VAN KAMPEN THEOREMS FOR DIAGRAMS OF SPACES*

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INTRODUCTION

LET X be a pointed space and $\{A, B\}$ an open cover of X such that A, B and $C = A \cap B$ are connected, and $(A, C), (B, C)$ are 1-connected. One of the corollaries of the main theorem of §5 is an algebraic description of the third triad homotopy group:

$$\pi_3(X; A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C) \qquad (*)$$

where $\otimes$ means "non-Abelian tensor product" of the two relative homotopy groups, each acting on the other via $\pi_1 C$. This new algebraic construction $M \otimes N$, which is defined for a pair of groups M, N each of which acts on the other, is studied in §2. As is well-known, a general determination of a triad homotopy group has consequences for certain absolute homotopy groups. Some of these are given in §3. For example, we prove that for any group G

$$\pi_3(SK(G, 1)) \cong \mathrm{Ker}\,(G \otimes G \xrightarrow{\kappa} G)$$

where $\kappa(g \otimes h) = ghg^{-1}h^{-1}, g, h \in G$ and G acts on itself by conjugation. As a further consequence we obtain in §4 new results on the low-dimensional homology of discrete groups, notably some new eight-term exact sequences. An immediate application is a formula for H_3 of a group Q in terms of a presentation of Q, this formula is analogous to the Hopf formula for H_2Q.

The description (*) of the triad group is a consequence of a special case of the case $n = 2$ of a Van Kampen-type theorem for n-cubes of spaces. Let Top^* be the category of pointed topological spaces, and let $\{0, 1\}^n$ be the n-fold product category with $0 < 1$. The functor category $\mathsf{Fun}(\{0, 1\}^n, \mathsf{Top}^*)$ is a proper closed model category in the sense of [3] (cf. also [12]) (it is Top^* for $n = 0$, and the category whose objects are pointed maps for $n = 1$). Its objects are called *n-cubes of spaces*.

In [20] the second author introduced a functor which we here write

$$\Pi : \mathsf{Fun}(\{0, 1\}^n, \mathsf{Top}^*) \to (\mathsf{cat}^n\text{-groups})$$

(in [20] this is $\mathcal{G}$ with values in n-cat-groups). If $n = 0$, this functor is the fundamental group functor. The functor Π was used in [20] to show that a cat^n-group is an algebraic equivalent for a path connected weak homotopy type X in Top^* with $\pi_r(X) = 0$ for $r > n + 1$. Thus such an X is a $K(\pi, 1)$ for $n = 0$.

For $n = 0, 1, 2$ cat^n-groups are equivalent to groups, crossed modules, crossed squares, respectively.

The main result of this paper (Theorem 5.4) is the fact that the functor Π carries certain colimits of "connected" n-cubes to colimits in (cat^n-groups). For $n = 0$, this is the Van Kampen theorem. For $n = 1$, this was proved by Brown and Higgins [5] by a different method. The case $n = 2$ is new. Applications for $n > 2$ are given in [9, §16].

In §5 we give the definition of cat^n-group, and construct the fundamental cat^n-group functor as in [20]. The proof of Theorem 5.4 is by induction on n and uses simplicial techniques. The case $n = 1$ contains the core of the proof, which is based on a property of simplicial groups G such that G_2 is generated by degenerate elements. We also rely on several results which are well known to experts, but are not easily available in the

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Standing assumptions: All spaces are assumed base pointed. All maps respect base,point.We abbreviate "path connected space" for "connected space". A simplicial space E is said to be *connected* if all the spaces E_n are connected. A sequence of maps $F \to E \to B$ is a fibration sequence if $E \to B$ is a Hurewicz fibration with fibre F.

literature. We are grateful to M. Zisman for supplying in the Appendix an account of these results, for example the spectral sequence of a simplicial space (cf. [31]). The main results of this paper for $n = 2$ were announced in [8].

1 SQUARES OF SPACES AND CROSSED SQUARES

1.1 A crossed module is a group homomorphism $\mu : M \to P$ together with an action of P on M, written ${}^{p}m$ for $p \in P$ and $m \in M$, and which satisfies the following conditions:

(a) $\mu({}^{p}m) = p\mu(m)p^{-1}, p \in P, m \in M$

(b) ${}^{\mu(m)}m' = mm'm^{-1}, m, m' \in M$.

This notion goes back to J. H. C. Whitehead [29] who-proved essentially the following. Let $F \to A \to X$ be a fibration sequence; then the homomorphism $\pi_1 F \to \pi_1 A$ equipped with the natural action of $\pi_1 A$ on $\pi_1 F$ is a crossed module.

Since any pointed map can be canonically converted into a fibration, there is a functor Π from the category of pointed maps to the category of crossed modules. The Van Kampen theorem for crossed modules [5] (see also Theorem 5.4) asserts that this functor Π commutes with certain amalgamated sums, and more generally with certain colimits.

1.2 A crossed square [17,20] is a commutative square of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow{\nu} & P \end{array}$$

together with actions of the group P on L, M, N (and hence actions of M on L and N via μ and of N on L and M via μ) and a function $h : M \times N \to L$. This structure shall satisfy the following axioms:

(i) the maps λ, λ' preserve the actions of P; further, with the given actions, the maps μ, ν and $K = \mu\lambda = \mu'\lambda'$ are crossed modules;

(ii) $\lambda h(m, n) = m{}^{n}m^{-1}, \lambda' h(m, n) = {}^{m}n\, n^{-1}$

(iii) $h(\lambda l, n) = l{}^{n}l^{-1}, h(m, \lambda' l) = {}^{m}l\, l^{-1}$

(iv) $h(mm', n) = {}^{m}h(m', n)h(m, n), h(m, nn') = h(m, n){}^{n}h(m, n')$;

(v) $h({}^{P}m, {}^{P}n) = {}^{P}h(m, n)$;

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

Note that in these axioms, a term such as ${}^{m}l$ is l acted on by m, and so ${}^{m}l = {}^{\mu(m)}l$. It is a consequence of (i) that λ, λ' are crossed modules. Further, by (iii), M acts trivially on $\operatorname{Ker}\lambda'$ and N acts trivially on $\operatorname{Ker}\lambda$.

A morphism of crossed squares is a morphism of squares of groups which is compatible with the actions and the functions h.

1.3 Suppose given a commutative square of spaces

$$\mathbf{X}: \quad \begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & X \end{array}$$

Let $F(f)$ be the homotopy fibre of f and let $F(\mathbf{X})$ be the homotopy fibre of $F(g) \to F(a)$.

Proposition 1.4 *[20]. The commutative square of groups*

$$\begin{array}{ccc} \pi_1 F(\mathbf{X}) & \longrightarrow & \pi_1 F(g) \\ \downarrow & & \downarrow a \\ \pi_1 F(f) & \xrightarrow[b]{} & \pi_1 C \end{array}$$

associated to $\mathbf{X}$ *is naturally equipped with a structure of crossed square.*

It is *this* crossed square which we denote by $\Pi\mathbf{X}$ and call the *fundamental crossed square* of $\mathbf{X}$.

In the case that $\mathbf{X}$ is a square of inclusions and $C = A \cap B$, the fundamental crossed square $\Pi\mathbf{X}$ may be identified with the square of groups

$$\begin{array}{ccc} \pi_3(X; A.B) & \longrightarrow & \pi_2(B, C) \\ \downarrow & & \downarrow \\ \pi_2(A, C) & \longrightarrow & \pi_1 C \end{array}$$

with the maps being boundary maps, the action of $\pi_1 C$ being the usual one, and with h-function given (up to sign) by the generalised Whitehead product (cf. [1, p. 107 and 2]).

The square $\mathbf{X}$ as above is said to be *connected* if all the spaces $X, F(a), F(b)$ and $F(\mathbf{X})$ are connected [which implies that $A, B, C, F(f), F(g)$ are also connected].

We say that a square of squares of spaces

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{f} & \mathbf{U} \\ g\downarrow & & \downarrow \\ \mathbf{V} & \longrightarrow & \mathbf{X} \end{array}$$

is a *homotopy amalgamated sum (or homotopy pushout)* if the canonical map of squares from the double mapping cylinder $\mathbf{M}(f, g)$ to $\mathbf{X}$ is a weak equivalence of the spaces at the four corners. If each of the maps of $f : W \to u$ is a cofibration, then $\mathbf{X}$ may be taken to be the square of pushout spaces from the four vertices.

Theorem 1.5 (Van Kampen theorem for squares of spaces) *Let* $\mathbf{X}$ *be the homotopy amalgamated sum of maps of squares* $\mathbf{U} \xleftarrow{f} \mathbf{W} \xrightarrow{g} \mathbf{V}$. *Suppose that* $\mathbf{U}, \mathbf{V}, \mathbf{W}$ *are connected squares. Then* $\mathbf{X}$ *is a connected square and the natural morphism*

$$\Pi\mathbf{U} *_{\Pi W} \Pi\mathbf{V} \to \Pi\mathbf{X}$$

is an isomorphism of crossed squares, where the first term is the amalgamated sum in the category of crossed squares.

The proof will follow from a generalised Van Kampen theorem for n-cubes of spaces (Theorem 5.4).

2 THE NON-ABELIAN TENSOR PRODUCT OF GROUPS

In this section we define and study a tensor product $M \otimes N$ for (not necessarily abelian) groups M, N. We also introduce a (non-Abelian) exterior product. Special cases of the tensor product have appeared elsewhere [11,21].

Let groups M, N be equipped with an action of M on the left of N, written ${}^m n, m \in M, n \in N$,and an action of N on the left on M, written ${}^n m$. It is always understood that a group acts on itself by conjugation: ${}^x y = xyx^{-1}$.

Definition 2.1 The *tensor product* $M \otimes N$ is the group generated by symbols $m \otimes n$, with relations
(a) $mm' \otimes n = (mm'm^{-1} \otimes {}^{m}n)(m \otimes n)$
(a') $m \otimes nn' = (m \otimes n)({}^{n}m \otimes nn'n^{-1})$
for all $m, m' \in M, n, n' \in N$.

Definition 2.2 Let M, N be as above, and let L be a group. A *crossed pairing (or biderivation)* from $M \times N$ to L is a function $h : M \times N \to L$ such that
(a) $h(mm', n) = h({}^{m}m', {}^{m}n)h(m, n)$
(a')$h(m, nn') = h(m, n)h({}^{n}m, {}^{n}n')$ for all $m, m' \in M, n, n' \in N$.

Clearly the function $M \times N \to M \otimes N; (m, n) \to m \otimes n$, is the universal crossed pairing in the sense that any crossed pairing $h : M \times N \to L$ determines a unique homomorphism $h^* : M \otimes N \to L$ such that $h^*(m \otimes n) = h(m, n)$.

Given the actions of M, N as above, the free product $M * N$ acts on both M and N. If $m, m' \in M, n, n' \in N$, then

$$
\begin{aligned}
{}^{({}^{n}m)}m' &= {}^{n}mm'{}^{n}m^{-1} = {}^{n}(m^{n^{-1}}m'm^{-1}) \\
&= {}^{nmn^{-1}}m'
\end{aligned}
$$

and similarly

$$
{}^{({}^{m}n)}n' = {}^{mnm^{-1}}n'.
$$

In all our applications, the actions will be compatible in the sense that

$$
{}^{({}^{m}n)}m' = {}^{mnm^{-1}}m', \quad {}^{({}^{n}m)}n' = {}^{nmn^{-1}}n'
$$

for all $m, m', \in M, nn' \in N$.

