Motion, space, knots, and higher dimensional algebra

William J. Spencer Lecture

Kansas State University

Manhattan

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April 17, 2012

Space

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thus space encodes motion.

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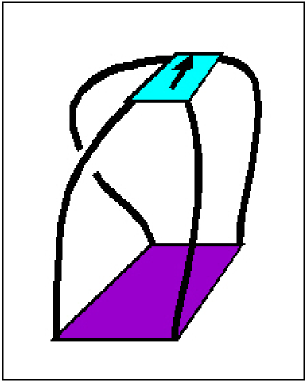

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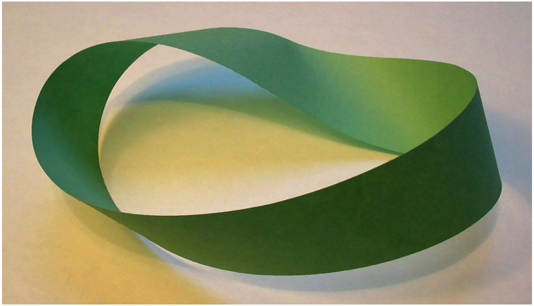

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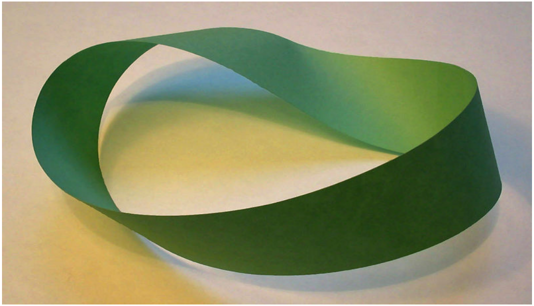

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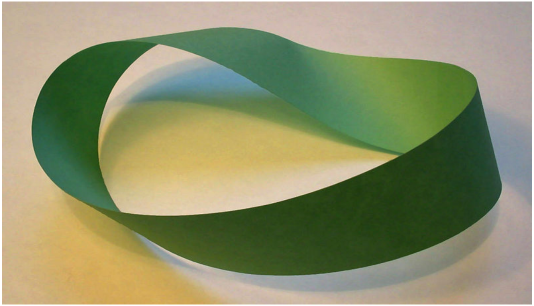

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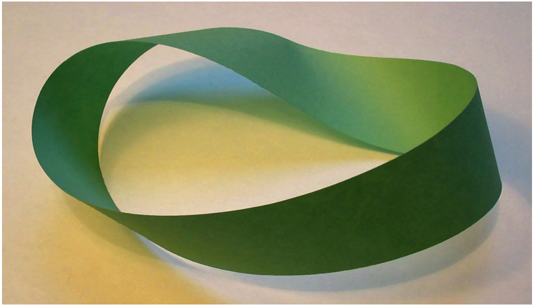

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So in principle, you can sew a disc onto the Möbius Band!
But if you do try, you get yourself quite tangled!

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The job of maths is to make difficult things easy.

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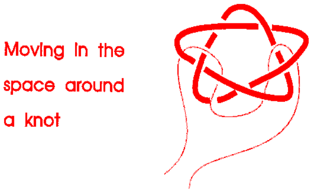

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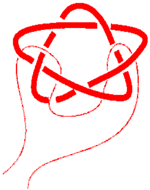
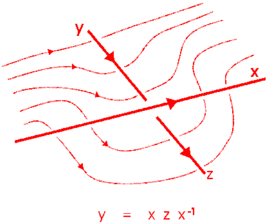
Moving in the space around a knot





Relation at a crossing

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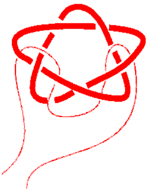

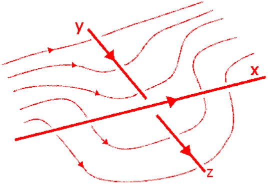

$y = x z x^{-1}$

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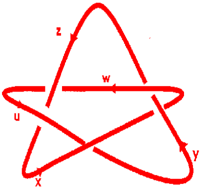

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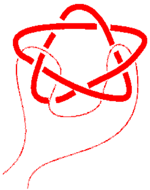

