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Dear Ronnie,

This is the last chapter of my thesis, which contains most of the work on commutators. I hope it is reasonably clear. The procedure used to write down a presentation of ρ_2 of the n-torus is essentially just:

generators = 2-cells
relators = boundaries of 3-cells.

The rest is self-explanatory, I think.

Yours sincerely,
Jim Howie

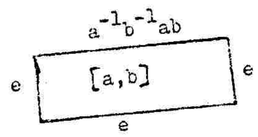
CHAPTER 9

COMMUTATORS

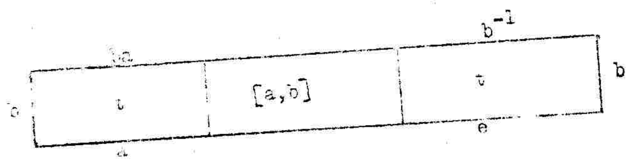
In this chapter we use double groupoids to give a new proof of a result ([10]) about commutator laws in Group Theory. Essentially, the result states four laws, and says that all commutator laws are consequences of these four laws (in some suitable abstract system).

The idea of our proof is that we express a commutator as the boundary of a torus (i.e. a square with each pair of opposite edges identified). The commutator laws have a natural geometrical interpretation, which we can express using squares in an appropriate double groupoid.

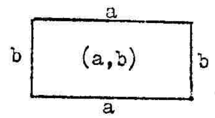
If Γ is a group, let $[\Gamma,\Gamma]$ denote the commutator subgroup of Γ. Then $[\Gamma,\Gamma] \longrightarrow \Gamma$ is a crossed module. Let $\gamma(\Gamma,\Gamma)$ denote the corresponding 1-vertex double groupoid. The commutator $[a,b] \in [\Gamma,\Gamma] = \gamma(\Gamma,\Gamma)$ has edges



Let (a,b) denote its conjugate

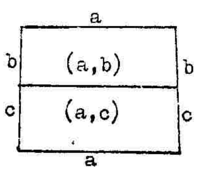


Then (a,b) is the unique square of (Γ,Γ) with edges



We can "translate" expressions involving commutators in Γ to expressions involving the squares (a,b) of (Γ,Γ). For example, the equation $[a,bc] = [a,c].[a,b]^c$ becomes

(a,bc) =



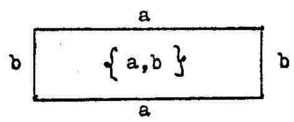
i.e. $(a,bc) = (a,b) \circ (a,c)$.

From relation (6) of [10] , the group $\langle\Gamma,\Gamma\rangle/B(\Gamma)$ (in the notation of [10]) is a Γ-crossed-module, and the "associated group" $H(\Gamma)$ is the kernel of the boundary map of this crossed module. We will give a Γ-presentation for the Γ-double-groupoid $\{\Gamma,\Gamma\}$ corresponding to $\langle\Gamma,\Gamma\rangle/B(\Gamma)$, so that

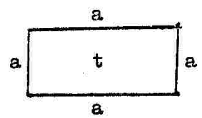
$\langle\Gamma,\Gamma\rangle/B(\Gamma) = \gamma(\{\Gamma,\Gamma\})$, and $H(\Gamma) = \pi_2(\{\Gamma,\Gamma\})$.

We can regard the crossed module $\gamma(\{\Gamma,\Gamma\})$ as a sort of non-commutative analogue of the Universal Lie Structures of [8] , and from this point of view it is not surprising that $\pi_2(\{\Gamma,\Gamma\})$ turns out to be a homological invariant of th[illegible]

Our Γ-presentation of $\{\Gamma,\Gamma\}$ has generators

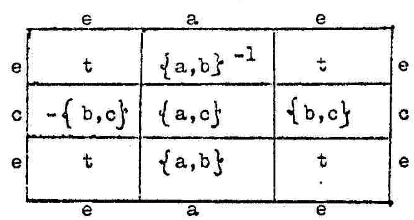


$(a,b \in \Gamma)$, and defining relations

$\{a,a\}$ =  $(a \in \Gamma)$ --- (24)

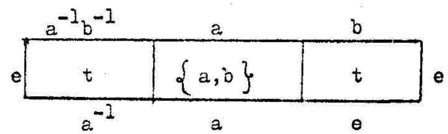
$\{ab,c\} = \{a,c\} + \{b,c\}$ $(a,b,c \in \Gamma)$ --- (25)

$\{a,bc\} = \{a,b\} \circ \{a,c\}$ $(a,b,c \in \Gamma)$ --- (26)

$\{a,c\}$ =  --- (27)

$(a,b,c \in \Gamma)$.

If we set

$\langle a,b\rangle$ =  $\in \gamma(\{\Gamma,\Gamma\})$

for a, b ∈ Γ, the defining relations (24)-(27) "translate" to defining relations for a presentation of $\gamma(\{\Gamma,\Gamma\})$ as a Γ-crossed-module thus :

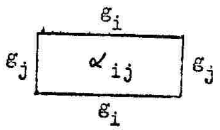
$$\langle a,a\rangle = \odot \qquad (a \in \Gamma) \qquad \text{--- (28)}$$
$$\langle ab,c\rangle = \langle a,c\rangle^{b}.\langle b,c\rangle \qquad (a,b,c \in \Gamma) \qquad \text{--- (29)}$$
$$\langle a,bc\rangle = \langle a,c\rangle.\langle a,b\rangle^{c} \qquad (a,b,c \in \Gamma) \qquad \text{--- (30)}$$
$$\langle a,b\rangle.\langle a,c\rangle^{b}.\langle b,c\rangle = \langle b,c\rangle^{a}.\langle a,c\rangle.\langle a,b\rangle^{c} \qquad \text{--- (31)}$$
$(a,b,c \in \Gamma)$.