Some special cases of the following consequences of the rules are essentially found in [11].

Proposition 2.3 *Let M, N, be groups equipped with compatible actions on each other.*

(a) The free product $M * N$ acts on $M \otimes N$ so that

$$
{}^{p}(m \otimes n) = {}^{p}m \otimes {}^{p}n, \quad m \in M, \quad n \in N, \, p \in M * N.
$$

(b) There are homomorphisms

$$
\lambda : M \otimes N \to M, \lambda' : M \otimes N \to N
$$

such that $\lambda'(m \otimes n) = m{}^{n}m^{-1}, \lambda(m \otimes n) = {}^{m}n\,n^{-1}$.

(c) The homomorphisms λ, λ' with the given actions, are crossed modules.

(d) If $l \in M \otimes N, m' \in M, n' \in N$, then

$$
\begin{aligned}
(\lambda l) \otimes n' &= l\,{}^{n'}l^{-1} \\
m' \otimes \lambda l &= {}^{m'}l l^{-1}.
\end{aligned}
$$

(e) The actions of M on $\mathrm{Ker}\ \lambda', N$ on $\mathrm{Ker}\ \lambda A$, are trivial.

(f) If $l, l' \in M \otimes N$, then

$$
[l, l'] = \lambda l \otimes \lambda' l'
$$

and in particular $[m \otimes n, m' \otimes n'] = (m\,{}^{n}m^{-1}) \otimes ({}^{m'}n'n'^{-1})$.

Proof The proofs of (a),(b) are straightforward, using the universal property. We emphasise that compatibility is required for both (a) and (b) (the first of these was pointed out to us by P. J. Higgins). The only non-trivial verification for (c) is the second axiom of crossed modules for λ (and λ'). The trick is to expand $mm' \otimes nn'$ in two ways, which gives

$$^{mn}(m' \otimes n')(m \otimes n) = (m \otimes n)^{nm}(m' \otimes n').$$

This implies $ll'l^{-1} = {}^{\lambda l}l'$ for $l = m \otimes n, l = m' \otimes n'$. The general case follows.

The general case of (d) follows from the case $l = m \otimes n$. For the proof of the first formula we note that

$$\begin{aligned}
\lambda(m \otimes n) \otimes n' &= (m^n m^{-1}) \otimes n' \\
&= {}^m({}^n m^{-1} \otimes n')(m \otimes n') \\
&= {}^{mn}(m^{-1} \otimes n^{-1}n'n)(m \otimes n') \\
&= {}^m(m^{-1} \otimes n)^{-1}\ {}^m(m^{-1} \otimes n'n)(m \otimes n') \\
&= (m \otimes n)^m(m^{-1} \otimes n')\ {}^{mn'}(m^{-1} \otimes n)(m \otimes n') \\
&= (m \otimes n)(m \otimes n')^{-1}\ {}^{mn'}(m^{-1} \otimes n)(m \otimes n') \\
&= (m \otimes n)\ {}^{n'm}(m^{-1} \otimes n) \\
&= (m \otimes n)\ {}^{n'}(m \otimes n)^{-1}
\end{aligned}$$

The proof of the second formula is similar.

The proofs of (e),(f) are now trivial. □

The Abelianisation of a group G is written G^{ab}.

Proposition 2.4 *If* M *acts trivially on* N *and* N *acts trivially on* M*, then*

$$M \otimes N = M^{ab} \otimes_{\mathbb{Z}} N^{ab},$$

where $\otimes_{\mathbb{Z}}$ *is the usual tensor product of Abelian groups.*

Proof From (b)and (e) of Proposition 2:3, one deduces that M and N act trivially on $M \otimes N$. It is well known that in this case the presentation of Definition 2.1 gives the group $M^{ab} \otimes_{\mathbb{Z}} N^{ab}$. □

As a special case, consider a group G acting on itself by conjugation, as usual.

Proposition 2.5 (11) . *The commutator map* $G \times G \to G$ *defines a homomorphism* $\kappa : G \otimes G \to G$. *Then* $G \otimes G$*, with the given action of* G*, becomes a crossed* G*-module in which* G *acts trivially on* $\operatorname{Ker} \kappa$. *If* G *is perfect,* $(G = [G, G])$*, then* $\kappa : G \otimes G \to G$ *is the universal central extension.*

Proof The first two statements are immediate from the definitions and Proposition 2.3. Let $p : E \to G$ be a central extension, and let $g \mapsto g'$ be a set theoretic section of p. The function $G \times G \to E, (g, h) \mapsto [g', h']$, is a crossed pairing and so defines a homomorphism $\phi : G \otimes G \to E$ such that $p\phi = \kappa$. If G is perfect, then so also is $G \otimes G$, by (f) of Proposition 2,3, and then the map ϕ of extensions over G is unique. □

There are well-known calculations of universal central extensions of perfect groups. We now state some calculations of tensor products.

Example 2.6 (a) Let D_m be the dihedral group with generators x, y and relations $x^2 = y^m = xyxy = 1$. We include the case $m = 0$, when y is of infinite order, and write $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Let $T_m = D_m \otimes D_m$.

m odd: then T_m is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_m$ with factors generated by $x \otimes x, x \otimes y$ respectively.

m even: then T_m is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with factors generated by $x \otimes x, x \otimes y, y \otimes y$ and $(x \otimes y)(y \otimes x)$ respectively. (Details of this and other calculations may be found in [7].)

2.7 We now relate $G \otimes G$ to more familiar constructions. Let $G \wedge G$ be obtained from $G \otimes G$ by imposing the additional relations $g \otimes g = 1$ for all $g \in G$. The image of $g \otimes h$ in $G \wedge G$ is written $g \wedge h$. Proposition 2.3 shows that $G \wedge G$ is isomorphic to the group written (G, G) in [23]. The commutator map induces a homomorphism $\kappa' : G \wedge G \to G$, and the results of [23] show that there is an exact sequence

$$0 \to H_2(G) \to G \wedge G \xrightarrow{\kappa'} G. \tag{2.8}$$

In order to analyse the kernel of $G \otimes G \to G \wedge G$ we use Whitehead's Γ-functor [30] which is the "universal quadratic functor" from Abelian groups to Abelian groups. Let A be an Abelian group. Then $\Gamma(A)$ is the Abelian group with generators $\gamma a, a \in A$, and the following relations:

(i) $\gamma(-a) = \gamma(a), a \in A$
(ii) if $\beta(a,b) = \gamma(a+b) - \gamma a - \gamma b, a, b \in A$, then $\beta : A \times A \to \Gamma(A)$ is biadditive.

Let $J_2(G) = \mathrm{Ker}\,(\kappa : G \otimes G \to G)$. Proposition 4.3 below implies that there is an exact sequence

$$H_3(G) \to \Gamma(G^{ab}) \xrightarrow{\psi} J_2(G) \to H_2(G) \to 0, \tag{2.9}$$

where ψ is determined by the map $G \to G \otimes G, g \mapsto g \otimes g$. This result, together with (2.8), shows that if G is finite, or is a p-group, then so also is $G \otimes G$.

If G^{ab} is free Abelian, then a basis for G^{ab} determines a basis for $\Gamma(G^{ab})$ which is mapped by $\Gamma(G^{ab}) \xrightarrow{\psi} G \otimes G \to G^{ab} \otimes G^{ab}$ into part of a basis for $G \otimes G$; so in this case $\psi : \Gamma(G^{ab}) \to G \otimes G$ is injective. This remark, with (2.8), shows that if G is a free group, then $G \otimes G \cong [G,G] \times \Gamma(G^{ab})$.

2.10 For applications in §4 we need a generalisation of $G \wedge G$ and of the exactness of (2.9) at $J_2(G)$. These applications involve a pair of crossed modules $\mu : M \to P, \nu : N \to P$ over the same group P. Then M and N act on each other via P, and the first crossed module rule implies that these actions are compatible. The second crossed module rule says precisely that the actions of M, N on themselves by conjugation are also given by action via P.

We consider the fibre product $M \times_P N = \{(m,n) \in M \times N : \mu m = \nu n\}$.

Definition 2.11 The (non-Abelian) exterior product $M \wedge^P N$ is obtained from the tensor product $M \otimes N$ by imposing the additional relations $m \otimes n = 1$ for all $(m,n) \in M \times_P N$. The image of a general element $m \otimes n$ in $M \wedge^P N$ is written $m \wedge n$.

In order to analyse the kernel of $M \otimes N \to M \wedge^P N$, we consider the morphism $(\lambda, \lambda') : M \otimes N \to M \times_P N$. The latter group acts on $M \otimes N$ via P, and (λ, λ') is a crossed module. The image of (λ, λ') is written $\langle M, N \rangle$. It is normal in $M \times_P N$, and the quotient $(M \times_P N)/\langle M, N \rangle$ is Abelian (and is even a P-module).

Theorem 2.12 *There is an exact sequence*

$$\Gamma(M \times_P N/\langle M, N \rangle) \xrightarrow{\psi} M \otimes N \to M \wedge^P N \to 1,$$

where $\psi(\gamma(m,n)) = m \otimes n$. *Also* ψ *has central image.*

Proof Let $\Psi : M \times_P N \to M \otimes N$ be the function $(m,n) \mapsto m \otimes n$; then $X = \mathrm{Ker}\,(M \otimes N \to M \wedge_P N)$ is the subgroup generated by $\mathrm{Im}\Psi$. Also $X \subseteq (\mathrm{Ker}\,\lambda) \cap (\mathrm{Ker}\,\lambda')$. By Proposition 2.3, (i) X is central in $M \otimes N$ and (ii) X is acted on trivially by M and by N. It follows easily that $\Psi(m,n) = \Psi(m^{-1}, n^{-1}), (m,n) \in M \times_P N$.

We now start the proof that Ψ induces a homomorphism on $\Gamma(M \times_P N/\langle M, N \rangle)$. Let $(m,n), (u,v) \in M \times_P N$. Expanding $mu \otimes nv$ in two ways and using (i), (ii) and $\mu m = \gamma n$ gives

$$mu \otimes nv = {}^{m}\{(u \otimes n)(m \otimes v)\}(m \otimes n)(u \otimes v).$$

It follows that:
(iii) $(u \otimes n)(m \otimes v) \in X$;
(iv) $mu \otimes nv = (u \otimes n)(m \otimes v)(m \otimes n)(u \otimes v)$; and
(v) $(mu \otimes nv)(m \otimes n)^{-1}(u \otimes v)^{-1} = (u \otimes n)(m \otimes v)$.

Suppose also $l \in M \otimes N$. Then by (iv) and Proposition 2.3

$$m\lambda l \otimes n\lambda' l = l^{n} l^{-1}\, {}^{m}l l^{-1}(m \otimes n)[l, l] = m \otimes n.$$

This proves that Ψ induces a function on $(M \times_P N)/\langle M, N \rangle$. We now have to prove that the defining relations for Γ are annihilated. This follows for the first from (i) and (ii) as above and for the second from (v) and

Lemma 2.13 *The function*

$$W : (M \times_P N) \times (M \times_P N) \to M \otimes N$$
$$((m, n), (u, v)) \mapsto (u \otimes n)(m \otimes v)$$

is bimultiplicative.

