Some strict higher homotopy groupoids: intuitions, examples, applications, prospects.

Ronnie Brown

Transpennine Topology Triangle- TTT74
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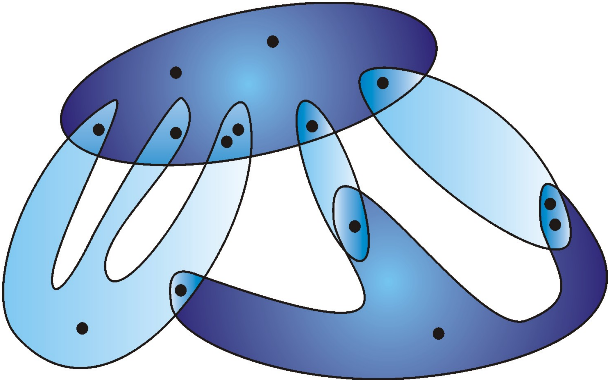
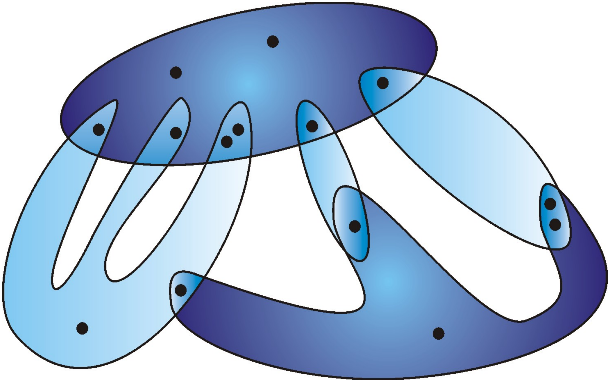
Such computation involves choices and may not be algorithmic.





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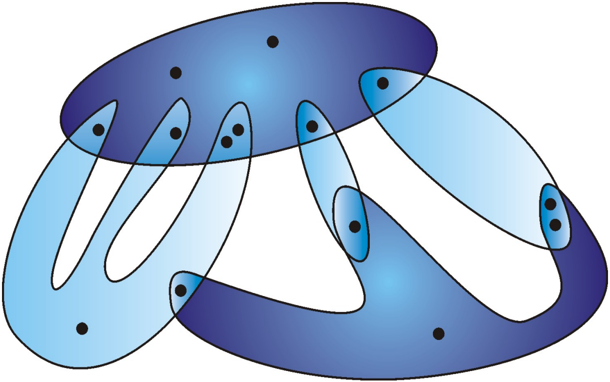

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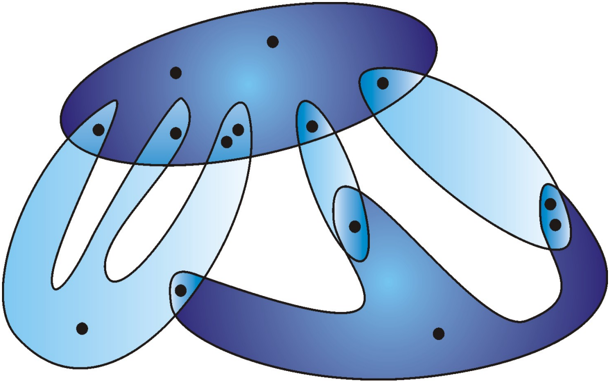

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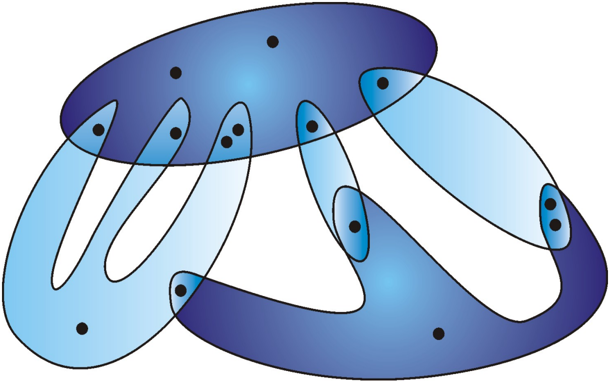

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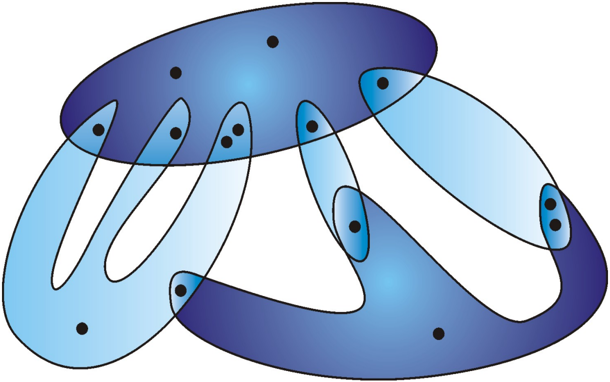

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Alexander Grothendieck

......people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care - or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids.

