains an element of I, then M is in I.

The set E is called the <u>extent</u> of the closed ideal. The closed ideal whose extent E is finite or whose $I=\emptyset$ is called an <u>empty closed ideal</u>, and the closed ideal (E,M_E), where M_E is the set of all infinite subsets of E, is the <u>largest closed ideal</u> whose extent is E.

If I is non-empty then any e in E is in some set of I. For this let M be a set of I; thus $M \cup \{e\}$ satisfies 2.

<u>Proposition 5.2.9</u> [J2] (i) The subobjects of $\hom(-,N^+)$ in Proc with ∞ as a point of the underlying set are bijective with the set of closed ideals of subsets of N.

(ii) The subobjects of $\hom(-,N^+)$, in Proc, which do not contain the point ∞ are in one to one correspondence with the power set of N.

<u>Proof</u>. Let C be the set of closed ideals of subsets of N, and $\mathrm{sub}^\infty(N^+)$ the set of subobjects R of N^+ with $\infty \in R(1)$. Define

$$\theta\colon \mathcal{C} \to \mathrm{Sub}^{\infty}(N^{+})$$
$$E \to \bar{E}$$

as $\quad \theta(E,I)(1) = \bar{E}(1) = E \cup \{\infty\}$

$\bar{E}(N^{+}) = \theta(E,I)(N^{+}) = \{g\colon N^{+} \to N^{+} \mid g$ factors through f_T for some $T \in I\}$

and $\quad \phi\colon \mathrm{Sub}^{\infty}(N^{+}) \to \mathcal{C}$

as $\quad \phi(R) = (E,I)$ where $E = R(1) \setminus \{\infty\}$,

$$I = \{T \subseteq N \mid f_T \in R\,N^{+}\}$$

Then $\quad \phi(\theta(E,I)) = \phi(\bar{E})$

$$= (E,\bar{I}) \quad , \text{ where } \bar{I} = \{T \subseteq N \mid f_T \in \bar{E}\,N^{+}\} \quad .$$

To show $\bar{I} = I$, clearly $f_T \in \bar{E}N^{+}$ for all $T \in I$. So if $T \in \bar{I}$, then $f_T \in \bar{E}N^{+}$. But f_T factors through f_T only, so $T \in I$. Also, $f_T \in \bar{E}(N^{+})$ for each $T \in I$, so $I \subseteq \bar{I}$. Hence $\phi\theta(E,I) = (E,I)$. Similarly $\theta\phi(R) = R$. So θ and ϕ are inverses of each other.

(iii) Any subset M of N determines a unique subsheaf of the representable sheaf N^{+} which can be described as follows. Define a subsheaf M as

$$M(1) = M$$
$$M(N^{+}) = \hom(N^{+},M)$$
$$= \{p \mid p \text{ is a process in } N^{+}(N^{+}) \text{ such that the underlying sequence } \bar{p} \text{ is in } M(1)\} \quad .$$

Then it is easy to see that the inclusions $M(1) \to N^{+}(1)$ and $M(N^{+}) \to N^{+}(N^{+})$ form the components of a monic morphism in Proc . $\square$

Proposition 5.2.10 [J2] The subobject classifier Ω in Proc can be described as follows.

$\Omega(1) = \{t,f\}$, and for any sequence (ω_n) of $\Omega(1)$ there is a unique process p such that $\bar{p} = ((\omega_n),f)$, and the processes q with $\bar{q} = ((\omega_n),t)$ are indexed by the closed ideals whose extent is $\{n \mid \omega_n = t\}$.

Proof. By the definition of Ω , $\Omega(N^+)$ is the set of subobjects of N^+ in Proc . So, by 5.2.9 , Ω is the subobject classifier.