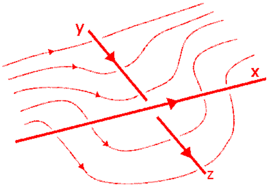

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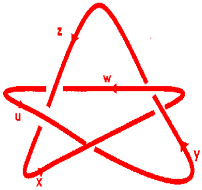

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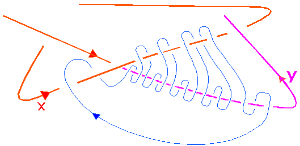

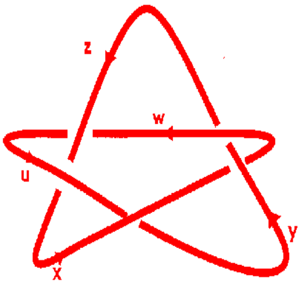

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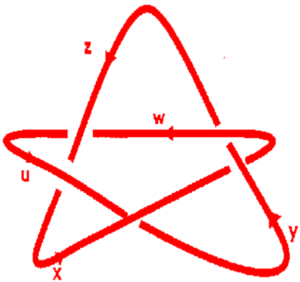

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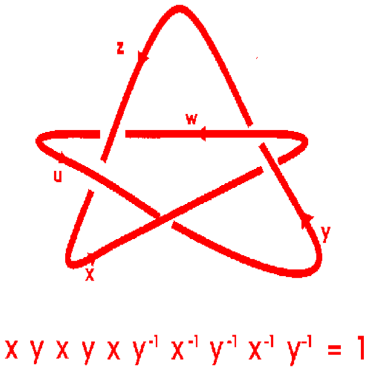

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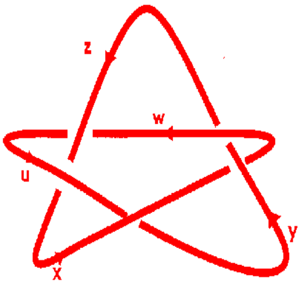

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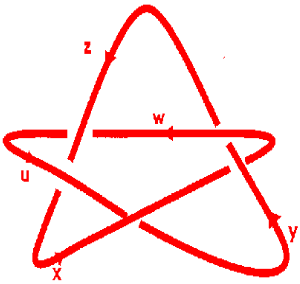

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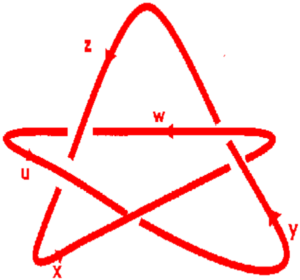

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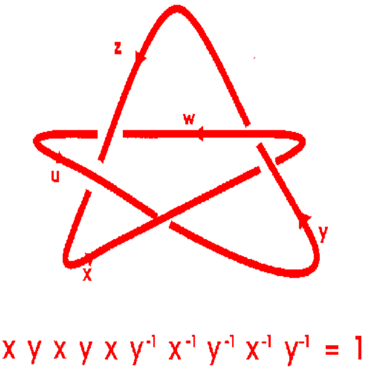

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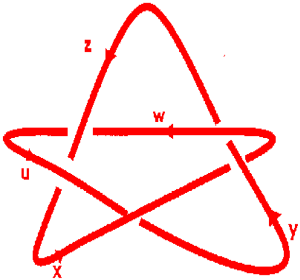

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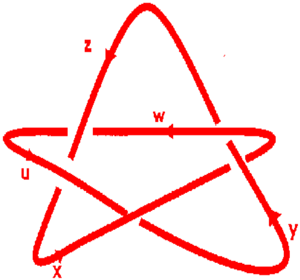

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The area of mathematics which has grown up since the 1950s to talk about varieties of structure, and to compare them, is that of category theory.