Proof Suppose also $(m', n') \in M \times_P N$. Then

$$\begin{aligned}(u \otimes nn')(mm' \otimes v) &= (u \otimes n)^n (u \otimes n')^m (m' \otimes v)(m \otimes v) \\ &= (u \otimes n)^m \{(u \otimes n)(m' \otimes v)\}(m \otimes v) \\ &= \{(u \otimes n)(m \otimes v)\}\{(u \otimes n')(m' \otimes v)\}\end{aligned}$$

by (i), (ii) and (iii). The other rule is proved similarly. This completes the proof of Theorem 2.12. □

2.14 In order to use the tensor product in applications of the Van Kampen Theorem for crossed squares (Theorem 1.5) we need an alternative characterisation of $M \otimes N$ when $\mu : M \to P, \nu : N \to P$ are crossed modules, and M, N act on each other via P.

Proposition 2.15 *Let $\mu : M \to P, \nu : N \to P$ be crossed modules, so that M, N act on both M and N via P. Then there is a crossed square*

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

where $\lambda(m \otimes n) = m\,{}^n m^{-1}, \lambda'(m \otimes n) = {}^m n\, n^{-1}$, and $h(m, n) = m \otimes n$. This crossed square is 'universal' in the sense that it satisfies the following two equivalent conditions:

(1)If $\begin{pmatrix} L & M \\ N & P \end{pmatrix}$ is another crossed square (with the same μ, ν), then there is a unique morphism

$$\begin{pmatrix} M \otimes N & M \\ N & P \end{pmatrix} \to \begin{pmatrix} L & M \\ N & P \end{pmatrix}$$

of crossed squares which is the identity on M, N, P.

(2) The following diagram of inclusions of crossed squares is a pushout in the category of crossed squares:

$$\begin{array}{ccc} \begin{pmatrix} 1 & 1 \\ 1 & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} 1 & M \\ 1 & P \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} 1 & 1 \\ N & P \end{pmatrix} & \longrightarrow & \begin{pmatrix} M \otimes N & M \\ N & P \end{pmatrix} \end{array}$$

where 1 denotes the trivial group.

Proof The crossed square properties are immediate from Proposition 2.3. To prove the equivalence of (1) and (2) one notes that the functor (crossed squares)→(crossed modules),

$$\begin{pmatrix} L & M \\ N & P \end{pmatrix} \mapsto (M \quad P),$$

preserves colimits since it has a right adjoint of the form $(MP) \mapsto \begin{pmatrix} M & M \\ P & P \end{pmatrix}$ with h-function of this last crossed square given by $(m, p) \mapsto m\,{}^p m^{-1}$. Similarly, $\begin{pmatrix} L & M \\ N & P \end{pmatrix} \mapsto (NP)$ preserves colimits. (These functors have left adjoints and so also preserve limits.)

The verification of (1) is easy, since the morphism of crossed squares is uniquely determined by the crossed pairing $h : M \times N \to L$. □

Remark 2.16 If M and N are groups which act on each other compatibly, then there is always a group P and crossed modules $M \to P$ and $N \to P$ such that the actions between M and N are obtained via P. The group P may be taken to be the free product $M * N$ divided by the relations ${}^n m = nmn^{-1}$, ${}^m n = mnm^{-1}$, for all $m \in M, n \in N$. It is also a quotient of the semi-direct product $M \rtimes N$. If M, N are already crossed Q-modules, with actions on each other via Q, then P is isomorphic to the co-product crossed Q-module defined in [4].

3 OBSTRUCTIONS TO HOMOTOPICAL EXCISION IN LOW DIMENSIONS

We now give some immediate applications of Theorem 1.5 and the non-Abelian tensor product.

Theorem 3.1 *Suppose given a commutative square of maps*

$$\mathbf{X}: \quad \begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & X \end{array}$$

and let $F(f), F(g), F(\mathbf{X})$ *be the homotopy fibres of* f, g *and* $\mathbf{X}$ *respectively. Suppose that* $\mathbf{X}$ *is a homotopy pushout and* f, g *are connected maps. Then the space* $F(\mathbf{X})$ *is connected and the crossed square*

$$\Pi\mathbf{X} = \begin{pmatrix} \pi_1 F(\mathbf{X}) & \pi_1 F(g) \\ \pi_1 F(f) & \pi_1 C \end{pmatrix}$$

is universal. Hence the canonical map

$$\pi_1 F(f) \otimes \pi_1 F(g) \to \pi_1 F(\mathbf{X})$$

is an isomorphism.

Proof This is immediate from Theorem 1.5 applied to the homotopy pushout of squares of maps

$$\begin{array}{ccc} \begin{pmatrix} C & C \\ C & C \end{pmatrix} & \longrightarrow & \begin{pmatrix} C & A \\ C & A \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} C & C \\ B & B \end{pmatrix} & \longrightarrow & \begin{pmatrix} C & A \\ B & X \end{pmatrix} \end{array}$$

and Proposition 2.15. □

In terms of triad homotopy groups this result can be phrased as follows.

Corollary 3.2 *Suppose in the homotopy pushout* $\mathbf{X}$ *of Theorem 3.1 that all the maps are inclusions and* $C = A \cap B$. *Let* C, A, B *be connected and let* $(A, C), (B, C)$ *be* 1*-connected. Then the triad* $(X; A, B)$ *is* 2*-connected and the canonical morphism given by the generalised Whitehead product*

$$\pi_2(A, C) \otimes \pi_2(B, C) \to \pi_3(X; A, B)$$

is an isomorphism.

For the generalised Whitehead product, see [1,2]. In another paper [9] we will generalise Corollary 3.2 to all dimensions and prove a general form of the Blakers-Massey triad connectivity theorem, with a determination of the critical group. In the particular case of the suspension triad $(SX; C_+X, C_-X)$ we find the following (compare with [32]).

Proposition 3.3 *Let* X *be connected and let* $G = \pi_1 X$. *There is a commutative diagram with exact rows and in which the maps marked are isomorphisms.*

$$\begin{array}{ccccccccc} \pi_2 X & \longrightarrow & \pi_3(SX) & \longrightarrow & \pi_2(\Omega SX, X) & \longrightarrow & \pi_1 X & \longrightarrow & \pi_2(SX) \\ & & \uparrow{\scriptstyle j} & & \cong\uparrow & & =\uparrow & & \cong\uparrow \\ & & \Gamma(G^{ab}) & \longrightarrow & G \otimes G & \longrightarrow & G & \longrightarrow & G^{ab} \end{array}$$

Also $\operatorname{Coker} j \cong H_2 X$.

If $\pi_2 X = 0$*, then*

$$\pi_3(SX) \cong \operatorname{Ker}(\kappa : G \otimes G \to G).$$

Proof This follows from Theorem 3.1 except for the facts involving $\Gamma(G^{ab})$, which are special cases of (4.1), (4.3) in the next section. □

We will show elsewhere that if $\pi_2 X = 0$, then the map $\Gamma(G^{ab}) \xrightarrow{\psi} G \otimes G$ determines the first k-invariant of SX.

For our next result, recall that it is well known [28] that an amalgamation of $K(\pi, 1)$-spaces is still a $K(\pi, 1)$ when the morphisms of groups involved are injective. Here we study the opposite case, when the morphisms of groups are surjective, thus continuing work of [4].

Corollary 3.4 *Let* M, N *be normal subgroups of the group* P*, and form the homotopy amalgamated sum*

$$\mathbf{X} : \quad \begin{array}{ccc} K(P,1) & \xrightarrow{\quad f \quad} & K(P/M,1) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ K(P/N,1) & \longrightarrow & X \end{array}$$

Then the first homotopy groups of X *are given by*

$$\pi_1 X = P/MN, \quad \pi_2 X = (M \cap N)/[M, N] \quad \textit{and}$$

$$\pi_3 X = \operatorname{Ker}(M \otimes N \xrightarrow{[\,,\,]} P).$$

Proof The first equality follows from the classical Van Kampen theorem. The second equality was proved in [4;3.2] as a consequence of the Van Kampen theorem for maps. The third equality follows from the homotopy exact sequences of the fibrations

$$F(a) \to K(P/M, 1) \to X,$$

$$F(X) \to F(g) \to F(a),$$

and Theorem 3.1 since $F(f) \simeq K(M, 1), F(g) \simeq K(N, 1)$. □

Example 3.5 If $X = S^1 = K(\mathbb{Z}, 1)$, then either of these corollaries show that $\pi_3 S^2 \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$. It is interesting to note that this well-known computation is obtained without using the Hopf fibration.

Example 3.6 More generally, let X be a 1-dimensional CW-complex, so that $\pi_1 X$ is a free group F. Then, from Corollaries 2.12 and 3.2 we obtain the well-known result $\pi_3 SX \cong \Gamma F^{ab}$.

Example 3.7 As an example of a range of results not previously available, we give for the dihedral group D_m the formula

$$\pi_3 SK(D_m, 1) \cong \begin{cases} \mathbb{Z}_2 & \text{if m is odd} \\ (\mathbb{Z}_2)^4 & \text{if m is even.} \end{cases}$$

This follows from example 2.6(a). Notice also that we have explicit generators for this homotopy group, namely $x \otimes x$ if m is odd, and $x \otimes x, (x \otimes y)^{m/2}, y \otimes y, (x \otimes y)(y \otimes x)$ if m is even. It will be shown elsewhere that $(x \otimes y)(y \otimes x$ is the only non-trivial Whitehead product element.

4 APPLICATIONS TO THE HOMOLOGY OF DISCRETE GROUPS

In this section, we obtain applications to the homology of discrete groups by applying the Mayer-Vietoris homology exact sequence to the homotopy amalgamated sum of $K(\pi, 1)$s used in Corollary 3.4. The transition from homotopy to homology is given by Whitehead's exact sequence30] for a connected space

$$H_4\tilde{X} \to \Gamma_3 X \to \pi_3 X \xrightarrow{\omega} H_3\tilde{X} \to 0$$

where $\tilde{X}$ is the universal cover of X and ω is the Hurewicz map. Recall that Whitehead gives an isomorphism $\Gamma_3 X \to \Gamma\pi_2 X$ induced by composition with the Hopf map $\eta \in \pi_3 S^2$.