Now to illustrate how the classifier works. Let $m: Y \to X$ be a monic in Proc. Then we can assume m gives inclusions $Y(1) \to X(1)$, $Y(\mathbb{N}^+) \to X(\mathbb{N}^+)$ (cf. 6.1.3) . Then the classifying map $\phi: X \to \Omega$ can be described as follows: $\phi_1: X(1) \to \Omega(1)$ is the characteristic function of $Y(1) \to X(1)$. For $\phi^+: X(N^+) \to \Omega(N^+)$, let $p \in X(N^+)$ such that $\bar{p} = ((n_x),x_\infty)$. If $x_\infty \notin Y(1)$, then $\phi^+(p)$ is the unique process with $\overline{\phi^+(p)} = ((\phi_1(x_n)),f)$. Also if $x_\infty \in Y(1)$ and $x_n \in Y(1)$ finitely often, then $\phi^+(p)$ is the process indexed by the empty closed ideal. Finally if $x_\infty \in Y(1)$ and $x_n \in Y(1)$ infinitely often then $\phi^+(p)$ is the process indexed by the set of infinite subsets of $\{n \mid x_n \in Y(1)\}$ for which the corresponding restriction of p is in $Y(N^+)$. □

CHAPTER 6

The Relations Between SEQ , SuSEQ and Proc

Introduction

We have seen, Chapter 4, that the problem of finding a space which contains $P_c(X,Y)$ and $P_0(X,Y)$ as subspaces cannot be solved in SuSEQ in general.

In a topos all notions of monic are equivalent so

$$Y^{\sim} \longrightarrow Y^{!}$$

and

$$\hat{Y} \longrightarrow Y^{!}$$

are strong monics, subobjects, in Proc . That is $Y^{\sim}$ and $\hat{Y}$ are "subspaces" of $Y^{!}$ in Proc .

However the extension of the space to work in Proc allows "subspaces" to include, intuitively, all continuous injections. The implications of this are not so easy to perceive. One aim of this chapter is an attempt to understand this, by studying the subobject classifier in detail.

Johnstone proved that the embedding SEQ → Proc has a good colimits preservation [J2] , he argued that all topological construction involving CW-complexes may be formed in Proc and still get the same results as in SEQ . We will give further properties of the embedding, for instance we show that SEQ → Proc preserves function spaces.

6.1 The subobject classifier

In this section our main concern is a detailed study of the subobject classifier and an illustration of how it works.

For example let $i: Y \to X$ be a strong monic in the category of sequential spaces SEQ, so Y is a subspace of X. Now the embedding $H: \text{SEQ} \to \text{Proc}$ has a left adjoint, so H preserves limits, in particular, H preserves equalisers. But in SEQ strong monic is an equaliser, Chapter 4. So $H(i): HY \to HX$ is monic, since in Proc all notions of monic are equivalent. That is the embedding H preserves subspaces. In fact Proposition 6.1.4 shows that the image of each monic in SEQ is a monic in Proc. But monic in SEQ is just an injective. So non-subspaces in SEQ become "subspaces" in Proc. It is also the case for SuSEQ that monics in SuSEQ become strong monics.

We now give an example to illustrate the classifying map, cf. Proposition 5.2.10.

<u>Example 6.1.1</u> Let (X,I) be the indiscrete space and let τ be any sequential topology on X. Now $id: (X,\tau) \to (X,I)$ is a monic in SEQ. So $H(id): H(X,\tau) \to H(X,I)$ is monic in Proc (6.1.3(i)) and the classifying ϕ of $H(id\ )$ can be described as follows:

ϕ is the characteristic function of $H(id)$, and for $p \in H(X,I)(N^+)$

$$\phi^*(p) = \begin{cases} (N, M_N) & \text{if } p \in H(X,\tau)N^+) \\ (N, \emptyset) & \text{if } p \notin H(X,\tau)(N^+) \end{cases} .$$

$$\begin{tikzcd} H(X,\tau)(1) \arrow[r, "H(id)'"] \arrow[d] & H(X,I)(1) \arrow[d, "\phi"] \\ 1 \arrow[r, "true"'] & \Omega(1) \end{tikzcd}$$

$$\begin{tikzcd}
H(X,\tau)(N^+) \arrow[r, "H(id)^+"] \arrow[d] & H(X,I)(N^+) \arrow[d, "\phi^+"] \\
1 \arrow[r, "true^+"'] & \Omega(N^+)
\end{tikzcd}$$

The following fact gives a necessary and sufficient condition for a morphism in the functor category to be monic, epic. The proof may be found in [S] .

<u>Proposition 6.1.2</u> [S] Let $\mathcal{C}$ be finitely complete and finitely cocomplete, and let $\mathcal{D}$ be any category. Let $\beta\colon S \to T$ be a natural transformation of functors $\mathcal{D} \to \mathcal{C}$. Then

(i) β is monic in $\mathrm{Fun}[\mathcal{D},\mathcal{C}]$ if and only if $\beta_D\colon SD \to TD$ is monic in $\mathcal{C}$ for all $D \in |\mathcal{D}|$.