Relations (28)-(31) can be shown to be equivalent to the set of relations marked (1)-(4) in [10]. We can now prove the result which is essentially the same as Theorem 1 of [10].

THEOREM 9.1 If Γ is free, then $\pi_2(\{\Gamma,\Gamma\}) = \{\odot\}$.

PROOF If A is a set, let F(A) denote the free group on A.

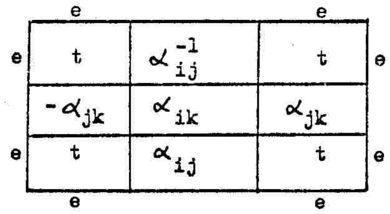
Suppose first of all that A is finite, say $A = \{g_1,\ldots\ldots,g_n\}$. Let X be the n-dimensional torus regarded as an n-dimensional CW-complex as in the example at the end of Chapter 5. Then, from the same example, we have an F(A)-presentation of $\rho_2(X,X^{(1)},X^{(0)})$ with generators



$(1 \leqslant i < j \leqslant n)$

and defining relations

α_{ik} =



$(1 < i < j < k \leqslant n)$

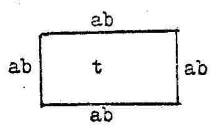
Define an F(A)-morphism $\theta : \rho_2(X,X^{(1)},X^{(0)}) \longrightarrow \{F(A),F(A)\}$ by $\theta(\alpha_{ij}) = \{g_i,g_j\}$. θ is regular, since it is an F(A)-morphism. Also
$\operatorname{Ker}\theta \subseteq \pi_2(\rho_2(X,X^{(1)},X^{(0)}) \cong \pi_2(X) = 0$, so θ is injective.

The defining relations (24)-(26) imply that

$$\{a,e\} = 1_a ; \quad \{e,a\} = 0_a \quad (a \in \Gamma); \text{ and}$$

$$\{a,b^{-1}\} = \{a,b\}^{-1} ; \quad \{a^{-1},b\} = -\{a,b\} \quad (a,b \in \Gamma).$$

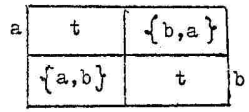
Also, since



$= \{ab,ab\}$

$$= \begin{array}{|c|c|} \hline \{a,a\} & \{b,a\} \\ \hline \{a,b\} & \{b,b\} \\ \hline \end{array}$$

=



it follows that $\{b,a\} = -\tau(\{a,b\})$. From these we can deduce that $\{g_i^{\mp 1}, g_j^{\mp 1}\} \in \operatorname{Im}\theta$ for all i,j, and hence that $\{a,b\} \in \operatorname{Im}\theta$ for all $a,b \in F(A)$. It follows that θ is surjective, and so an isomorphism. Thus $\pi_2(\{F(A),F(A)\}) = \pi_2(\rho_2(X,X^{(1)},X^{(0)}) = 0$.

In general, A may be infinite. If B is a subset of A, then the inclusion $B \hookrightarrow A$ induces a morphism $\theta_B : \{F(B),F(B)\} \longrightarrow \{F(A),F(A)\}$ of double groupoids such that $S_1(\theta_B) : F(B) \hookrightarrow F(A)$ is the inclusion. Any square of $\{F(A),F(A)\}$ can be expressed as a composite of finitely many squares, each of which is either thin or of the form $\{a,b\}$ for some $a,b \in F(A)$. Thus any square α is in the image of θ_B for some <u>finite</u> subset B of A, say $\alpha = \theta_B(\beta)$.

But $S_1(\theta_B)$ is injective, so if $\alpha \in \pi_2(\{F(A),F(A)\})$, it follows that $\beta \in \pi_2(\{F(B),F(B)\})$, which is trivial, and so $\alpha = \theta_B(\odot) = \odot$, and $\pi_2(\{F(A),F(A)\})$ is also trivial. /

<u>COROLLARY</u> For any group Γ, $\pi_2(\{\Gamma,\Gamma\}) \cong H_2(\Gamma)$.

<u>PROOF</u> Let $\Gamma \cong F/R$ where F is free. Then by the theorem, $\gamma(\{F,F\}) \cong [F,F]$, so we have a presentation of $[F,F]$ as an F-crossed-module. That is, F is characterized by a certain universal property. Use this to check that $[F,F]/[F,R]$ satisfies the corresponding universal property for Γ, i.e. that $\gamma(\{\Gamma,\Gamma\}) \cong [F,F]/[F,R]$. The result then follows from the Hopf formula

$$H_2(\Gamma) \cong ([F,F] \cap R)/[F,R] \quad . /$$

References

[8] P.J. HIGGINS "Baer Invariants and the Birkhoff - Witt Theorem" J. Alg 11 (1969) 469-482

[10] C. MILLER "The Second Homology Group of a Group - Relations among Commutators" Proc. A.M.S. 3 (1952) 588-595

Notation

$\boxed{t}$ = thin square

~~Γ - dou~~ $S_1 G$ = 1-skeleton of G

Γ - double-groupoid = d.g. G with $S_1 G = \Gamma$

Γ - morphism = morph. θ with $S_1 \theta = id_\Gamma$