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★**Nonabelian algebraic topology.**
Filtered spaces, crossed complexes, cubical homotopy groupoids.
With contributions by Christopher D. Wensley and Sergei V. Soloviev.
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The main theme of this book is the role of strict ω-groupoids in algebraic topology via the cubical homotopy groupoid associated to a filtered topological space, giving an exposition of the work of the first two authors and their collaborators over the past decades. The theory generalizes that of the homotopy crossed module associated to a pair of topological spaces, studied by Whitehead and others. For motivation, the authors review the latter in Part I, explaining the relationship between crossed modules and double groupoids and proving a 2-dimensional form of the Seifert–van Kampen theorem.

Part II indicates how to pass to higher dimensions using the fundamental homotopy crossed complex associated to a filtered space, which is built from relative homotopy groups. (A crossed complex combines a crossed module with a chain complex over the fundamental group of the crossed module.) The higher homotopy Seifert–van Kampen theorem is introduced as a form of local-to-global principle and is shown to encompass certain classical theorems. Applications to the abelian setting are indicated, via passage to the chain complex with groupoid operators associated to a crossed complex (termed the derived complex here).

Cubical sets make their first appearance in Part II, so as to construct the cubical classifying space of a crossed complex. It is explained how this represents a version of nonabelian cohomology with coefficients in a crossed complex, which for CW-complexes can be calculated via the associated homotopy crossed complex, by the homotopy classification theorem.

Part III is the core of the volume, built on the foundations of strict ω-groupoids. The important notion of cubical sets with connections (imposing the existence of certain degeneracies) is introduced; these are known to form a strict test category in the sense of Grothendieck, and hence provide a model for homotopy types [G. Maltsiniotis, Homology Homotopy Appl. **11** (2009), no. 2, 309–326; MR2591923]. Thereafter compositions (which are viewed as allowing an algebraic inverse to subdivision) are defined, leading to the definition of ω-groupoids. The main results of the book are established using this category, which is shown to be equivalent to the category of crossed complexes.

The cubical homotopy groupoid ρX_* associated to a filtered space X_* is described using the filtered singular cubical set RX_* (a variant of the singular cubical set, taking into account the filtration), which has connections and compositions; ρX_* is an ω-groupoid, defined as a natural quotient $RX_* \to \rho X_*$, which is a fibration of cubical sets.

Properties of the cubical homotopy groupoid are explained, for instance, the compatibility of ρ with appropriate closed monoidal structures on filtered spaces and ω-groupoids. The proof of the higher homotopy Seifert–van Kampen theorem is given for ρ, from which the result stated in Part II is deduced. The theory is illustrated, for example, by showing that the cubical fibration $RX_* \to \rho X_*$ gives rise to Whitehead's exact sequence for the Hurewicz morphism.

The book seeks to cater to many tastes, one aim being that it should be readable by a graduate student (the authors provide appendices to aid students, introducing fibrations and cofibrations of categories in Appendix B, for instance). Many readers may wish to follow the 'pure logical order' suggested by the authors, by starting with Part III, thus putting the cubical methods at the heart of the theory.

The wide scope of the work and inherent choices mean that there are points which have been omitted or are treated briefly, such as the general formation of colimits in the categories considered or the underlying model category structures. This reader would have welcomed more detail on the respective roles of cubical sets, filtered cubical sets and their versions with connections.

The theory developed here is essentially linear: homotopy crossed complexes do not see quadratic structure. Whereas, by results of Mac Lane and Whitehead, crossed modules model homotopy 2-types, to establish algebraic models for homotopy n-types, more general higher categorical structures have to be studied; this is a rich field. This presentation of what can be thought of as the semi-abelian model is a valuable contribution to the literature; as indicated in the final chapter, there are abundant areas for future study.

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