Lemma 4.1 *Suppose given a homotopy pushout*

$$\mathbf{X}: \quad \begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow[b]{} & X \end{array}$$

such that (a) f and g are connected maps, and (b) the maps $\pi_2 F(f) \to \pi_2 C, \pi_2 F(g) \to \pi_2 C$ are surjective. Let $M = \pi_1 F(f), N = \pi_1 F(g)$, both considered as crossed modules over $P = \pi_1 C$. Then there is an isomorphism

$$\pi_2 X \cong M \times_P N/\langle M, N\rangle$$

which leads to a commutative diagram

$$\begin{array}{ccc} \Gamma_3 X & \longrightarrow & \pi_3 X \\ {\scriptstyle \theta}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \xi} \\ \Gamma(M \times_P N/\langle M, N\rangle) & \xrightarrow{\psi} & M \otimes N \end{array}$$

where ξ is the composition

$$\pi_3 X \xrightarrow{\partial} \pi_2 F(a) \xrightarrow{\partial'} \pi_1 F(X) \xrightarrow{\cong} M \otimes N.$$

Proof The isomorphism $\delta : \pi_2 X \cong M \times_P N/\langle M, N\rangle$ is proved in [4; 3.2] under the assumption that $\pi_2 C = O$. However, it is easy to see that the proof remains valid under the weaker assumption (b). The isomorphism θ is given by Whitehead's isomorphism $\Gamma_3 X \cong \Gamma\pi_2 X$ and our isomorphism δ. The map ψ is determined by $(m, n) \mapsto m \otimes n$ as in Theorem 2.11. The maps ∂, ∂' are boundaries in the exact sequences of fibrations,and the isomorphism $\pi_1 F(X) \cong M \otimes N$ is given by Theorem 3.1. So the lemma follows from the more detailed

Lemma 4.2 *Assume the conditions (a) and (b'): the map $\pi_2 C \to \pi_2 X$ is trivial. Then there is a commutative diagram*

$$\begin{array}{ccccc} \pi_2 X & \xleftarrow{\delta'} & M \times_P N & \xrightarrow{\psi} & M \otimes N \\ {\scriptstyle \eta^*}\downarrow & & & & \downarrow{\scriptstyle \cong} \\ \pi_3 X & \xrightarrow{\partial} & \pi_2 F(a) & \xrightarrow{\partial'} & \pi_1 F(\mathbf{X}) \end{array}$$

in which η^* is given by composition with η and Ψ is $(m, n) \mapsto m \otimes n$.

Proof The function δ' is a difference construction. It is well-defined by condition (b') which is implied by (b).

The proof is obtained by working in the universal example. So let $(m, n) \in M \times_P N$, and let $\mathbf{S}$ denote the suspension square

$$\begin{array}{ccc} S^1 & \xrightarrow{i_+} & C_+ S^1 \\ {\scriptstyle i_-}\downarrow & & \downarrow{\scriptstyle j_+} \\ C_- S^1 & \longrightarrow & S^2 \end{array}$$

Then there is a map of squares $\mathbf{t} : \mathbf{S} \to \mathbf{X}$ whose restrictions $t_+ : i_+ \to f, t_- : i_- \to g, t : S^2 \to X$ represent m, n and $\delta'(m, n)$ respectively. Since $\mathbf{t}$ is a map of squares, the following diagram is commutative

$$\begin{array}{ccccccc} \pi_3 S^2 & \xrightarrow{\partial} & \pi_2 F(j_+) & \xrightarrow{\partial'} & \pi_1 F(S) & = & \mathbb{Z} \otimes \mathbb{Z} \\ \downarrow{t_*} & & \downarrow & & \downarrow{t_*} & & \downarrow \\ \pi_3 X & \xrightarrow{\partial} & \pi_2 F(a) & \xrightarrow{\partial'} & \pi_1 F(\mathbf{X}) & = & M \otimes N \end{array}$$

In the upper line of isomorphisms, $\partial\partial'(\eta) = 1 \otimes 1 \in \mathbb{Z} \otimes \mathbb{Z}$. Hence in the lower line

$$\begin{aligned} \partial'\partial\eta^*(\delta'(m, n)) &= \partial'\partial(t_*(\eta)) \\ &= t_*(1 \otimes 1) \\ &= t_{+*}(1) \otimes t_{-*}(1) \\ &= m \otimes n. \end{aligned}$$

□

Proposition 4.3 *Suppose given the homotopy pushout* $\mathbf{X}$ *in which* f, g *are connected maps and* $\pi_2 C = 0, \pi_2 A = \pi_3 A = 0, \pi_2 F(g) = 0$. *Then*

$$\begin{aligned} H_2\tilde{X} &= M \times_P N/\langle M, N\rangle, \\ H_3\tilde{X} &= \operatorname{Ker}([\,,] : M \wedge^P N \to N). \end{aligned}$$

Proof The assumptions imply that

$$\pi_3 X \cong \pi_2 F(a) \cong \operatorname{Ker}(\pi_1 F(X) \to \pi_1 F(g)),$$

so that in Lemma 4.1, ξ maps $\pi_3 X$ isomorphically to $\operatorname{Ker}(M \otimes N \to N)$. The result follows from Lemma 4.1 and Whitehead's exact sequence. □

4.4 The previous results give immediately the exact sequence (2.9), which yields information on $G \otimes G$ and $G \wedge G$. We now generalise the method.

Consider a homotopy pushout

$$\mathbf{X} : \quad \begin{array}{ccc} K(P, 1) & \longrightarrow & K(P/M, 1) \\ \downarrow & & \downarrow \\ K(P/N, 1) & \longrightarrow & X \end{array}$$

where M, N are normal subgroups of a group P. We obtain a number of new results in the homology of groups by considering the Mayer-Vietoris sequence of $\mathbf{X}$, and applying Proposition 4.3; we make the assumption $P = MN$ to ensure that $\tilde{X} = X$. This gives us:

Theorem 4.5 *Suppose given extensions of groups*

$$1 \to M \to P \to Q \to 1,$$

$$1 \to N \to P \to R \to 1,$$

such that $P = MN$. *Let V be the kernel of the commutator map* $M \wedge^P N \to P$. *Then there is an exact sequence*

$$\begin{aligned} H_3 P \to H_3 Q \oplus H_3 R \to V \to H_2 P \to H_2 Q \oplus H_2 R \\ \to (M \cap N)/[M, N] \to P^{ab} \to Q^{ab} \oplus R^{ab} \to 0. \end{aligned}$$

This theorem extends by three terms an exact sequence of [4]. An application is given in [14].

Corollary 4.6 *If* $1 \to M \to P \to Q \to 1$ *is an extension of groups, then there is an exact sequence*

$$H_3P \to H_3Q \to V \to H_2P \to H_2Q \to M/[M,P] \to P^{ab} \to Q^{ab} \to 0,$$

where $V = \operatorname{Ker}([\,,\,] : M \wedge^P P \to P)$. *Further, if the extension is central then* $V = M \wedge_{\mathbb{Z}} P^{ab}$, *the quotient of* $M \otimes_{\mathbb{Z}} P^{ab}$ *by the relations* $m \otimes m = 0, m \in M$.

Corollary 4.7 *Let* $1 \to M \to P \to Q \to 1$ *be a group extension for which* $H_2P = H_3P = 0$, *for example* P *is free. Then there are isomorphisms*

$$H_2Q = \operatorname{Ker}(M/[M,P] \to P^{ab}),$$
$$H_3Q = \operatorname{Ker}([\,,\,] : M \wedge^P P \to P).$$

Remarks 1. The first formula in this corollary is essentially the Hopf formula. The second formula has been shown by J.-L. Loday to be related to Igusa's "pictures" [19].

2. An exact sequence of a type similar to that in Corollary 4.6 is given in [18, Theorem 2.3]. The sequence there is more general, in that it deals with homology with coefficients. On the other hand, a specific formula for the group V in terms of M and P is not given. An algebraic derivation of this formula for V is given in [13].

Finally, we give an exact sequence which generalises (2.9).

Corollary 4.8 *Let* $1 \to M \to P \to Q \to 1$ *be an extension of groups such that* $H_2P = H_3P = H_4P = 0$, *for example* P *is free. Then there is an exact sequence*

$$H_4Q \to \Gamma(M/[M:P]) \xrightarrow{\psi} \operatorname{Ker}([\,,\,] : M \otimes P \to M) \to H_3Q \to 0.$$

Proof We consider the homotopy pushout X of (4.4) with $N = P, Q = P/M$. The assumptions on P and the Mayer–Vietoris sequence of X, imply that $H_4Q = H_4X, H_3Q = H_3X$. Whitehead's Γ-sequence for X and Corollary 3.4 give the result. □

4.9 Let G be a group. Then G gives rise to the tensor product $G \otimes G$ and its quotient $G \wedge G$ discussed in 2.7. There is an intermediate group $G\tilde{\wedge}G$ defined as being the quotient of $G \otimes G$ by the (normal) subgroup generated by $(g \otimes h)(h \otimes g)$ for all $g, h \in G$.

Because $K(G,1)$ is connected, the homotopy groups $\pi_{2+k}(S^kK(G,1))$ stabilise from $k = 2$; in particular $\pi_4(S^2K(G,1)) = \pi_2^s(K(G,1))$.

Proposition 4.10 *In the following commutative diagram, the rows are exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_3(SK(G,1)) & \longrightarrow & G \otimes G & \longrightarrow & [G,G] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_2^s(K(G,1)) & \longrightarrow & G\tilde{\wedge}G & \longrightarrow & [G,G] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2G & \longrightarrow & G \wedge G & \longrightarrow & [G,G] & \longrightarrow & 1
\end{array}$$

Proof The first row is a particular case of Corollary 3.3. The third row [cf. (2.8)] is a special case of Corollary 4.6 with $M = P = G$.

For connected X, the commutative diagram

$$\begin{array}{ccccccccc}
H_3X & \longrightarrow & \Gamma_3SX & \longrightarrow & \pi_3SX & \longrightarrow & H_2X & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} & & \downarrow & & \\
H_3X & \longrightarrow & H_1(X,\pi_1^s) & \longrightarrow & \pi_2^sX & \longrightarrow & H_2X & \longrightarrow & 0
\end{array}$$

is obtained by comparing Whitehead's sequence for SX with an exact sequence deduced from the Atiyah-Hirzebruch spectral sequence of stable homotopy. When $X = K(G,1)$, the map $\alpha : \Gamma(G^{ab}) \to G^{ab} \otimes \mathbb{Z}_2$ is given by $\gamma g \mapsto g \otimes 1$. It is surjective with kernel generated by $\beta(g,h) = \gamma(g+h) - \gamma g - \gamma h$. Therefore α' is surjective with kernel generated by the image of $\beta(g,h)$. As the image of γg in $G \otimes G$ is $g \otimes g$, then $\beta(g,h)$ maps to $(h \otimes g)(g \otimes h)$ by ψ. This proves the exactness of the middle row. □

5 THE GENERALISED VAN KAMPEN THEOREM

In this section we state and prove the Van Kampen theorem for n-cubes of spaces in full generality (Theorem 5.4). The proof of this theorem is by induction on n, assuming the case $n = 0$, which is the classical theorem for the fundamental group of the union of connected spaces. In the general case, the role of the fundamental group is taken by the *fundamental cat^n-group functor*, and so we start by recalling from [20] the notion of cat^n-group.

At the end of this section it is shown that the case $n = 2$ of theorem 5.4 implies our earlier Theorem 1.5, on which all our previous applications depend.

Definition 5.1 [20]. A cat^n-group $(G; N_1, \dots, N_n)$ is a group G together with n subgroups $N_1, \dots, N_n$ and $2n$ homomorphisms s_i and $b_i : G \to N_i$ satisfying
(a) s_i and b_i restrict to the identity on N_i,
(b) $[\operatorname{Ker} s_i, \operatorname{Ker} b_i] = 1$,
(c) $s_i s_j = s_j s_i, b_i b_j = b_j b_i, s_i b_j = b_j s_i$ for $i \neq j$.

The group G is called the *big group* of the cat^n-group. If condition (b) is not fulfilled then we call such a structure a *pre-cat^n-group*. To any pre-cat^n-group there is canonically associated a cat^n-group, obtained by quotienting the big group by the commutator subgroups $[\operatorname{Ker} s_i, \operatorname{Ker} b_i], i = 1, \dots, n$. The corresponding functor is denoted

$$\textbf{ass}\colon \text{(pre-cat}^n\text{-groups)} \to \text{(cat}^n\text{-groups)}.$$

This functor is clearly the identity when restricted to cat^n-groups.

