Some intuitions of Higher Dimensional Algebra, and potential applications
Askloster

Ronnie Brown

July 23, 2009

Consilience, John Robinson

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www.popmath.org.uk

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www.bangor.ac.uk/r.brown/hdaweb2.htm

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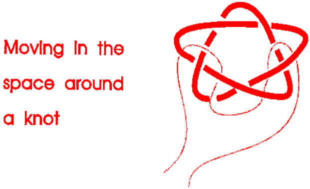

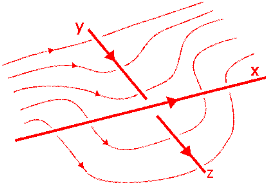

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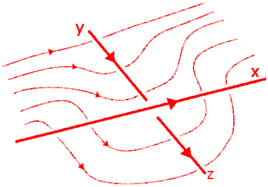
Relation at a crossing



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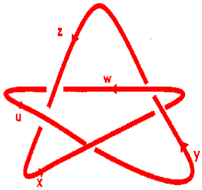

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Algebra structuring space

Groupoids in topology

Higher dimensional algebra

Higher homotopy theory

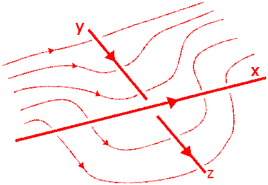
Commutative cubes

Connections

Rotations

Algebraic topology of filtered spaces

Potential applications



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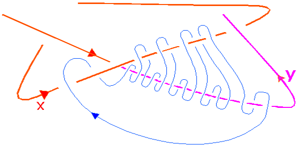

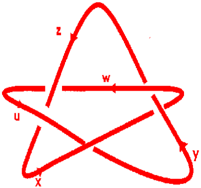

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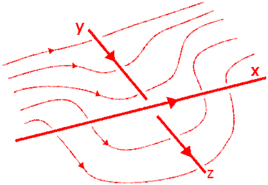

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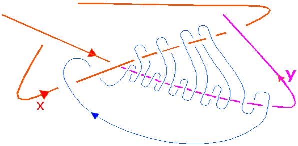

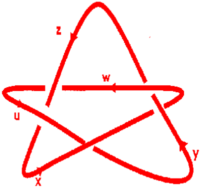

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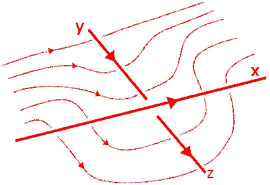

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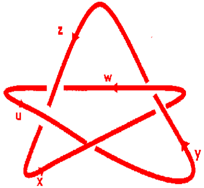

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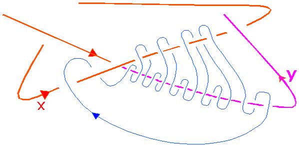

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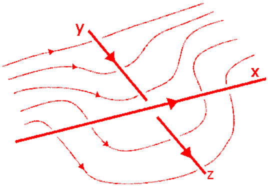

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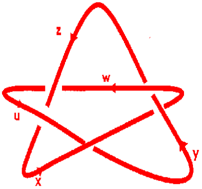

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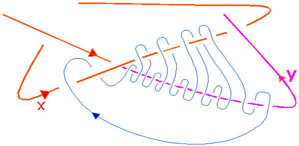

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Use rewriting of relations.
Classify the ways of pulling the loop off the knot!

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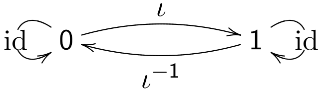

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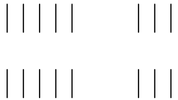

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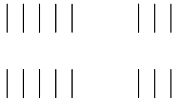

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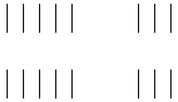

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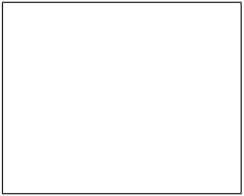