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That argument does not apply to partial compositions.

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This freeness looks like a universal property in dimension 2!

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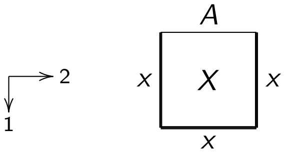

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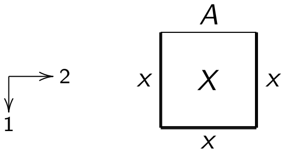

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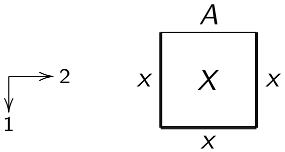

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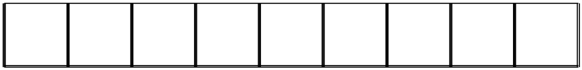

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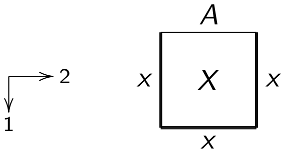

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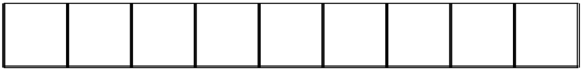

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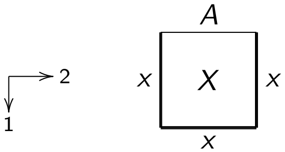

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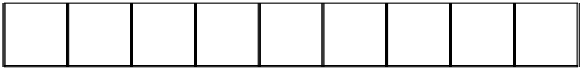

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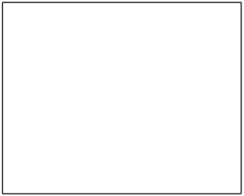

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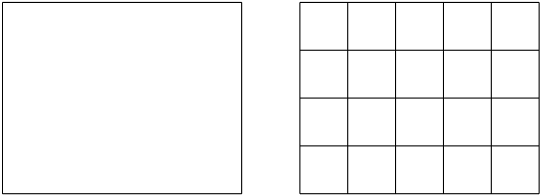

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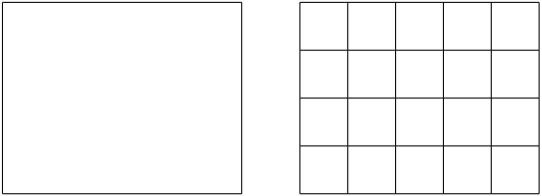

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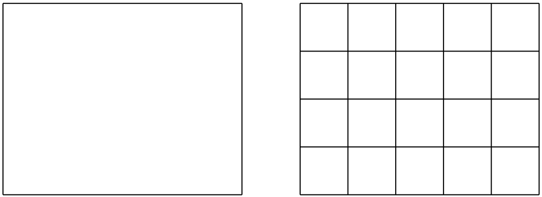

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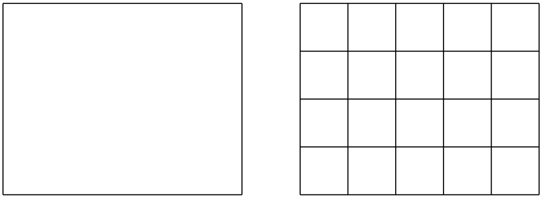

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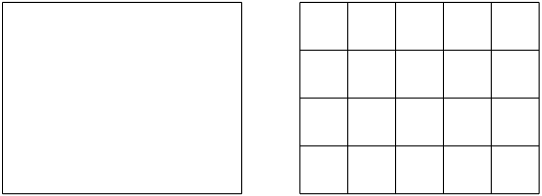

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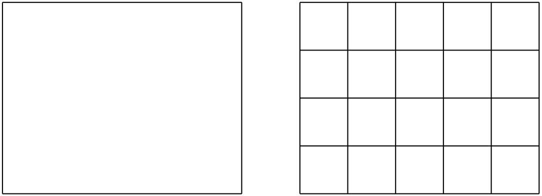

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We know how to cut things up, but how to control algebraically putting them together again?