(ii) β is epic in $\mathrm{Fun}[\mathcal{D},\mathcal{C}]$ if and only if $\beta_D\colon SD \to TD$ is epic in $\mathcal{C}$ for all $D \in |\mathcal{D}|$.

We now give a necessary and sufficient condition for a morphism in Proc to be monic.

<u>Proposition 6.1.3</u> Let $f\colon Y \to X$ be a morphism in Proc . Then
(i) f is such that $f'\colon Y(1) \to X(1)$ and $f^+\colon Y(N^+) \to X(N^+)$ are monics if and only if f is monic in Proc.
(ii) If f is such that $f'\colon Y(1) \to X(1)$ and $f^+\colon Y(N^+) \to X(N^+)$ are epics then f is epic in Proc .

<u>Proof</u>. (i) Let f be a monic in Proc . Then f is an equaliser. But the inclusion $\mathrm{Proc} \to \mathrm{Set}^{\Sigma^{op}}$ preserves limits. So f is monic in $\mathrm{Set}^{\Sigma^{op}}$. Hence f' and f^+ are monics.

The converse is trivial, since any monic in $\mathrm{Set}^{\Sigma^{op}}$ is a monic in Proc .

(ii) If f is such that f' and f^+ are epics then f is epic in $\mathrm{Set}^{\Sigma^{op}}$. So f is epic in Proc . □

We now state the result that relates monic and epic in SEQ to their images in SuSEQ and in Proc .

<u>Proposition 6.1.4</u> (i) The following are equivalent:

1) m is monic in SEQ .

2) km is monic in SuSEQ .

3) $Hm = hkm$ is monic in Proc , (cf. next section for definition of h) .

(ii) m is epic in SEQ if and only if km is epic in SuSEQ .

<u>Proof</u>. (i) We will show that if m is monic then Hm is monic. The other implications follow easily from 6.1.3 and 4.5.2 (iii) .

Let f and f' be morphisms in Proc such that f' , $f: Z \to HY$ and $(Hg)f' = (Hg)f$.

Clearly $f_1 = f'_1 : Z(1) \to (HY)(1)$, since g is injective.

To show $f_{N^+} = f'_{N^+}$, note that $(HY)(N^+)$ contains exactly one process to each convergent sequence. Now let $p \in Z(N^+)$ with $\bar{p} = ((x_n), x_\infty)$. Then

$$f_{N^+}(p) = ((f(x_n)), f(x_\infty)) = ((f'(x_n)), f'x_\infty) = f'_{N^+}(p) .$$

Hence $f = f'$.

(ii) The proof follows from 4.5.2 (iii) . □

We now give an example to show that the functor $H: SEQ \to Proc$ does not in general preserve epics.

Example 6.1.5 Consider the identity $id:(X,D) \to (X,I)$ as an epic-monic in SEQ, where X has more than one point. Then $H(id)$ is not epic. For this consider g and $\phi_{id}: H(X,I) \to \Omega$ where ϕ_{id} is the classifying map and g is defined by

$$g(1)(x) = t \quad \text{for all} \quad x \in X$$

and $$g^+(p) = (N,M_N) \quad \text{for all} \quad p \in H(X,I)(N^+) .$$

Then trivially $\phi_{id}(1) \circ H(id)(1) = g(1) \circ H(id)(1)$. Also for $p \in H(X,D)(N^+)$

$$\begin{aligned}\phi^+_{id} \circ (H(id))^+(p) &= \phi^+_{id}(p) \\ &= \text{the process indexed by } (N,M_N) \\ &= g^+(p) \\ &= g^+(H(id))^+(p).\end{aligned}$$

Hence $\phi_{id} \circ H(id) = g \circ H(id)$. However $g \neq \phi_{id}$, consider a process $p \notin H(X,D)(N^+)$. □

Similar arguments to that of Example 5.3.4 shows that the functor $h: SuSEQ \to Proc$, h is defined in the next section, does not preserve epics.

We now give a sufficient condition on epic i in SEQ, or in SuSEQ, for the image $H(i)$ to be epic in Proc.

Proposition 6.1.5 Let $i: X \to Y$ be an epic in SEQ, or SuSEQ, such that X is initial with respect to (i,Y). Then $H(i): HX \to HY$, or $h(i)$, is epic in Proc.