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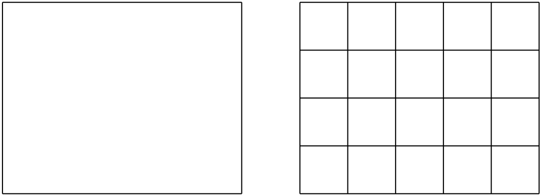

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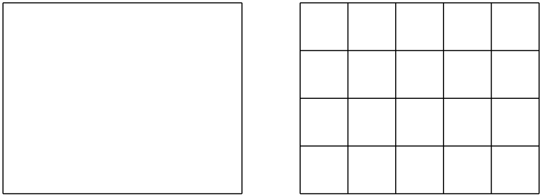

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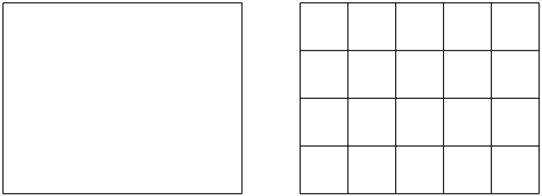

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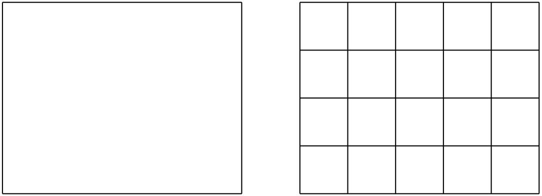

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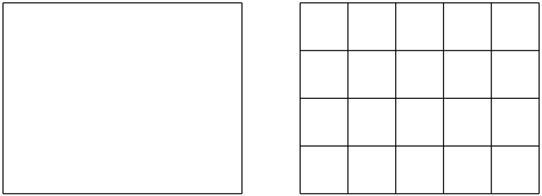

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i.e., we know how to cut things up, but how to control algebraically putting them together again?

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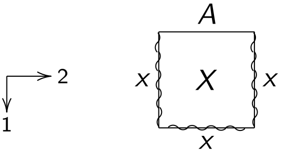

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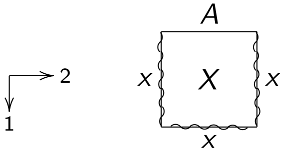

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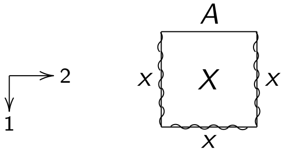

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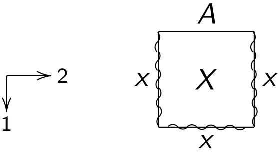

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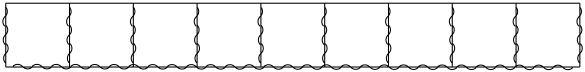

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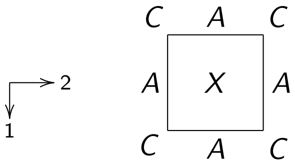

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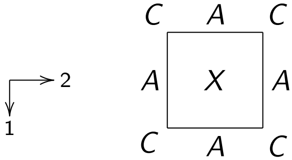

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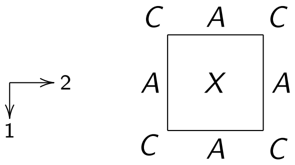

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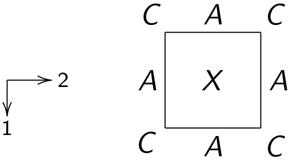

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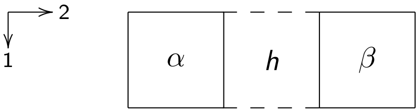

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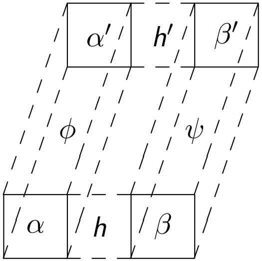

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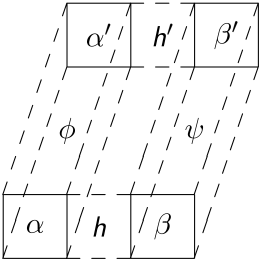