5.2 In order to define formally an n-cube of spaces, we introduce the category $\{0,1\}$ associated to the ordered set $0 < 1$. Its n-fold product is written $\{0,1\}^n$. We denote by **1** the multi-index $(1, \dots, 1)$. We will use also the similarly defined category $\{-1, 0, 1\}^n$.

An *n-cube of spaces* $\mathbf{X}$ is an object of the functor category $\mathsf{Fun}(\{0,1\}^n, \mathsf{Top}^*)$. This category is a proper, closed model category in the sense of [3]. Thus, as shown in [12, Chap.3], for each n-cube of spaces X there is a natural embedding $\mathbf{X} \to \bar{\mathbf{X}}$ such that (i) each map $\mathbf{X}(\alpha) \to \bar{\mathbf{X}}(\alpha)$ is a homotopy equivalence with natural homotopy inverse; (ii) $\bar{\mathbf{X}}$ is fibrant in the sense that for each $\alpha \in \{0,1\}^n$ the canonical map $\bar{\mathbf{X}}(\alpha) \to \lim_{\sigma > \alpha} \bar{\mathbf{X}}(\sigma)$ is a fibration. It is shown in [27] that such a fibrant n-cube $\bar{\mathbf{X}}$ may be extended to an n-cube of fibrations [20], that is a functor from $\{-1,0,1\}^n$ to pointed spaces, also written $\bar{\mathbf{X}}$, and such that for all k and $\alpha \in \{-1,0,1\}^n$ $\bar{\mathbf{X}}(\alpha_1, \dots, \alpha_{k-1}, -1, \alpha_{k+1}, \dots, \alpha_n)$ is the fibre of the fibration $\bar{\mathbf{X}}(\alpha_1, \dots, \alpha_{k-1}, 0, \alpha_{k+1}, \dots, \alpha_n) \to \bar{\mathbf{X}}(\alpha_1, \dots, \alpha_{k-1}, 1, \alpha_{k+1}, \dots, \alpha_n)$.

Thus a 1-cube of spaces, which is just a map $f : A \to X$, is converted into the fibration $F(f) \to \bar{A} \to X$.

5.3 For the statement of the generalised Van Kampen theorem we need three more ingredients. First the methods of [20] give a functor Π from fibrant n-cubes of spaces, and so composing with the functor $\mathbf{X} \mapsto \bar{\mathbf{X}}$ gives a functor Π from n-cubes of spaces to cat^n-groups; the definition of Π is recalled below. Second, an n-cube of spaces $\mathbf{X}$ is said to be *connected* if all the spaces $\bar{\mathbf{X}}(\alpha), \alpha \in \{-1,0,1\}^n$ are connected. Third, for any non-empty set Λ, let Λ_{fin} denote the category of non-empty finite subsets of Λ, with maps the inclusions. Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a covering of X such that each U_λ contains the base point of X. If σ is a non-empty finite subset of Λ, then U_σ denotes the intersection of the U_λ for $\lambda \in \sigma$. If $\sigma \subset \tau$ then $U_\tau \subset U_\sigma$, and so $\mathcal{U}$ determines a contra variant functor on Λ_{fin}. Hence $\operatorname{colim}^\sigma U_\sigma$ makes sense, as does $\operatorname{colim}_\sigma \phi(U_\sigma)$ for any functorial construction ϕ on the U_σ.

Theorem 5.4 (Van Kampen theorem for cubical diagrams of spaces) *Let $\mathbf{X}$ be an n-cube of spaces and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of $\mathbf{X}(1, \dots, 1)$. Each U_σ for $\sigma \in \Lambda_{\text{fin}}$ determines by inverse image an n-cube of spaces $\mathbf{U}_\sigma$. Suppose that each such $\mathbf{U}_\sigma$ is a connected n-cube. Then the following hold:*

(C) *the n-cube $\mathbf{X}$ is connected, and*
(I) *the natural homomorphism of cat^n-groups*

$$\operatorname{colim}^\sigma_{\mathsf{cat}} \Pi \mathbf{U}_\sigma \to \Pi \mathbf{X} = \Pi \operatorname{colim}^\sigma \mathbf{U}_\sigma$$

is an isomorphism.

The colimit on the left, denoted $\mathrm{colim}_{\mathsf{cat}}$, is taken in the category of cat^n-groups, while that on the right is in the category of n-cubes of spaces. The proof of the theorem occupies most of the rest of §5.

5.5 As promised, we recall now the construction of the fundamental cat^n-group functor.

$$\Pi : (n - \text{cubes of spaces}) \to (\mathrm{cat}^n\text{-groups}).$$

Let $f : A \to X$ be a map, and let $\bar{f} : \bar{A} \to X$ be the associated fibration. The fibre product of $n+1$ copies of $\bar{A}$ over X is a space denoted E^f_n. These E^f_n form a simplicial space

$$E^f_\bullet : \cdots \to \bar{A} \times_X \bar{A} \times_X \bar{A} \to \bar{A} \times_X \bar{A} \to \bar{A} = E^f_o.$$

Let $\mathbf{X}$ be an n-cube of spaces, and let $\bar{\mathbf{X}}$ be the associated fibrant n-cube of spaces. For $\varepsilon \in \{-1, 0, 1\}$, let $\partial^\varepsilon_n \bar{X}$ denote the $(n-1)$-cube $\alpha \mapsto \bar{\mathbf{X}}(\alpha, \varepsilon)$. Then $\bar{\mathbf{X}}$ determines a map of $(n-1)$-cubes $\partial^0_n \bar{\mathbf{X}} \to \partial^1_n \bar{\mathbf{X}}$ in direction n and at each index $\alpha \in \{0,1\}^{n-1}$ this fibration is replaced by the simplicial space constructed as above by taking iterated fibre products. This gives an $(n-1)$-cube of simplicial spaces, written $\mathcal{S}_n \bar{\mathbf{X}}$ such that for each $m \geqslant 0$ the $(n-1)$-cube of spaces $(\mathcal{S}_n \bar{\mathbf{X}})_m$ is fibrant. For each $m \geqslant 0$ we can do the same process in direction $n-1$ to obtain an $(n-2)$-cube of simplicial spaces denoted $(\mathcal{S}_{n-1}\,\mathcal{S}_n \bar{\mathbf{X}})_{\bullet m}$, i.e., an $(n-2)$-cube of bisimplicial spaces. By iterating this process, we end up with an n-simplicial space

$$\mathsf{Simp}\,\mathbf{X} = \mathcal{S}_1 \ldots \mathcal{S}_n \bar{\mathbf{X}}.$$

Taking the fundamental group π_1 gives an n-simplicial group

$$\pi_1 \mathsf{Simp}\,\mathbf{X}.$$

We focus our attention on the particular group $G = (\pi_1 \mathsf{Simp}\,\mathbf{X})_1$ because it is the big group of a cat^n-group $(G; N_1, \ldots, N_n)$ such that

$$N_i = (\pi_1 \mathsf{Simp}\,\mathbf{X})_{1\ldots 101\ldots 1}$$

with 0 at the ith place and with $s_i, b_i : G \to N_i$ determined by the face maps ∂_0, ∂_1 of the ith simplicial structure. We denote this cat^n-group by ${}^\wedge\pi_1 \mathsf{Simp}\,\mathbf{X}$, and call it the *fundamental cat^n-group* of the n-cube $\mathbf{X}$, written

$$\Pi\mathbf{X} = {}^\wedge\pi_1 \mathsf{Simp}\,\mathbf{X}.$$

Remark. *A priori*, $^\wedge$ should be considered as a functor from simplicial groups to precat^ngroups. However, in the particular case of $\mathsf{Simp}\,\mathbf{X}$ this precat^n-group, is indeed a cat^n-group as proved in [20]. For $n = 1$ this fact is equivalent to a result of Whitehead [see(1.1) and the remark after (5.7) below]. The advantage of fibrant n-cubes is that the fibrant condition is preserved under the various pull-back constructions used in [20].

The main point of the proof of Theorem 5.4 relies on some preliminary results which we now give.

Lemma 5.6 . *Let $f : A \to X$ be a connected map and let $E^f_\bullet$ be the associated simplicial space. Then*
(a) the Moore complex of the simplicial group is

$$\cdots 1 \to 1 \to \pi_1 F \xrightarrow{\mu} \pi_1 A,$$

(b) the group $\pi_1 E^f_2$ is generated by degenerate elements, i.e. by the images of $\pi_1 E^f_1$ by the two degeneracy operators s_0 and s_1.

Proof The fundamental group $\pi_1 A$ of the total space of the fibration $\bar{A} \to X$ acts on the fundamental group $\pi_1 F$ of the fibre. Then $\pi_1(E^f_n) = \pi_1(\bar{A} \times_X \bar{A})$ is isomorphic to the semi-direct product $\pi_1 F \rtimes \pi_1 A$. Similarly $\pi_1(E^f_n) = (\pi_1 F)^n \rtimes \pi_1 A$. In low dimensions the face and degeneracy operators from $\pi_1 E^f_1$ are given by $d_1(m,n) = n, d_0(m,n) = \mu(m)n, s_1(m,n) = (m,1,n), s_0(m,n) = (1,m,n)$.

The Moore complex $(\bar{G}_\bullet, \partial_n)$ (cf. [10,22]) of a simplicial group is defined by

$$\bar{G}_n = \bigcap_{i=1}^{n} \mathrm{Ker}\, d_i$$

and ∂_n is the restriction of d_0 to $\bar{G}_n$.

The two statements are immediate from the above description of the simplicial structure of $\pi_1 E^f_\bullet$. □

Lemma 5.7 *Let* $G_\bullet$ *be a simplicial group such that* G_2 *is generated by degenerate elements. Then in the Moore complex of* $G_\bullet$ *we have* $\mathrm{Im}\,\partial_2 = [\mathrm{Ker}\, d_1, \mathrm{Ker}\, d_0]$.

Proof There are canonical isomorphisms $G_1 = \bar{G}_1 \rtimes s_0G_0$, where $\bar{G}_1 = \mathrm{Ker}\, d_1$ and $G_2 = (\bar{G}_2 \rtimes s_1\bar{G}_1) \rtimes (s_0\bar{G}_1 \rtimes s_0s_0G_0)$ where $\bar{G}_2 = \mathrm{Ker}\, d_1 \cap \mathrm{Ker}\, d_2$. Hence any element in G_2 can be written uniquely $z = \bar{z}s_1 a a_0 a' s_0 u$ with $\bar{z} \in \bar{G}_2$, and $u \in s_0G_0$ (note that $s_0u = \overline{s_0u}$). Let C be the normal subgroup of $\bar{G}_2$ generated by the commutators $[s_0x, s_1y(s_0y)^{-1}]$, with x and y in $\bar{G}_2$. The image of C by ∂_2 (which is the restriction of d_0) is exactly $[\mathrm{Ker}\, d_1, \mathrm{Ker}\, d_0]$ because any element of $\mathrm{Ker}\, d_0$ can be written $(d_0s_1y)y^{-1}$ with $y \in \bar{G}_1$. Therefore it suffices to prove that $C = \bar{G}_2$, or equivalently that in G_2/C any element can be written $s_1as_0a's_0u$ with $a, a' \in \bar{G}_1$ and $u \in s_0G_0$. As G_2 is generated by degenerate elements it is sufficient to prove that this is true for the products of $s_1as_0a's_0u$ with s_0v, s_0b' and s_1b, for $v \in s_0G_0, b, b' \in \bar{G}_1$.