Proof. The proof follows easily from 6.1.3. □

However the condition that X must be initial with respect to (i,Y), in order for the image Hi of $i: X \to Y$ to be epic in Proc , is not necessary.

Example 6.1.7 Let X be a non-indiscrete space with more than one point, and let $Y = \{*\}$. Then $i: X \to Y$ is not initial with respect to (i,Y) but $H(i)$ is epic.

We now state the result that will illustrate how the classifying map can be used to classify particular kinds of monics.

Proposition 6.1.8 Let X and Y be objects of Proc such that both X and Y are sequential spaces, and $Y(1) \subseteq X(1)$ and $Y(N^+) \subseteq X(N^+)$. Consider the inclusions $Y(1) \to X(1)$ and $Y(N^+) \to X(N^+)$ as components of a monic $i: Y \to X$ of Proc . Let ϕ_i be the classifying map of i. Then

(i) Y is an open subspace of X, as sequential space, if and only if whenever a process p converges to t in $\phi_i^+(XN^+)$ then there is a cofinite set K in N such that p is indexed by (K,M_K).

(ii) Let Y be a subspace of X. Y is a closed subspace of the sequential space X if and only if whenever a process p converges to f in $\phi_i(XN^+)$, then $\{n \mid p_n = t\}$ is finite.

Proof. Let Y be an open subspace of X. Then the following diagrams

$$\begin{tikzcd} Y(1) \arrow[r, "i"] \arrow[d] & X(1) \arrow[d, "\phi_i'"] \\ 1 \arrow[r, "\text{true}"'] & \Omega(1) \end{tikzcd} \qquad \begin{tikzcd} YN^+ \arrow[r, "i^+"] \arrow[d] & XN^+ \arrow[d, "\phi_i^+"] \\ 1 \arrow[r, "\text{true}^+"'] & \Omega(N^+) \end{tikzcd}$$

commute. So, if p is a process in $\phi_i^+(XN^+)$ that converges to t, then there exists a process q in $X(N^+)$ with $\phi_i'(q_\infty) = p_\infty = t$. So q_∞ is in $Y(1)$, that is $E = \{n \mid p_n = f\}$ is finite. So p is indexed by the largest closed ideal where the extent is $K = N \setminus E$.

Conversely let x_n be a sequence in X converging to x_∞ in Y. Consider the unique process q with $\bar{q} = ((x_n), x_\infty)$. Then $\phi_i^+(q)$ is a process converging to t, so $\phi_i^+(q)$ is indexed by the largest closed ideal whose extent is $N \setminus \{n_1, \ldots, n_k\}$ for some finite k. That is finitely many terms of (x_n) are in $X \setminus Y$. Hence Y is an open subspace of X.

(iii) Let p be a process convering to f, p is in $\phi_i^+(XN^+)$ and $E = \{n \mid p_n = t\}$ is infinite. Then the restriction of p to E determines a sequence of $Y(1)$ that converges to a point not in $Y(1)$. That is Y is not a closed subspace of X.

Conversely let (x_n) be a sequence in Y converging to a point x_∞ not in Y. If q is the unique process in $X(N^+)$ with $\bar{q} = ((x_n), x_\infty)$, then $\phi_i^+(q)$ is a process in $\phi_i^+(XN^+)$ with $\{n \mid \phi(q_n) = t\}$ infinite. $\square$

6/2 Properties of the embedding SEQ → SuSEQ → Proc

The aim of this section is to show that the embedding H: SEQ → Proc preserves function spaces. We will factor the adjunction $(H' \dashv H)$: SEQ → Proc as

$$\begin{tikzcd}
\mathrm{SEQ} \arrow[rr, shift right, "H"'] \arrow[dr, shift right, "k"'] & & \mathrm{Proc} \arrow[ll, shift right, "H'"'] \arrow[dl, shift right, "h'"'] \\
& \mathrm{SuSEQ} \arrow[ul, shift right, "k'"'] \arrow[ur, shift right, "h"'] &
\end{tikzcd}$$

with $h' \dashv h$. So for sequential spaces X and Y

$$H\,[\,P_c(X,Y)\,] \cong \mathrm{Proc}(HX, HY^{\sim})$$
$$\cong h\,[\,\mathrm{SuSEQ}(kX, kY^{\sim})\,] \;,$$

and
$$H\,[\,P_0(X,Y)\,] \cong \mathrm{Proc}(HX, H\hat{Y})$$
$$\cong h\,[\,\mathrm{SuSEQ}(kX, k\hat{Y})\,] \;.$$

Define a functor h: SuSEQ → Proc as follows:

For $X \in |\mathrm{SuSEQ}|$,

$$h(X) = h_X = \mathrm{hom}(-,X) : \Sigma^{op} \to \mathrm{Set}$$

and h is defined in the obvious way on morphisms.