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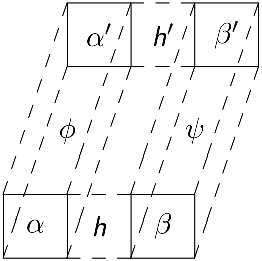

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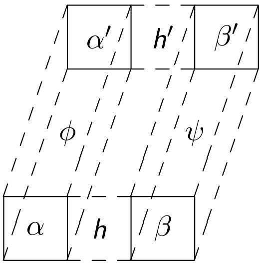

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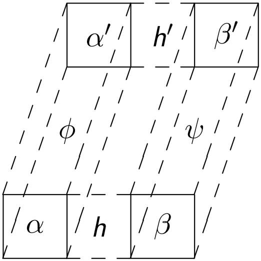

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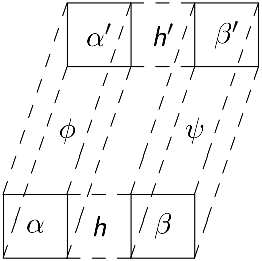

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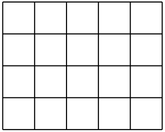

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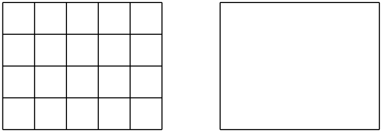

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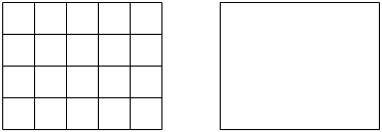

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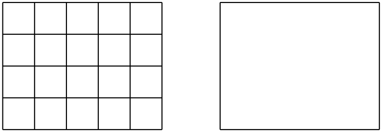

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Aesthetic implies power!!

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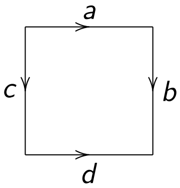

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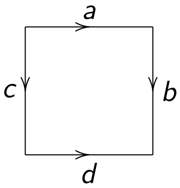

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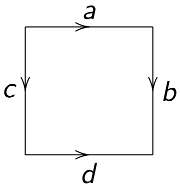

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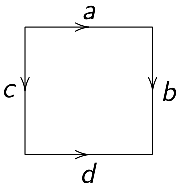

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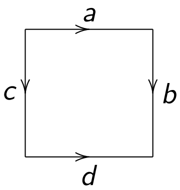

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Compare Stokes' theorem! Local Stokes implies global Stokes.

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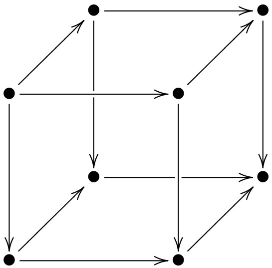

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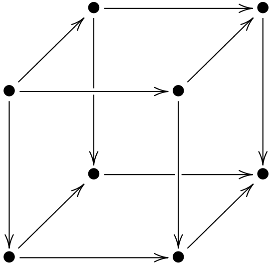

We want the faces to commute!

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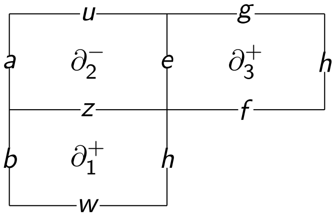

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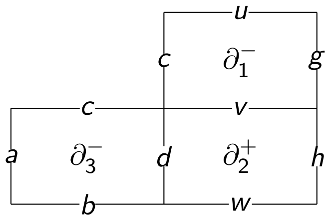

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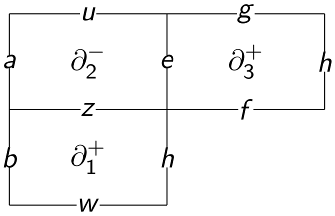