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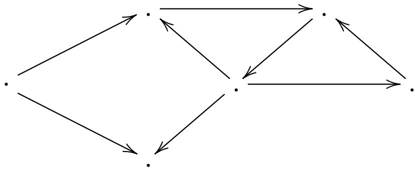
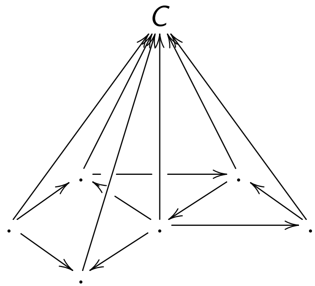
Aim: Describe constructions common to many mathematical situations.
Developed from a useful notation for a function: moving from $y = f(x)$ to $f : X \to Y$. The composition of functions then suggests the first step in the notion of a **category** C, which consists of a class $Ob(\mathrm{C})$ of 'objects' and a set of 'arrows', or 'morphisms' $f : x \to y$ for any two objects x, y, and a composition $fg : x \to z$ if also $g : y \to z$. The only rules are associativity and the existence of identities 1_x at each object x.

A colimit has 'input data', a 'cocone', and output from the 'best' cocone (when it exists).
Example: $X \cup Y$ has input data the two inclusions $X \cap Y \to X, X \cap Y \to Y$; the cocone is functions $f : X \to C, g : Y \to C$ which agree on $X \cap Y$. The output is a function $(f, g) : X \cup Y \to C$.

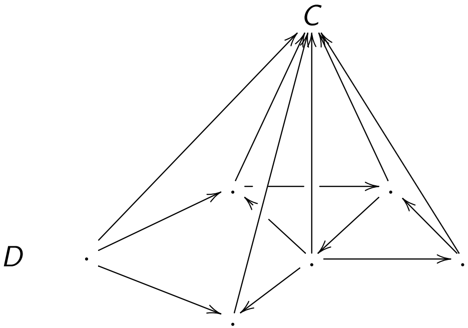
'Input data' for a colimit: a diagram D, that is a collection of some objects in a category C and some arrows between them, such as:

$D =$ 

'Functional controls': cocone *with base D and vertex an object C.*

D= 

such that each of the triangular faces of this cocone is commutative.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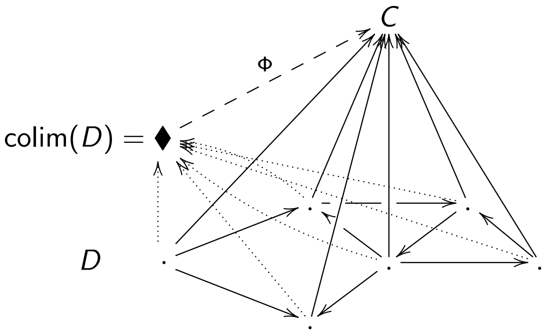
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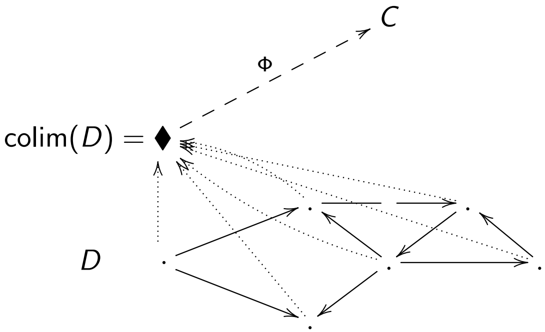
C
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D

Intuitions:

The object colim(D) is 'put together' from the constituent diagram D by means of the colimit cocone. From beyond (or above our diagrams) D, an object C 'sees' the diagram D 'mediated' through its colimit, i.e. if C tries to interact with the whole of D, it has to do so via colim(D). The colimit cocone is a kind of program: given any cocone on D with vertex C, the output will be a morphism

$$\Phi : \mathrm{colim}(D) \to C$$

constructed from the other data. How is this done?

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Compare: Ehresmann, A. and Vanbremeersch. *Memory Evolutive Systems: Hierarchy, Emergence, Cognition*, *Studies in Multidisciplinarity*, Volume 4. Elsevier, Amsterdam (2008).

