# Applications of higher order Seifert–van Kampen Theorems for structured spaces

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**Abstract**

The purpose of this note is to give clear references to the many applications of higher order Seifert–van Kampen theorems, in terms of specific calculations in homotopy theory and related areas; such theorems refer to all involve spaces with structure, either a filtration, or an $n$-cube of spaces. Numbers [xx] refer to my publication list `http://pages.bangor.ac.uk/~mas010/publicfull.htm`. Numbers [[xx]] at the end of an item refer to the number of citations given on MathSciNet.

**Applications of a 2-dimensional Seifert van Kampen Theorem for pairs of spaces, with values in crossed modules**

[25]. (with P.J. HIGGINS),"On the connection between the second relative homotopy groups of some related spaces", *Proc. London Math. Soc.* (3) 36 (1978) 193–212. [[19]]

[43]. "Coproducts of crossed $P$-modules: applications to second homotopy groups and to the homology of groups", *Topology* 23 (1984) 337–345. [[10]]

V. G. Bardakov, R. Mikhailov, V. V. Vershinin, J. Wu, 'Brunnian Braids on Surfaces', arXiv:0909.3387 (This refers to [51] but really gives an application of [43].)

[92]. (with C.D.WENSLEY), "On finite induced crossed modules and the homotopy 2-type of mapping cones", *Theory and Applications of Categories* 1(3) (1995) 54–71.

[95]. (with C.D.WENSLEY), "Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types", *Theory and Applications of Categories* 2(1996) 3–16.

[124]. (with C.D.WENSLEY), "Computation and homotopical applications of induced crossed modules", *J. Symbolic Computation* 35 (2003) 59–72.

**Applications of an $n$-dimensional Seifert van Kampen Theorem for filtered spaces, with values in crossed complexes**

[32]. (with P.J. HIGGINS), "Colimit theorems for relative homotopy groups", *J. Pure Appl. Algebra* 22 (1981) 11–41. [[58]]

This paper, which generalises [25] to all dimensions, gives a new approach to the border between homotopy and homology by working with filtered spaces, and, using homotopical constructions, gives what we now call a *Higher Homotopy Seifert van Kampen Theorem.* From this, without setting up singular homology, or using simplicial approaximation, one proves:

A. The Brouwer Degree Theorem (the $n$-sphere $S^n$ is $(n-1)$-connected and the homotopy classes of maps of $S^n$ to itself are classified by an integer called the *degree* of the map);

B. The Relative Hurewicz Theorem, which is seen here as describing the morphism

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x) \xrightarrow{\cong} \pi_n(X \cup CA, x)$$

when $(X, A)$ is $(n-1)$-connected, and so does not require the usual involvement of homology groups.

The following book contains a comprehensive survey of much of the above, and more, including in Part I results in dimensions 1 and 2 which should be seen as an aspect of low dimensional topology.

[BHS] R. Brown, P.J. Higgins, R. Sivera, *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics Vol. 15, 703 pages. (August 2011).

Faria Martins, J. and Kauffman, L. 'Invariants of welded virtual knots via crossed module invariants of knotted surfaces'. *Comp. Math.* 144 (2008), no. 4, 1046-1080.

**Applications of a higher dimensional Seifert–van Kampen Theorem for $n$-cubes of spaces, with values in crossed squares, cat$^n$-groups, and crossed $n$-cubes of groups**

[42]. (with J.-L. LODAY), "Excision homotopique en basse dimension", *C.R. Acad. Sci. Paris Sér.* I 298 (1984) 353–356. [[11]]

This paper introduced a *nonabelian tensor product of groups which act on each other.* A current bibliography on this subject with 120 items is at
`http://pages.bangor.ac.uk/~mas010/nonabtens.html` .

[51]. (with J.-L. LODAY), "Van Kampen theorems for diagrams of spaces", *Topology* 26 (1987) 311-334. [[72]]

[49]. (with J.-L. LODAY), "Homotopical excision, and Hurewicz theorems, for $n$-cubes of spaces", *Proc. London Math. Soc.* (3) 54 (1987) 176-192.[[9]]

Ellis, G. J. and Steiner, R. 'Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads'. *J. Pure Appl. Algebra* **46** (2-3) (1987) 117–136. [[21]]

Brown, R. and Ellis, G. J. 'Hopf formulae for the higher homology of a group'. *Bull. London Math. Soc.* **20** (2) (1988) 124–128. [[17]]

Ellis, G. J. 'The group $K_2(\Lambda; I_1, \cdots, I_n)$ and related computations'. *J. Algebra* **112** (2) (1988) 271–289.

Ellis, G. 'Crossed squares and combinatorial homotopy'. *Math. Z.* **214** (1) (1993) 93–110. [[9]]

[60]. "Triadic Van Kampen theorems and Hurewicz theorems", *Algebraic Topology, Proc. Int. Conf. March 1988*, Edited M.Mahowald and S.Priddy, Cont. Math. 96 (1989) 39–57.

Ellis, G. J. and Mikhailov, R. ‘A colimit of classifying spaces’. *Advances in Math.* 223 (2010), no. 6, 2097-2113.

Mikhailov, Roman and Wu, Jie, ‘On homotopy groups of the suspended classifying spaces’. *Algebr. Geom. Topol.* 10 (2010), no. 1, 565-625.