The first case is immediate:

$$s_1as_0a's_0us_0v = s_1as_0a's_0(uv).$$

For the second case we have

$$\begin{aligned} s_1as_0a's_0us_0b &= s_1as_0a's_0(ub'u^{-1})s_0u \\ &= s_1as_0(a'ub'u^{-1})s_0u. \end{aligned}$$

To prove the third case we need the identity $s_0xs_1y \equiv s_1ys_0(y^{-1}xy)$ in G_2/C, which is another way of writing that the commutator $[s_0x, s_1y(s_0y)^{-1}]$ is trivial. Then we have

$$\begin{aligned} s_1as_0a's_0us_1b &= s_1as_0a's_1(ubu^{-1})s_0u \\ &\equiv s_1as_1(ubu^{-1})s_0(ub^{-1}u^{-1}a'ubu^{-1})s_0u \\ &= s_1(aubu^{-1})s_0(ub^{-1}u^{-1}a'ubu^{-1})s_0u. \end{aligned}$$

Hence $C = \bar{G}_2$ and the lemma is proved. □

Remark. These two lemmas imply that $(\pi_1E_1^f, \pi_1A)$ is a cat[1]-group.

Lemma 5.8 *Let* $E_\bullet$ *be a connected simplicial space (i.e.* E_n *is connected for all* n*). Then there is an exact sequence of groups*

$$\pi_0\pi_2E_\bullet \to \pi_2\|E_\bullet\| \to \overline{\pi_1E_1}/\mathrm{Im}\partial_2 \to \pi_1E_0 \to \pi_1\|E_\bullet\|.$$

Proof From [31] (see the Appendix) we know that to any connected simplicial space E there is associated a spectral sequence

$$E^2_{pq} = \pi_p\pi_qE_\bullet = \pi_{p+q}\|E_\bullet\|.$$

The expression $\pi_p\pi_qE_\bullet$ means that we first take the homotopy groups π_qE_n. For fixed q this gives a simplicial group from which we take π_p. The connectivity of the spaces E_n implies that the first non-trivial row of the E^2-plane is for $q = 1$. This gives $\pi_0\pi_1E_\bullet = \pi_1\|E_\bullet\|$ and the exact sequence

$$\pi_0\pi_2E_\bullet \to \pi_2\|E_\bullet\| \to \pi_1\pi_1E_\bullet \to 1. \qquad (*)$$

To compute $\pi_p\pi_1E_\bullet$ we consider the simplicial group $\pi_1E_\bullet$ and its Moore complex (same notations as in 5.7)

$$\overline{\pi_1E_2} \xrightarrow{\partial_2} \overline{\pi_1E_1} \xrightarrow{\partial_1} \pi_1E_0$$

where $\overline{\pi_1E_1} = \mathrm{Ker}\, d_1 \subset \pi_1E_1$ and $\overline{\pi_1E_2} = \mathrm{Ker}\, d_1 \cap \mathrm{Ker}\, d_2 \subset \pi_1E_2$. With this notation the spectral sequence. gives the following exact sequence

$$1 \to \pi_1\pi_1E_\bullet \to \pi_1E_1/\mathrm{Im}\partial_2 \to \pi_1E_0 \to \pi_0\pi_1E_\bullet \to 1. \qquad (**)$$

Splicing (*) and (**) together gives the result. □

5.9 (KEY PROPOSITION) Let $f : A \to X$ be a connected map and let $E_\bullet \to E^f_\bullet$ be a map of simplicial spaces satisfying:

(a) $E_\bullet$ is connected;

(b) $E_0 \to E_0^f$ is a weak homotopy equivalence;

(c) $\|E_\bullet\| \to \|E_\bullet^f\|$ is a weak homotopy equivalence;

(d) $\pi_1 E_2$ is generated by degenerate elements.

Then $\pi_1 E_1^f = \pi_1 E_1/[\mathrm{Ker}\, s, \mathrm{Ker}\, b]$ where s and $b : \pi_1 E_1 \to \pi_1 E_0$ are induced by the face maps d_1 and d_0.

Proof By Lemma 5.8 and the hypotheses on $E_\bullet$ and $E_\bullet^f$ there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} \pi_0\pi_2 E_\bullet & \longrightarrow & \pi_2\|E_\bullet\| & \longrightarrow & \overline{\pi_1 E_1}/\mathrm{Im}\partial_2 & \longrightarrow & \pi_1 E_0 & \longrightarrow & \pi_1\|E_\bullet\| \\ \downarrow & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ \pi_0\pi_2 E_\bullet^f & \longrightarrow & \pi_2\|E_\bullet^f\| & \longrightarrow & \overline{\pi_1 E_1^f} & \longrightarrow & \pi_1 E_0^f & \longrightarrow & \pi_1\|E_\bullet^f\| \end{array}$$

Here we use the fact that in the case of $E_\bullet^f$ the group $\mathrm{Im}\partial_2$ is trivial (cf. Lemma 5.6). The vertical isomorphisms follow from conditions (b) and (c). Also from (b) we deduce that $\pi_0\pi_2 E_\bullet \to \pi_0\pi_2 E_\bullet^f$ is surjective. By the five lemma we deduce that $\overline{\pi_1 E_1}/\mathrm{Im}\partial_2 \to \pi_1\overline{E_1^f}$ is an isomorphism, and so (because $\pi_1 E_0 = \pi_1 E_0^f$) that $\pi_1 E_1/\mathrm{Im}\partial_2 \to \pi_1 E_1^f$ is an isomorphism. Condition (d) and Lemma 5.7 imply that $\mathrm{Im}\partial_2 = [\mathrm{Ker}\, s, \mathrm{Ker}\, b]$, whence the proposition. □

5.10 (Proof of Theorem 5.4.)

This will be done by induction on n. Let $(C_n), (I_n)$ be the statements that the connectivity of the n-cubes $\mathbf{U}_\sigma, \sigma \in \Lambda_{\mathrm{fin}}$, implies the connectivity and isomorphism statements (C),(I) respectively of Theorem 5.4. Then (C_0) is immediate and (I_0) is the classical Van Kampen theorem for the fundamental group. From now on we assume (C_{n-1}) and (I_{n-1}).

5.11 (Proof of the connectivity statement (C_n).)

The n-cube of spaces X determines a map of $(n-1)$-cubes of spaces $\partial_n^0\mathbf{X} \to \partial_n^1\mathbf{X}$ in direction n.

Lemma 5.12 *Let* $\mathbf{X}$ *be an* n*-cube of spaces. Then the following are equivalent.*
(i) $\mathbf{X}$ *is connected.*
(ii) $\partial_n^0\mathbf{X}$ *and* $\partial_n^1\mathbf{X}$ *are connected for all* $i = 1, \ldots, n$ *and the induced map of cat*$^{n-1}$*-groups*

$$\Pi\partial_n^0\mathbf{X} \to \Pi\partial_n^1\mathbf{X}$$

is surjective when restricted to the big groups.

Proof. Let F be the space $\bar{\mathbf{X}}(-1, \ldots, -1)$. Then $\mathbf{X}$ is connected is equivalent to $\partial_i^0\mathbf{X}, \partial_i^1\mathbf{X}, i = 1, \ldots, n$, and F are all connected. We use the $(n-1)$-simplicial spaces $\mathsf{Simp}\,\partial_i^\varepsilon\mathbf{X}$ for $\varepsilon = -1, 0, 1$. The construction of these spaces shows that if $\partial_i^0\mathbf{X}, \partial_i^1\mathbf{X}, i = 1, \ldots, n$ are all connected, then

$$\pi_0 F = \pi_0((\mathsf{Simp}\,\partial_n^{-1}\mathbf{X})_1).$$

The big group of $\Pi\partial_n^\varepsilon\mathbf{X}$ is defined to be $\pi_1((\mathsf{Simp}\,\partial_n^\varepsilon\mathbf{X})_1)$. So the result follows from the fibration sequence

$$(\mathsf{Simp}\,\partial_n^{-1}\mathbf{X})_1 \to (\mathsf{Simp}\,\partial_n^0\mathbf{X})_1 \to (\mathsf{Simp}\,\partial_n^1\mathbf{X})_1$$

□

By the induction hypothesis (I_{n-1}) we know that for $\varepsilon = 0, 1$ and $i = 1, \ldots, n$, the cat^{n-1}group $\Pi\partial_i^\varepsilon\mathbf{X}$ is $\mathrm{colim}^\sigma_{\mathsf{cat}}\, \partial_i^\varepsilon\mathbf{U}_\sigma$. As $\mathbf{U}_\sigma$ is connected, the map $\Pi\partial_n^0\mathbf{U}_\sigma \to \Pi\partial_n^1\mathbf{U}_\sigma$ is surjective by Lemma 5.12, and hence the map on the colimit is surjective. Thanks to hypothesis (C_{n-1}), $\partial_i^\varepsilon\mathbf{X}$ is connected, and so, by Lemma 5.12 again, X is connected. This verifies (C_n). □

5.13 (Proof of the isomorphism statement (I_n).) The isomorphism $\mathrm{colim}^\sigma_{\mathsf{cat}}\,\Pi\mathbf{U} \cong \Pi\,\mathrm{colim}^\sigma\,\mathbf{U}_\sigma$ will be the composite of several isomorphisms:

$$
\begin{aligned}
\mathrm{colim}^{\sigma}_{\mathsf{cat}}\,\Pi\mathbf{U}_\sigma &\overset{\textbf{(a)}}{=} \mathbf{ass}\,(\mathrm{colim}^{\sigma}_{\mathrm{pre}}\,\Pi\mathbf{U}_\sigma)\\
&\overset{\text{def.}}{=} \mathbf{ass}\,(\mathrm{colim}^{\sigma}_{\mathrm{pre}}(^\wedge\pi_1\mathsf{Simp}\mathbf{U}_\sigma))\\
&\overset{\textbf{(b)}}{=} \mathbf{ass}\,(^\wedge\pi_1\,\mathrm{colim}^{\sigma}\,\mathsf{Simp}\mathbf{U}_\sigma)\\
&\overset{\textbf{(c)}}{=} \mathbf{ass}\,^\wedge\pi_1(\mathsf{Simp}\,\mathrm{colim}^{\sigma}\,\mathbf{U}_\sigma)\\
&\overset{\textbf{(d)}}{=} \mathbf{ass}\,^\wedge\pi_1(\mathsf{Simp}\,\mathrm{colim}^{\sigma}\,\mathbf{U}_\sigma)\\
&\overset{\text{def.}}{=} \Pi\,\mathrm{colim}^{\sigma}\,\mathbf{U}_\sigma
\end{aligned}
$$

(a) follows from a simple algebraic lemma (5.14) comparing the colimit in the category of cat^n-groups ($\mathrm{colim}_{\mathsf{cat}}$) with the colimit in the category of pre-cat^n-groups ($\mathrm{colim}_{\mathrm{pre}}$); (b) follows from the classical Van Kampen theorem (Lemma 5.15); (c) is the main step in the proof and uses the key Proposition 5.9; (d) is immediate because $^\wedge\pi_1(\mathsf{Simp}\,\mathbf{X})$ is a cat^n-group.