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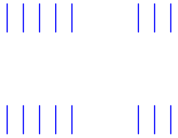

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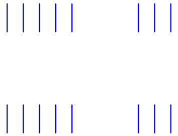

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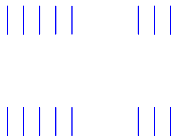

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$$a \times (b+c) = a \times b + a \times c.$$

Flatland

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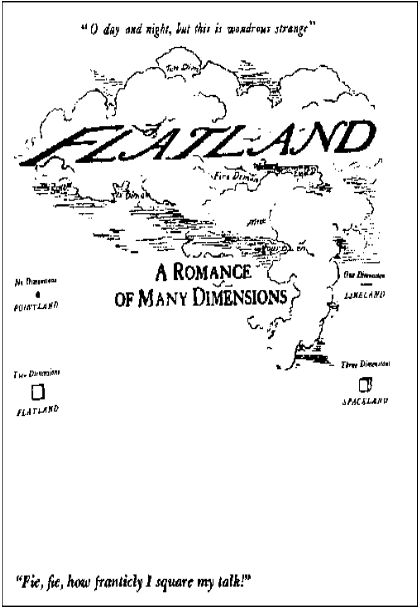
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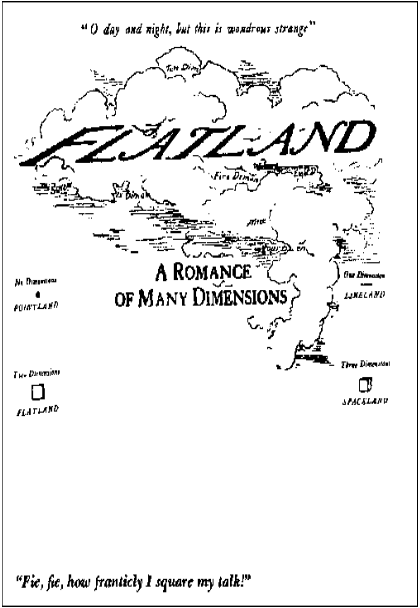
POINTLAND
FLATLAND
LINELAND
SPACELAND
"Fie, fie, how franticly I square my talk!"

Flatland



By
Rev. A. Abbott

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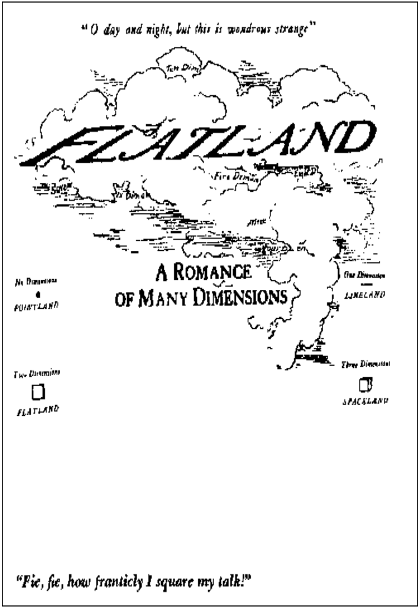

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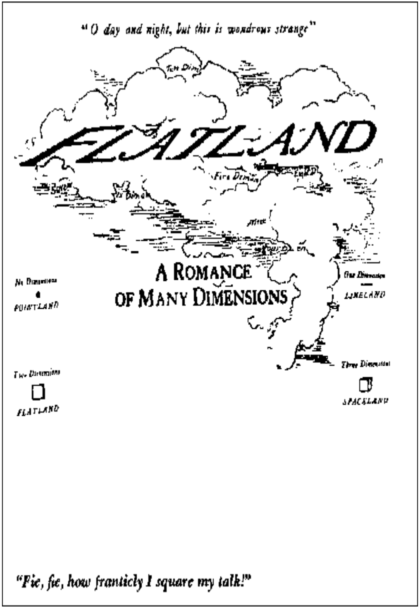

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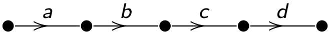

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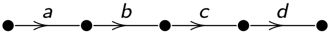

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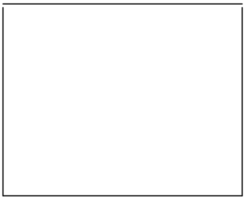

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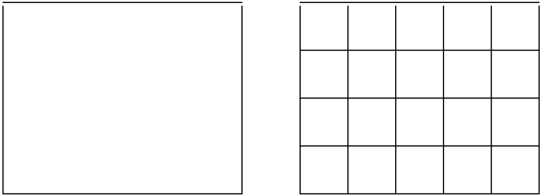