<u>Proposition 6.2.1</u> The functor h: SuSEQ → Proc has a left adjoint h' .

<u>Proof</u>. Define a functor h': Proc → SuSEQ as follows:

For $Y \in |\mathrm{Proc}|$, $h'(Y) = (\underline{Y}, C_{\underline{Y}})$ where

$$\underline{Y} = Y(1)$$

and $C_{\underline{Y}} = \{\bar{p} \mid p$ is a process in $Y(N^+)\}$.

And for a Proc-morphism

$\eta: \; Y \to Z$

$h'(\eta) \; = \; \eta_1$.

Then h' is a functor (cf the remark after 5.2.7).

Now let $X \in |\text{SuSEQ}|$. We will show that

$\theta_{Y,X}: \; \text{Proc}(Y,h(X)) \; \to \text{SuSEQ}(h'(Y),X)$

is an isomorphism, where $\theta_{Y,X}(\eta) = \eta_{(1)}$.

Let η , $\tau: Y \to h_X$ be morphisms such that

$\eta_{(1)} = \tau_{(1)} : Y(1) \to \hom(1,X)$.

For $p \in Y(N^+)$ with $\bar{p} = ((x_n),x_\infty)$, consider $n: 1 \to N^+$.
Then

$$\begin{aligned} n^*(\eta^+(p)) &= \eta_{(1)}(Y_n(p)) \\ &= \eta_{(1)}(x_n) \\ &= \tau_{(1)}(x_n) \\ &= \tau_{(1)}(Y_n(p)) \\ &= n^*(\tau^+(p)) \end{aligned}$$

$$\begin{tikzcd} Y(N^+) \arrow[r, "\eta^+"] \arrow[r, bend left, "\tau^+"] \arrow[d, "Y_n"'] & \hom(N^+,X) \arrow[d, "n^*"] \\ Y(1) \arrow[r, "\tau_{(1)} = \eta_{(1)}"'] & \hom(1,X) \end{tikzcd}$$

So $\tau^+(p) = \eta^+(p)$, since X is a subsequential space.

Hence $\theta_{Y,X}$ is injective.

Also each $f: Y(1) \to \hom(1,X)$ induces a morphism $\bar{f}: Y \to \hom(-,X)$ such that

$$\bar{f}_1 = f : Y(1) \longrightarrow \hom(1,X)$$

and

$$\bar{f}^+ : Y(N^+) \longrightarrow \hom(N^+,X)$$

$$p \longmapsto ((fx_n),fx_\infty)$$

where $\bar{p} = ((x_n),x_\infty)$. □

Notation: Let X be a process space. And let p be a process in $X(N^+)$ with $\bar{p} = ((x_n),x_\infty)$.

Write $X_{\bar{p}}$ for the set of all processes whose underlying sequence is $((x_n),x_\infty)$. That is

$$X_{\bar{p}} = \{ q \in X(N^+) \mid \bar{q} = \bar{p} = ((x_n),x_\infty) \} .$$

Proposition 6.2.2 The functor $h': \mathrm{Proc} \to \mathrm{SuSEQ}$, preserves products.

Proof. Let X and Y be objects of Proc , let $Z = X \times Y$, then

$$Z(1) = X(1) \times Y(1)$$

$$Z(N^+) = X(N^+) \times Y(N^+) .$$

Now observe that for $p = (q,r) \in Z(N^+)$ with $\bar{p} = ((x_n,y_n) , (x_\infty, y_\infty))$, that is $\bar{g} = ((x_n),x_\infty)$ and $\bar{r} = ((y_n),y_\infty)$, $Z_{\bar{p}} = X_{\bar{q}} \times Y_{\bar{r}}$.

So it follows easily that $C_{\underline{Z}} = C_{\underline{X}} \times C_{\underline{Y}}$. Hence h' preserves products. □

Now the functor h satisfies the assumption of (Proposition 4.6.5), so we have the following result.