Lemma 5.14 *If $(\mathbf{G}_\sigma)$ is a diagram of cat^n-groups, then the colimit $\mathbf{G} = \mathrm{colim}^{\sigma}_{\mathsf{cat}}\,\mathbf{G}_\sigma$ is given by $\mathbf{G} = \mathbf{ass}\,\mathrm{colim}^{\sigma}_{pre}\,\mathbf{G}$, and the big group of $\mathbf{G}$ is the quotient of the big group of $\mathrm{colim}^{\sigma}_{pre}\,\mathbf{G}$ by the subgroups $[\mathrm{Ker}\, s_i,\ \mathrm{Ker}\, b_i], i = 1, \dots, n$.*

Proof We first note that the big group of the pre-cat^n-group $\mathrm{colim}^{\sigma}_{\mathrm{pre}}\,\mathbf{G}$ is simply the colimit, in the category of groups, of the big groups of the $\mathbf{G}_\sigma$s (and similarly for the N_is). The associated cat^n-group is obtained by quotienting by the commutator subgroups $[\mathrm{Ker}\, s_i,\ \mathrm{Ker}\, b_i], i = 1, \dots, n$, and it is easily verified that this new cat^n-group satisfies the universal property required for a colimit of cat^n-groups. □

Lemma 5.15 *If the n-cube $\mathbf{U}_\sigma$ is connected for all $\sigma \in \Lambda_{fin}$, then the natural map*

$$\mathrm{colim}^{\sigma}(^\wedge\pi_1\,\mathsf{Simp}\,\mathbf{U}_\sigma) \to {}^\wedge\pi_1\ \mathrm{colim}^{\sigma}\ \mathsf{Simp}\,\mathbf{U}_\sigma.$$

is an isomorphism.

Proof Because of functoriality, it is sufficient to prove this isomorphism at the big group level:

$$\mathrm{colim}^{\sigma}_{\mathsf{Gp}}(\pi_1(\mathsf{Simp}\,\mathbf{U}_\sigma)_1) = \pi_1\ \mathrm{colim}^{\sigma}((\mathsf{Simp}\,\mathbf{U}_\sigma)_1).$$

Note that the spaces $V_\sigma = (\mathsf{Simp}\,\mathbf{U}_\sigma)_1$ for $\sigma \in \Lambda_{\mathrm{fin}}$ may be identified with open subsets of $(\mathsf{Simp}\,\mathbf{X})_1$ and that V_σ is the intersection of the V_λ for $\lambda \in \sigma$. Thus $\mathrm{colim}^{\sigma}\,V_\sigma$ means simply union. So the above equality follows from the classical Van Kampen theorem provided that V_σ is connected for all $\sigma \in \Lambda_{\mathrm{fin}}$.

By hypothesis, $\mathbf{U}_\sigma$ is connected. Therefore $\mathcal{S}\mathbf{U}_\sigma$ [see (5.5)] is a simplicial object of connected, fibrant $(n-1)$-cubes of spaces. Continuing this process $n-1$ more times, we obtain $\mathsf{Simp}\,\mathbf{U}_\sigma$ which is thus a connected n-simplicial space. In particular, the space V_σ is connected. □

5.16 (End of the proof of Theorem 5.4: the isomorphism (c).)

This isomorphism is obtained by applying the functor $\mathbf{ass}\,^\wedge\pi_1$ to the map

$$E = \mathrm{colim}^{\sigma}\,\mathsf{Simp}\,\mathbf{U}_\sigma \to E^{\mathbf{X}} = \mathsf{Simp}\ \mathrm{colim}^{\sigma}\ \mathbf{U}_\sigma.$$

It obviously suffices to check this isomorphism on the big groups. By Lemma 5.14 the big group of $\mathbf{ass}\,^\wedge\pi_1 E$ is the quotient of $\pi_1(E_1)$ by the subgroups $[\mathrm{Ker}\, s_i,\ \mathrm{Ker}\, b_i], i = 1, \dots, n$. Therefore it is sufficient to prove that $\pi_1(E^{\mathbf{X}}_1)$ is obtained in the same way. In order to use the inductive hypothesis we introduce between E and $E^{\mathbf{X}}$ an intermediate n-simplicial space E' as in the diagram

$$
\begin{array}{ccccc}
\mathrm{colim}^{\sigma}\,\mathcal{S}_1\dots\mathcal{S}_n\bar{\mathbf{U}}_\sigma & \longrightarrow & \mathcal{S}_1\dots\mathcal{S}_{n-1}\,\mathrm{colim}^{\sigma}\,\mathcal{S}_n\bar{\mathbf{U}}_\sigma & \longrightarrow & \mathcal{S}_1\dots\mathcal{S}_n\,\mathrm{colim}^{\sigma}\,\bar{\mathbf{U}}_\sigma\\
\| & & \| & & \|\\
E & & E' & & E^{\mathbf{X}}
\end{array}
$$

where $\bar{\mathbf{U}}_\sigma$ is the fibrant n-cube associated to $\mathbf{U}_\sigma$. By induction, $\pi_1 E'_1$ is $\pi_1 E_1$ quotiented by the subgroups $[\mathrm{Ker}\, s_i,\ \mathrm{Ker}\, b_i]$, $i = 1, \dots, n-1$. Hence the isomorphism (c) will be a consequence of

Lemma 5.17 *The group* $\pi_1(E_1^{\mathbf{X}})$ *is* $\pi_1(E_1')$ *quotiented by* $[\operatorname{Ker} s_n, \operatorname{Ker} b_n]$, *where* s_n *and* b_n *are induced by the face maps in direction* n.

Proof This is an immediate consequence of the Key Proposition 5.9 applied to the map of simplicial spaces $E_1' \ldots {}_{1\bullet} \to E_1^{\mathbf{X}} \ldots {}_{1\bullet}$ once we have verified the hypotheses (a)–(d) of this proposition.

Let $\#$ be the $(n-1)$-multi-index with each value 1. We regard $\mathbf{X}$ as a map $\mathbf{X}_n : \partial_n^0 \mathbf{X} \to \partial_n^1 \mathbf{X}$ so that we have an induced map $\mathcal{S}_1 \ldots \mathcal{S}_{n-1} \bar{\mathbf{X}}_n$ of $(n-1)$-simplicial spaces and hence a map $f = (\mathcal{S}_1 \ldots \mathcal{S}_{n-1} \bar{\mathbf{X}})_{\#}$, of spaces. Note that $E_\bullet^f = E_{\#\bullet}^{\mathbf{X}}$ It is to this map f that we apply the key Proposition 5.9. Note that the map f is connected since we proved (C_n), that $\mathbf{X}$ is connected.

The simplicial $(n-1)$-cubes $\mathcal{S}_n \bar{\mathbf{U}}_\sigma$ are connected. Therefore the $(n-1)$-cubes $(\mathcal{S}_n \bar{\mathbf{U}}_\sigma)_j$ are connected, and the induction hypothesis (C_{n-1}) gives that $\operatorname{colim}_\sigma (\mathcal{S}_n \bar{\mathbf{U}}_\sigma)_j$ is connected. On applying $\mathcal{S}_1 \ldots \mathcal{S}_{n-1}$, we obtain that E' is connected.

The induction hypothesis (I_{n-1}) applied for $j = 2$ gives that $\pi_1 E'_{\#2}$ is a quotient of $\pi_1 E_{\#2} = \operatorname{colim}^\sigma \pi_1 E_{\#2}^{\mathbf{U}_\sigma}$ By Lemma 5.6, $\pi_1 E_{\#2}^{\mathbf{U}_\sigma}$ is generated by degenerate elements (in direction n). Hence so also is the colimit $\pi_1 E_{\#2}$ and hence so also is the quotient $\pi_1 E'_{\#2}$ This is condition (d) of 5.9.

There are natural weak homotopy equivalences (w.h.e.'s) of $(n-1)$-cubes $(\mathcal{S}_n \bar{\mathbf{X}})_0 \to \partial_1^0 \mathbf{X}$, $(\mathcal{S}_n \bar{\mathbf{U}}_\sigma) \to \partial_1^0 \mathbf{U}_\sigma$. So we have a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}^\sigma (\mathcal{S}_n, \bar{\mathbf{U}}_\sigma)_0 & \longrightarrow & (\mathcal{S}_n \bar{\mathbf{X}})_0 \\ \downarrow & & \downarrow \\ \operatorname{colim}^\sigma \partial_1^0 \mathbf{U}_\sigma & \xrightarrow{=} & \partial_1^0 \mathbf{X} \end{array}$$

The left hand map is a colimit of w.h.e.'s and so is a w.h.e. (cf. Proposition A4 of the Appendix). Hence the top map is a w.h.e. After taking $\mathcal{S}_1 \ldots \mathcal{S}_{n-1}$ we get a w.h.e. of spaces. This is condition (b) of 5.9.

As simplicialisation (i.e. applying the functor $\mathcal{S}_i$) commutes with geometric realisation, it suffices, to provide condition (c), to demonstrate it in the case $n = 1$. The key point here is the fact that, for a connected map $fA \to X$, the natural map $\|E_\bullet^f\| \to X$ is a weak homotopy equivalence (cf. Appendix). We apply this result to f and f_σ in the following diagram

$$\begin{array}{ccccccc} \|E_\bullet\| & \xrightarrow{=} & \| \operatorname{colim}^\sigma E_\bullet^{f\sigma} \| & \xleftarrow{\sim} & \operatorname{colim}^\sigma \|E_\bullet^{f\sigma}\| & \xrightarrow{\text{w.h.e.}} & \operatorname{colim}^\sigma U_\sigma \\ \downarrow & & & & & & \| \\ \|E_\bullet^f\| & & & \xrightarrow{\text{w.h.e.}} & & & X \end{array}$$

to prove that $\|E_\bullet\| \to \|E_\bullet^f\|$ is a w.h.e., so obtaining condition (c) of 5.9.

This completes the proof of (I_n), and so Theorem 5.4 has been proved by induction. □

5.18 (Van Kampen theorem for maps: n = 1.) For $n = 1$ a cat^n-group $(G; N)$ is equivalent to a crossed module $\mu : M \to N$ thanks to the following formulae: $M = \operatorname{Ker} s$, $\mu = b|_M$ and $G = M \rtimes N$, $s(m, n) = n$, $b(m, n) = \mu(m)n$.

Under this equivalence the fundamental cat^1-group of the fibration sequence $F \to A \to X$ corresponds to the crossed module $\pi_1 F \to \pi_1 A$ described in (1.1). Therefore Theorem 5.4 can be phrased for $n = 1$ in terms of crossed modules (as already proved in [5,6].

5.19 (Van Kampen theorem for squares of spaces: $n = 2$.)

Proposition 5.20 *[20] The category of cat^2-groups is equivalent to the category of crossed squares.*

Proof. This was proved in [20 §5] with a slightly different (but equivalent) definition of crossed squares. Therefore we just indicate how to go back and forth between cat^2-groups and crossed squares. Starting with a cat^2-group $(G; K, K')$ we put $L = \operatorname{Ker} s \cap \operatorname{Ker} s'$, $M = K \cap \operatorname{Ker} s'$, $N = K' \cap \operatorname{Ker} s$ and $P = K \cap K'$. Then λ (respectively λ', respectively μ, respectively ν) is the restriction of b (respectively b', respectively b, respectively

b'). This gives the commutative square. The actions of P are given by conjugation in G, and $h(m,n) = mnm^{-1}n^{-1}$ where the commutator, computed in G, is obviously in L.