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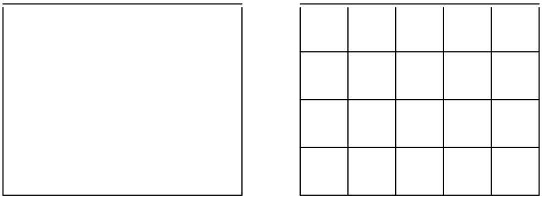

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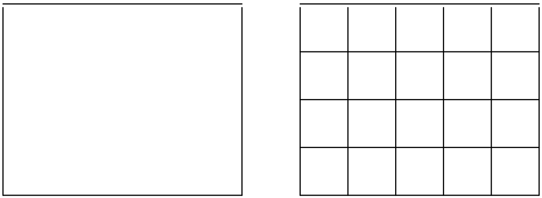

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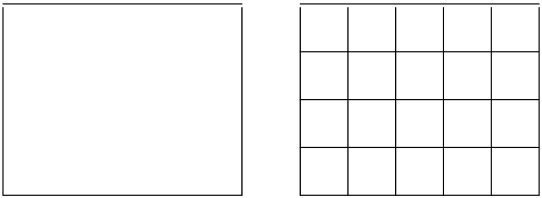

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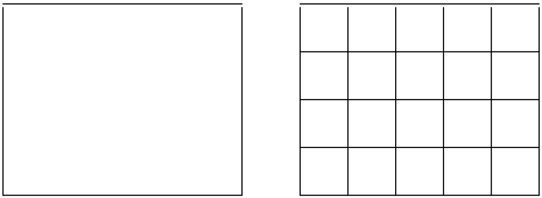

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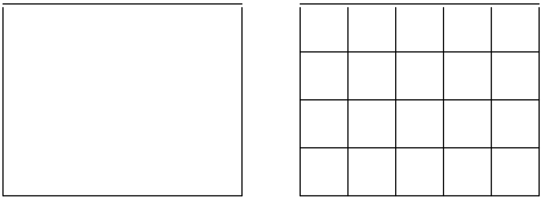

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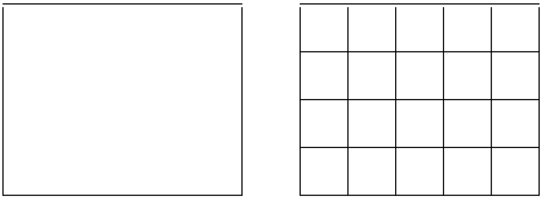

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What we need for local-to-global problems is:
Algebraic inverses to subdivision also in dimension 2.
We know how to cut things up, but how to control algebraically putting them together again?

Double Categories

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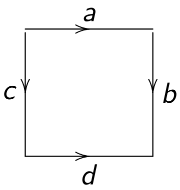

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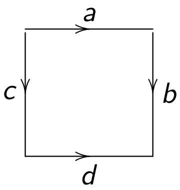

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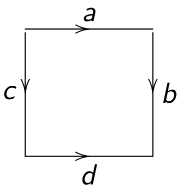

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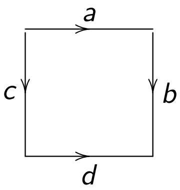

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The commutative squares in a category form a double category!

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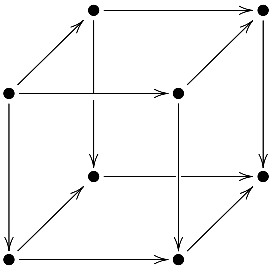

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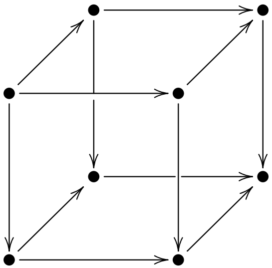

We want the faces to commute!