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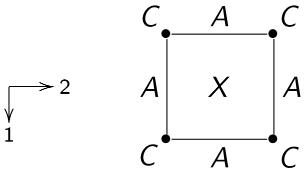

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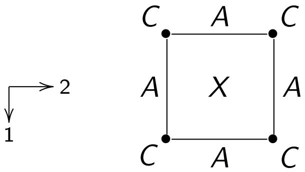

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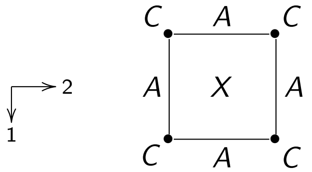

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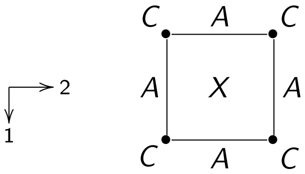

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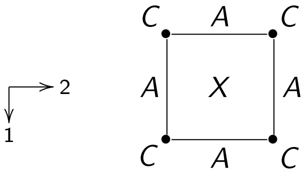

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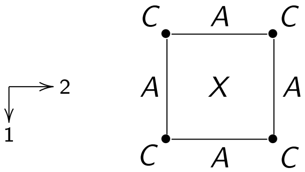

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That is my definition of higher dimensional algebra.

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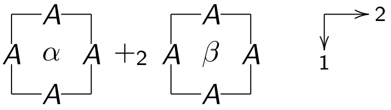

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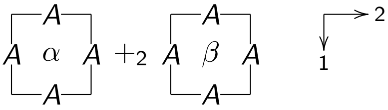

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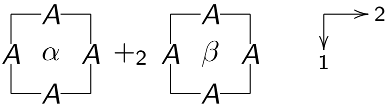

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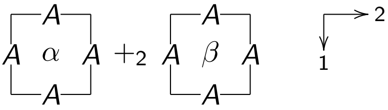

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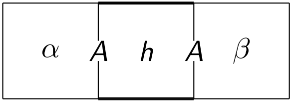

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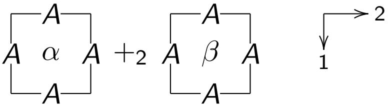

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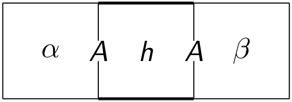

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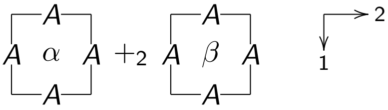

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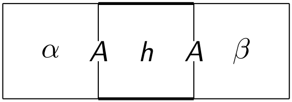

where thick lines show constant paths, and define

$$\langle\langle \alpha \rangle\rangle +_2 \langle\langle \beta \rangle\rangle = \langle\langle \alpha +_2 h +_2 \beta \rangle\rangle$$

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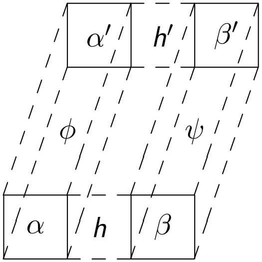

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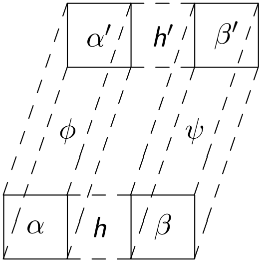

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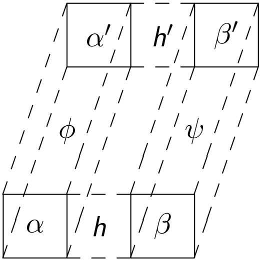

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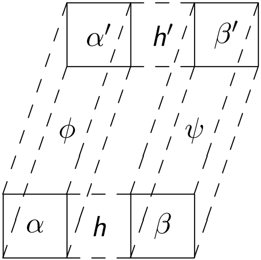

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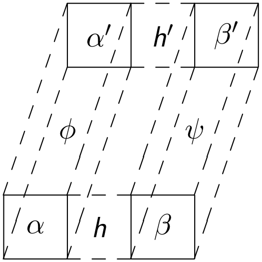