Proposition 6.2.3 The functor $h\colon \mathrm{SuSEQ} \to \mathrm{Proc}$ preserves function spaces.

We now state one main result of this section.

Theorem 6.2.4 The functor $H\colon \mathrm{SEQ} \to \mathrm{Proc}$ preserves function spaces.

Proof. Since H is the composite $h \circ k$, so the result follows. □

The next result gives a description of $P_c(X,Y)$ and $P_o(X,Y)$ in Proc .

Proposition 6.2.5 Let X and Y be sequential spaces. Then

(i) $H[P_c(X,Y)] \cong \mathrm{Proc}(HX, H\tilde{Y})$

(ii) $H[P_o(X,Y)] \cong \mathrm{Proc}(HX, H\hat{Y})$

Proof. (i) Since $P_c(X,Y) \cong \mathrm{SEQ}(X,\tilde{Y})$

so
$$H[P_c(X,Y)] \cong H[\mathrm{SEQ}(X,\tilde{Y})]$$
$$\cong \mathrm{Proc}[HX, H(\tilde{Y})]$$

(ii) The proof is similar to that of (i) . □

The aim of the rest of this section is to show that H preserves exponents, function spaces in $\mathrm{SEQ} \downarrow B$.

Proposition 6.2.6 Let B be an object of SEQ . Then the forgetful functor

$$h'\colon \mathrm{Proc} \to \mathrm{SuSEQ}$$

preserves pullbacks over $B = HB$.

Proof. Let Z and Y' be objects of Proc . Let $g: Z \to B$ and $f: Y' \to B$ be morphisms in Proc , that is objects of Proc $\downarrow B$.

Consider the pullback

$$\begin{tikzcd} X = Z \times_B Y' \arrow[r] \arrow[d] & Y' \arrow[d, "f"] \\ Z \arrow[r, "g"'] & B \end{tikzcd}$$

in Proc .

Then $X(1) = Z(1) \times_{B(1)} Y'(1)$

$$= \{(z,y) \mid gz = fy\} .$$

Also $X(N^+) = \{(p,q) \in Z(N^+) \times Y'(N^+) \mid g^+(p) = f^+(q)\}$

$$\begin{tikzcd} X(N^+) = Z(N^+) \times_{BN^+} Y'(N^+) \arrow[r] \arrow[d] & Y'(N^+) \arrow[d, "f^+"] \\ ZN^+ \arrow[r, "g^+"'] & B(N^+) \end{tikzcd}$$

More precisely, let $\bar{p} = ((z_n), z_\infty)$, $\bar{q} = ((y_n), y_\infty)$. Then each pair (p',q') , where $p' \in Z_{\bar{p}}$ and $\bar{q} \in Y'_{\bar{q}}$, is a process in $X(N^+)$ with $(\overline{p',q'}) = ((z_n,y_n), (z_\infty,y_\infty))$.

Now the functor h' is such that, for Z in Proc

$h'(Z) = (\underline{Z}, C_{\underline{Z}})$ where $\underline{Z} = Z(1)$ and

$$C_{\underline{Z}} = \{\bar{p} \mid p \in Z(N^+)\} .$$

Consider the pullback diagram

$$\begin{tikzcd} \underline{W} = \underline{Z} \times_{\underline{B}} \underline{Y'} \arrow[r] \arrow[d] & \underline{Y'} \arrow[d, "f"] \\ \underline{Z} \arrow[r, "g"'] & \underline{B} \end{tikzcd}$$

in SuSEQ , where $\underline{Y'} = (Y', C_{\underline{Y'}})$ and $\underline{Z} = (Z, C_{\underline{Z}})$.

Then $W = \{(z,y) \mid gz = fy\}$, and a pair $((z_n,y_n),(z_\infty,y_\infty))$, of a sequence (z_n,y_n) of W together with a point (z_∞, y_∞) of W , is in C_W if and only if (z_n,z_∞) is in $C_{\underline{Z}}$ and (y_n,y_∞) is in $C_{\underline{Y'}}$.

Claim: $C_W = C_{\underline{X}}$.

<u>Proof of the claim:</u> Let $((z_n,y_n) , (z_\infty,y_\infty)) \in C_W$.

Then there are $p \in Z(N^+)$ and $q \in Y'(N^+)$ such that $\bar{p} = (z_n,z_\infty)$ and $\bar{q} = (y_n,y_\infty)$.