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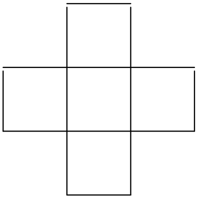

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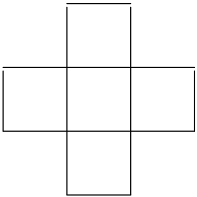

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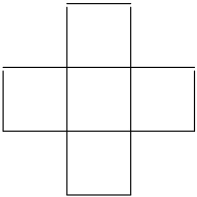

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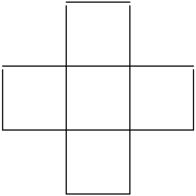

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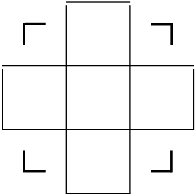

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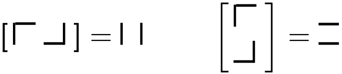

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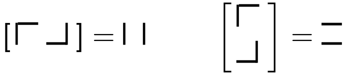

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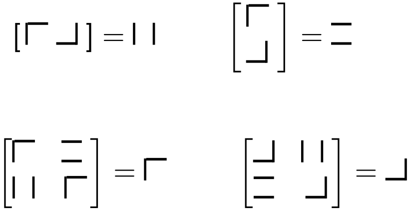

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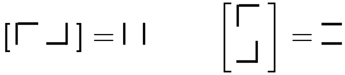

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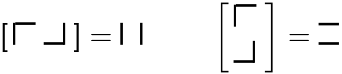

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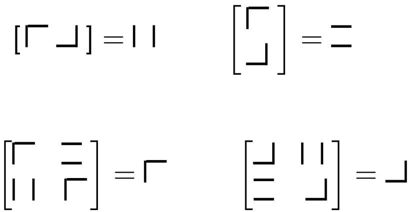

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These are equations on turning left or right, and so are a part of 2-dimensional algebra.
The term transport law and the term connections came from laws on path connections in differential geometry.
It is a good exercise to prove that any composition of commutative cubes is commutative.

Rotations in a double groupoid with connections

To show some 2-dimensional rewriting, we consider the notion of rotations σ, τ of an element u in a double groupoid with connections:

$$\sigma(u) = \begin{bmatrix} | \, | & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & | \, | \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & | \, | \\ \ulcorner & u & \lrcorner \\ | \, | & \llcorner & = \end{bmatrix}.$$

For any $u, v, w \in G_2$,

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u]$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.
Further $\sigma^2\alpha = -_1 -_2 \alpha$, and $\tau\sigma = 1$.

To prove the first of these one has to rewrite $\sigma(u +_2 v)$ until one ends up with an array, shown on the next slide, which can be reduced in a different way to $\sigma u +_2 \sigma v$. Can you identify σu, σv in this array? This gives some of the flavour of this 2-dimensional algebra of double groupoids.

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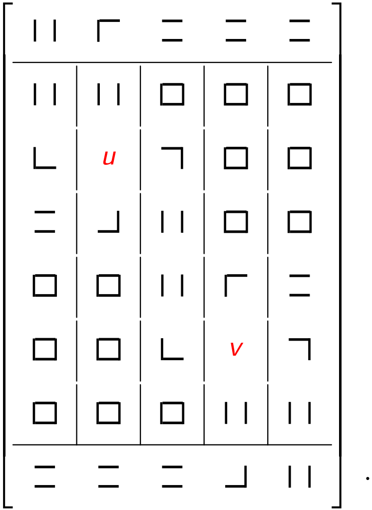
This has a homotopical interpretation.


u
v

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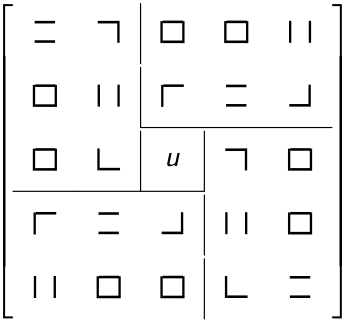

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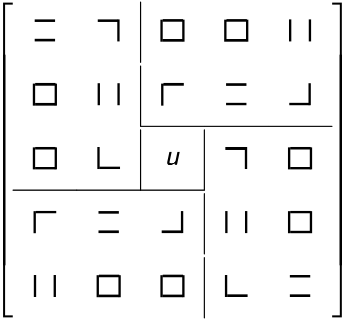

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Can you see the final steps?

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Some of these languages may be highly significant for the science and technology of the future.