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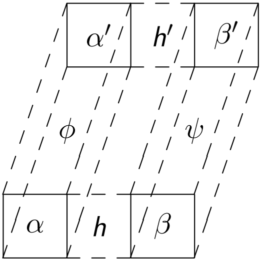

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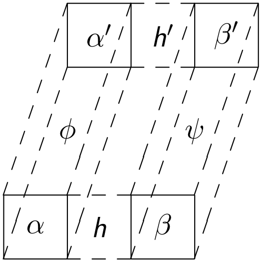

Can you see why the middle 'hole' can be filled appropriately? Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

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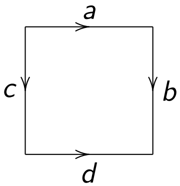

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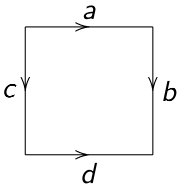

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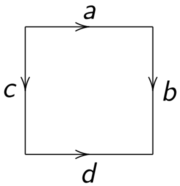

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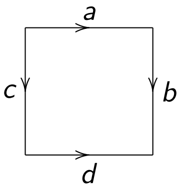

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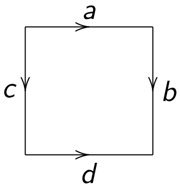

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Compare Stokes' theorem! Local Stokes implies global Stokes.

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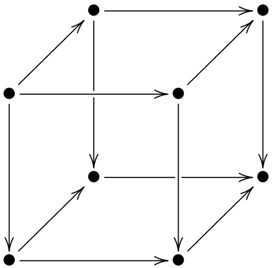

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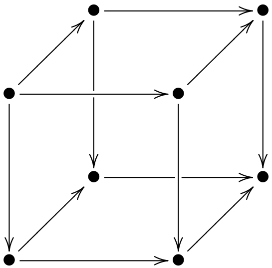

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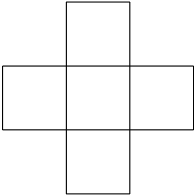

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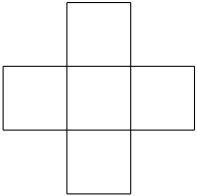

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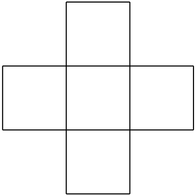

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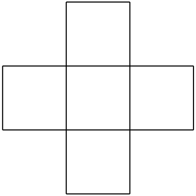

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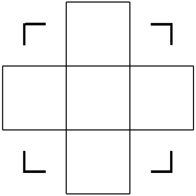

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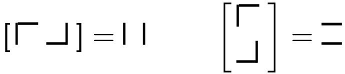

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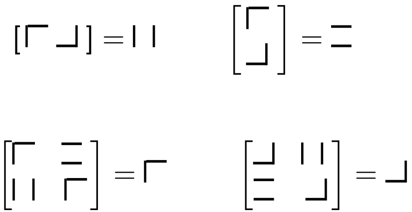

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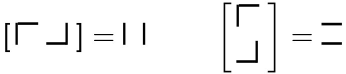

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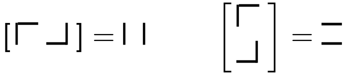

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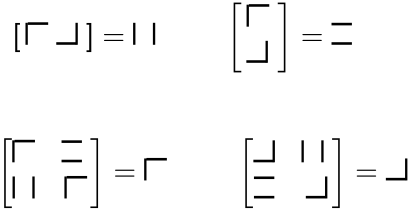

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It is a good exercise to prove that any composition of commutative cubes is commutative.

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Calculation of the corresponding $\pi_2(X, x)$ may be tricky!

Computer calculations of the induced crossed module $\delta : \iota_*(P) \to S_4$ representing the 2-type of the mapping cone Γ of $B\iota : BP \to BS_4$ for various subgroups P of S_4, and of the kernel $\pi_2(\delta) \cong \pi_2(\Gamma)$ of δ.

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S_3	$GL(2,3)$	C_2
C_2'	$C_2^3\ H_8^+$	$C_2^3 C_4$
C_2^2	$S_4 C_2$	C_2
C_4	$SL(2,3) \rtimes C_4$	C_4
D_8	$S_4 C_2$	C_2

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Computer calculations of the induced crossed module $\delta : \iota_*(P) \to S_4$ representing the 2-type of the mapping cone Γ of $B\iota : BP \to BS_4$ for various subgroups P of S_4, and of the kernel $\pi_2(\delta) \cong \pi_2(\Gamma)$ of δ.