Since (z_n,y_n) is in W for all $n \in N^+$, so $g(z_n) = f(y_n)$ for all n .

Thus $g^+(p) = f^+(q)$. That is (p,q) is in $X(N^+)$. So

$$\overline{(p,q)} = ((z_n,y_n) , (z_\infty,y_\infty)) \text{ is in } C_{\underline{X}} .$$

Hence $C_W \subseteq C_{\underline{X}}$.

Conversely, let $((z_n,y_n) , (z_\infty,y_\infty)) \in C_{\underline{X}}$.

Then there are $p \in Z(N^+)$ and $q \in Y'(N^+)$ such that $\bar{p} = ((z_n),z_\infty)$, i.e. $((z_n),z_\infty) \in C_{\underline{Z}}$,

$\bar{q} = ((y_n), y_\infty)$, i.e. $((y_n), y_\infty) \in C_{\underline{Y}'}$, and

$g^+(p) = f^+(q)$, i.e. $(p,q) \in X(N^+)$.

So $g(z_n) = f(y_n)$ for all $n \in N^+$. Hence

$(z_n, y_n) \in W$ for $n \in N^+$.

So $((z_n, y_n), (z_\infty, y_\infty)) \in C_W$.

That is $C_{\underline{X}} \subseteq C_W$. So $C_W = C_{\underline{X}}$. □

<u>Proposition 6.2.7</u> k': SuSEQ $\rightarrow$ SEQ preserves pullback over $B = kB$.

<u>Proof</u>. Let X and Y be objects of SuSEQ . Consider the pullback diagram

$$\begin{tikzcd} W = X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

in SuSEQ . Then $W = \{(x,y) \mid fy = gx\}$ and

$C_W = \{((x_n, y_n), (x_\infty, y_\infty)) \mid x_n \rightarrow x_\infty$ and $y_n \rightarrow y\}$.

Now consider the pullback diagram

$$\begin{tikzcd} Z = k'X \times_B k'Y \arrow[r] \arrow[d] & k'Y \arrow[d, "f"] \\ k'X \arrow[r, "g"'] & B \end{tikzcd}$$

in SEQ . Then $Z(1) = \{(x,y) \mid fy = gx\}$. The sequentialised topology on X is induced by the sequentially open sets. That is $V \subseteq X$ is open if and only if whenever $(x_n,y_n) \to (x_\infty,y_\infty) \in V$ then (x_n,y_n) is eventually in V . But $(x_n,y_n) \to (x_\infty,y_\infty)$ if and only if $x_n \to x_\infty$ and $y_n \to g_\infty$. Also $k'(W)$ has the sequential topology induced by the family C_W .

$$\text{So} \quad k'(W) = k'(X \underset{B}{\times} Y) = k'X \underset{B}{\times} k'Y . \quad \square$$

Corollary 6.2.8 H': Proc $\to$ SEQ preserves pullback over B .

Proof. H' is the composite $k' \circ h'$. $\square$

We conclude this section by stating the following result. Note that we will use (6.2.8) in the proof.

Proposition 6.2.9 Let B be a Hausdorff space. Then H: SEQ $\to$ Proc preserves exponents, function spaces in SEQ $\downarrow B$.

Proof. Let $X \in |\text{Proc}|$ and let Y and $Z \in |\text{SEQ}|$. Since B is a Hausdorff space, so SEQ $\downarrow B$ is cartesian closed and we have a natural bijection

$$\begin{aligned}
\text{Proc} \downarrow HB\,[X, H\{(YZ)_B\}] &\cong \text{SEQ} \downarrow B[H'X, (YZ)_B] \\
&\cong \text{SEQ} \downarrow B[H'X \underset{B}{\times} Y, Z] \\
&\cong \text{SEQ} \downarrow B[H'X \underset{B}{\times} H'(HY), Z] \\
&\cong \text{SEQ} \downarrow B[H'(X \underset{B}{\times} HY), Z] \\
&\cong \text{Proc} \downarrow HB[X \underset{B}{\times} HY,\ HZ] \\
&\cong \text{Proc} \downarrow HB[X, ((HY)(HZ))_B] .
\end{aligned}$$

Hence $H\{(YZ)_B\}$ is naturally isomorphic to $((HY)(HZ))_B$. $\square$

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