On the other hand, starting with a crossed square as in (3.1), we put $G = (L \rtimes N) \rtimes (M \rtimes P)$, $K = M \rtimes P$, $K' = N \rtimes P$, s is the projection and $b(l,n,m,p) = (\lambda(l)^n m\nu(n)p)$. (The h function is taken into account in the action of $M \rtimes P$ on $L \rtimes N$ used to construct G). As G is canonically isomorphic to $(L \rtimes M) \rtimes (N \rtimes P)$, s' and b' are defined similarly. □

Under this equivalence the fundamental cat^2-group of a square of spaces $\mathbf{X}$ corresponds to the crossed square associated to $\mathbf{X}$ and described in (1.3). Therefore Theorem,5.4 can be phrased for $n = 2$ in terms of crossed squares. The specific form of Theorem 1.5 in terms of homotopy amalgamated sums now follows in a standard way, by applying Theorem 5.4 for $n = 2$ to a covering of a double mapping cyc1inder by two open sets.

Finally, we remark that Theorem 5.4 also implies a major case of the Van Kampen theorem for filtered spaces proved in [6], namely the case of filtered spaces $\mathbf{X}$ for which X_0 is a single point.

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[1] There is an erratum to this paper available as a preprint from Graham Ellis' home page at http://hamilton.nuigalway.ie/. A connectivity condition needs to be imposed to apply the Generalised van Kampen Theorem.

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A APPENDIX BY MICHEL ZISMAN

We quote the existence of a homotopy spectral sequence for a connected bisimplicial set $\mathbf{X}_{\bullet\bullet}$ The spectral sequence for a connected simplicial space $E_\bullet$ is then derived, using an amalgamation theorem for weak homotopy equivalences which is required elsewhere in the main part of the paper.

Let $\mathbf{X}_{\bullet\bullet}$ be a bisimplicial set. A base point $* \in X_{00}$ is supposed given, and its degeneracies are taken as base points for the sets X_{qn}. For varying n, the groups $\pi_q X_{\bullet n}$ form a simplicial group $\pi_q \mathbf{X}_{\bullet\bullet}$ (we take $\pi_0 \mathbf{X}_{\bullet\bullet} = 1$) for which we may take the pth homotopy group $\pi_p\pi_q\mathbf{X}_{\bullet\bullet}$ The diagonal simplicial set of $\mathbf{X}_{\bullet\bullet}$ is written $\nabla\mathbf{X}_{\bullet\bullet}$.

Theorem A.1 . *Suppose that for $n \geqslant 0$, each simplicial set $\mathbf{X}_\bullet n$ is connected. Then there is a functorial convergent spectral sequence*

$$E^2_{p,q}(\mathbf{X}_{\bullet\bullet}) = \pi_p\pi_q\mathbf{X}_{\bullet\bullet} \Rightarrow \pi_{p+q}\nabla\mathbf{X}_{\bullet\bullet}.$$

This was proved in [31] by using standard simplicial constructions to reduce the theorem to Quillen’s spectral sequence for a bisimplicial group [25]. A proof has also appeared in [3] (under slightly weaker conditions than connectivity) and so we omit further details.

We now consider a simplicial space $E_\bullet$. As explained in §1 of the paper, by $\pi_p\pi_q E_\bullet$ we mean the pth homotopy group of the simplicial group $\pi_q E_\bullet$ (simplicial set if $q = 0$). Let $\|E_\bullet\|$ be the geometric realisation of $E_\bullet$ without degeneracies.

Theorem A.2 *Let $E_\bullet$ be a connected simplicial space, Then there is a convergent spectral sequence*

$$E^2_{pq}(E_\bullet) = \pi_p\pi_q E_\bullet \Rightarrow \pi_{p+q}\|E_\bullet\|.$$

Proof. Application of the simplicial functor S to each E_n gives a bisimp1icial set $(SE)_{pq} = S_p E_q$.

Let $D_\bullet$ be the simplicial space defined by $D_q = |SE_q|$, where $||$ is the usual geometric realisation and let $\theta_\bullet : D_\bullet \to E_\bullet$ be defined by the usual adjunction maps $|SE_q| \to E_q$. Since each θ_q is a weak homotopy equivalence, Proposition A.3 below shows that so also is $\|\theta_\bullet\| : \|D_\bullet\| \to \|E_\bullet\|$. The simplicial space $D_\bullet$ is nice enough to ensure that $\|D_\bullet\| \to |D_\bullet|$ is a homotopy equivalence [26], and for general reasons $|\nabla(SE)_{\bullet\bullet}| = |D_\bullet|$. So we have a natural isomorphism $\pi_*\nabla(SE)_{\bullet\bullet} \to \pi_*\|E_\bullet\|$. Theorem A.2 follows now from Theorem A.1 once we have proved the following proposition (which is well known [26] if “weak homotopy equivalence” is replaced by “homotopy equivalence”). □

Proposition A.3 *Let $\theta_\bullet : D_\bullet \to E_\bullet$ be a map between simplicial spaces such that for each $n, \theta_n : D_n \to E_n$ is a weak homotopy equivalence. Then*

$$\|\theta_\bullet\| : \|D_\bullet\| \to \|E_\bullet\|$$

is a weak homotopy equivalence.

This is proved by induction on the standard filtration of the realisations, using an amalgamation lemma on the pushout of weak homotopy equivalences. This amalgamation lemma follows in a standard way (using double mapping cylinders) from the following result, which is also used in the body of this paper.

Proposition A.4 *Let $g : X \to Y$ be a continuous map of spaces and let $U = \{U_\lambda\}_\lambda \in \Lambda V = \{V_\lambda\}_{\lambda\in\Lambda}$ be open coverings of X, Y respectively such that $g(U_\lambda) \subset V_\lambda$ for all $A \in A$. Suppose that for any non-empty finite subset $\sigma \subset \Lambda$, the restriction of g to $g_\sigma : U_\sigma \to V_\sigma$ is a weak homotopy equivalence. Then g is a weak homotopy equivalence.*

Proof As in §1, U_σ (respectively V_σ) is the intersection of the sets U_λ (respectively V_λ) for $\lambda \in \sigma$.

It is clear that it suffices to prove the proposition when Λ is finite, which we now assume. Now $g : X \to Y$ is a weak homotopy equivalence if and only if $\pi_0 g : \pi_0 X \to \pi_0 Y$ is surjective and for any commutative diagram

$$\begin{array}{ccc} \dot{I}^n & \xrightarrow{\nu} & X \\ \downarrow & & \downarrow g \\ I^n & \xrightarrow{u'} & Y \end{array} \qquad (*)$$

(where $\dot{I}^n$ is the boundary of the n-dimensional cube I^n) there exists $u : I^n \to X$ such that $u|I^n = \nu$ and $gu \simeq u' \operatorname{rel} \dot{I}^n$.

The fact that $\pi_0 g$ is surjective is clear. Suppose given the diagram (*). Subdivide I^n by hyperplanes parallel to the coordinate planes, into subcubes small enough so that for any subcube c there is a $\lambda \in \Lambda$ such that $u'(c) \subset V_\lambda$ and if $c \cap \dot{I}^n \neq 0$, then $\nu(c \cap \dot{I}^n) \subset U_\lambda$. This decomposition into subcubes defines a CW-structure K on I^n and we set $K_p = \dot{I}^n \cup K^p, p \geqslant -1$. Suppose $u_p : K_p \to X, u'_p I^n \to Y$, and a homotopy $h_p : u'(c) \simeq \operatorname{rel} \dot{I}^n$ are given such that (1) $u_p|\dot{I}^n = \nu$, (2) $gu_p = u'_p$, (3) if c is an n-cube of K and $\lambda \in \Lambda$ satisfies $u'(c) \subset V_\lambda$ then $h_p(I \times c) \subset V_\lambda$ and $u_p(c \cap K_p) \subset U_\lambda$. Let e be a $(p+1)$-cube of K. Let $\sigma \subset \lambda$ be the maximal subset for which $u'_p(e) \subset V_\sigma$. We have a commutative diagram □

Since g_σ is a weak homotopy equivalence, u_p extends to a map $u|e : e \to U_\sigma$ such that there is a homotopy $g_\sigma(u|e) \simeq u'_p|e \operatorname{rel} \dot{e}$. These maps and homotopies for all $(p+1)$-cubes e combine to give u_{p+1} and h_{p+1}. The construction of u_0 involves only the surjectivity of $\pi_0 g$.

(I am indebted to D. Puppe for this proof of Proposition A4.) Finally we give one other result which is used in the text and which is due to F. Waldhausen.

Proposition A.5 *Let* $fA \to X$ *be a connected map. Then the canonical map* $\|E^f_\bullet\| \to X$ *is a weak homotopy equivalence.*

Proof Let F be the homotopy fibre of f The diagram

$$\begin{array}{ccccc} F & \longrightarrow & A & \longrightarrow & X \\ \downarrow \mu & & \downarrow f & & \downarrow = \\ * & \longrightarrow & X & \xrightarrow[=]{} & X \end{array}$$

induces a diagram of simplicial spaces

$$E^u_\bullet \xrightarrow{\psi} E^f_\bullet \xrightarrow{\psi} E^{id}_\bullet.$$

Now $\|E^{id}_\bullet\| = X \times \|\Delta[0]\|$ has the homotopy type of X (in fact $\|E^{id}_\bullet\| = X$), and $\|E^u_\bullet\|$ is contractible (it is easy to construct a homotopy $1 \simeq *$ on this space). We then have the following commutative diagram

$$\begin{array}{ccc} F =^{\mu}_{0} & \longrightarrow & \|E^\mu_\bullet\| \simeq * \\ \downarrow & & \downarrow \|\emptyset\| \\ A = E^F_0 & \longrightarrow & \|E^f_\bullet\| \\ \downarrow & & \downarrow \|\psi\| \\ X = E^{id} & \longrightarrow & X \times \|\Delta[0]\| \simeq X \end{array}$$

The upper square is homotopy Cartesian, by Lemma A.6 below, and [26, Proposition 1.6] or [24]. It follows that the map $\|E^f_\bullet\| \to X$, which is the composition of $\|\psi\|$ with the projection onto the first factor, is a weak homotopy equivalence.

Lemma A.6 *Let* $\emptyset : X_\bullet \to Y_\bullet$ *be a morphism of simplicial spaces, and suppose that for any strictly increasing map* $\theta : [m] \to [n]$, *the commutative square*

$$
\begin{array}{ccc}
X_n & \xrightarrow{\theta^*} & X_m \\
{\scriptstyle \emptyset_n}\downarrow & & \downarrow{\scriptstyle \emptyset_m} \\
X_n & \xrightarrow[\theta^*]{} & Y_n
\end{array}
$$

is homotopy Cartesian. Then it is homotopy Cartesian for any $\theta : [m] \to [n]$.

Proof It suffices to suppose θ surjective. Then θ has a section σ, say, and the square constructed with θ^* is homotopy Cartesian by assumption. The lemma is then deduced from general facts about homotopy Cartesian squares: "a retract of an homotopy Cartesian square is itself homotopy Cartesian" (see [24, Lemma 1)]. □