P	$\iota_* P$	$\pi_2(\delta)$
C_2	$GL(2,3)$	C_2
C_3	$C_3\ SL(2,3)$	C_6
S_3	$GL(2,3)$	C_2
C_2'	$C_2^3\ H_8^+$	$C_2^3 C_4$
C_2^2	$S_4 C_2$	C_2
C_4	$SL(2,3) \rtimes C_4$	C_4
D_8	$S_4 C_2$	C_2

Here $C_2 = \langle(1,2)\rangle$, $C_2' = \langle(1,2)(3,4)\rangle$, $C_2^2 = \langle(1,2),(3,4)\rangle$; $SL(2,3)$, $GL(2,3)$ and $GL(3,2)$ are linear groups of orders 24, 48 and 168 respectively;
H_n is the holomorph of C_n, and H_n^+ is its positive subgroup in degree n.
π_1, π_2 give only a pale shadow of the 2-type, which is essentially nonabelian, but can be calculated in some cases.

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But this is not in the current 'canon' of algebraic/geometric topology.

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4) Strict cubical ω-groupoids with connections are equivalent to crossed complexes and ρX_* is in this equivalent to ΠX_*.
5) This gives a different foundation for algebraic topology whose full consequences have yet to be worked out. See 'Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids' R. Brown, P.J. Higgins, R. Sivera, EMS Tracts in Mathematics 15, xxxiii+640 pages, (autumn 2010).

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Philosophy: spaces often come with structure, or are replaced by spaces with structure, so it is reasonable to base algebraic topology on spaces with structure rather than just bare spaces.

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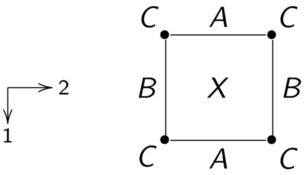

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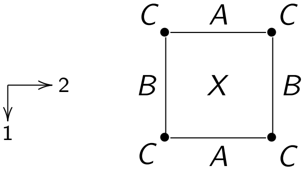

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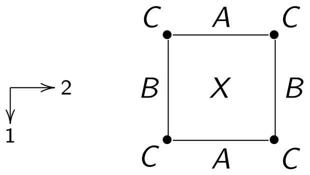

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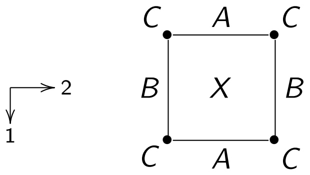

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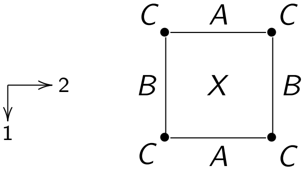

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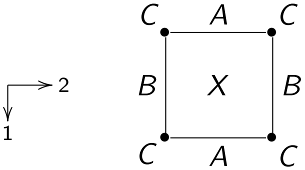

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Strict n-fold groupoids model weak homotopy n-types, so there is still a lot to be said for studying the relations between strict and non strict structures.

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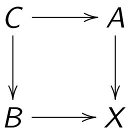

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This is how we got a strong generalisation of Whitehead's theorem involving induced crossed modules and so bifibrations of categories. Three papers by Brown-Wensley include some group computation to do the sums:

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where ε is the excision map. Applying Π gives an excision theorem for Π.

This is how we got a strong generalisation of Whitehead's theorem involving induced crossed modules and so bifibrations of categories. Three papers by Brown-Wensley include some group computation to do the sums: we obtain specific groups and numbers.

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Computes certain nonabelian triad homotopy groups $\pi_3(X; A, B; x)$ as built up by generalised Whitehead products from lower relative homotopy groups.

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which is a 3-pushout of squares of spaces!

This is how we got a totally new triadic Hurewicz Theorem, essentially conjectured by Loday, and proved as a consequence of our van Kampen theorem for n-cubes of spaces.

Theorem

Suppose for the pointed triad $(X; A, B)$ that $A, B, A \cap B$ are connected, $(A, A \cap B), (B, A \cap B)$ are 1-connected, and $(X; A, B)$ is 2-connected. Then $X \cup CA \cup CB$ is 2-connected and the Hurewicz map

$$\pi_3(X; A, B) \to H_3(X; A, B)$$

factors the action of $\pi_1(A \cap B)$ and the generalised Whitehead product.

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