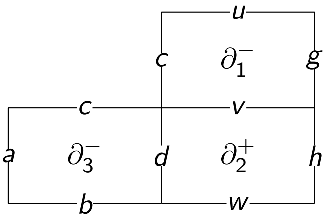

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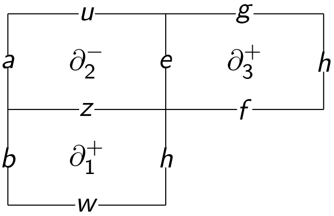

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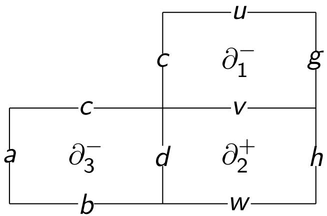

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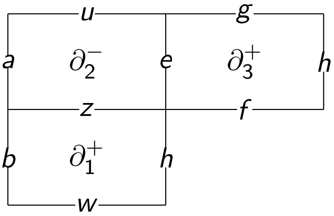

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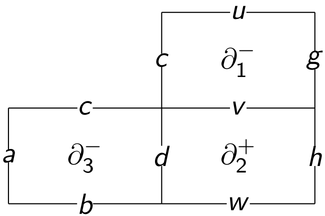

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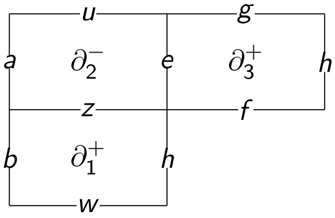

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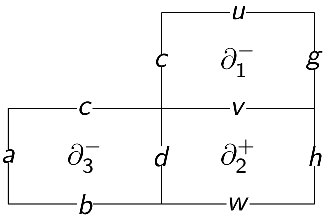

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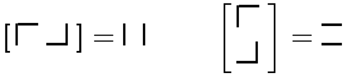

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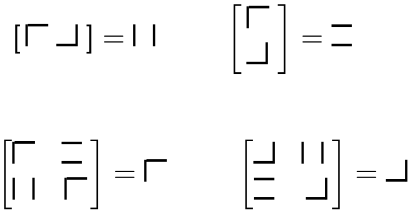

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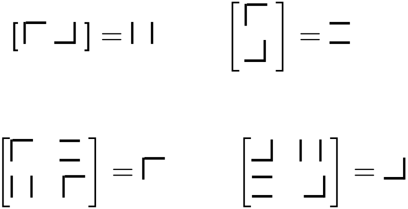

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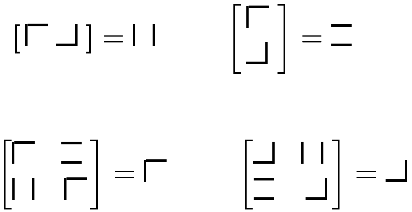

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The term transport law and the term connections came from laws on path connections in differential geometry.

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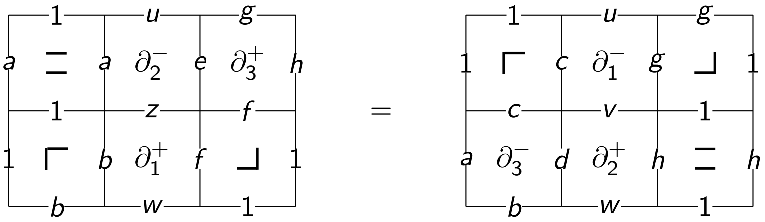

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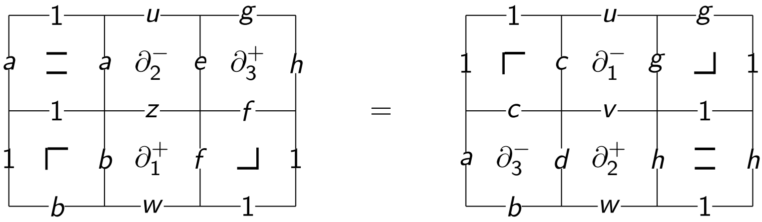

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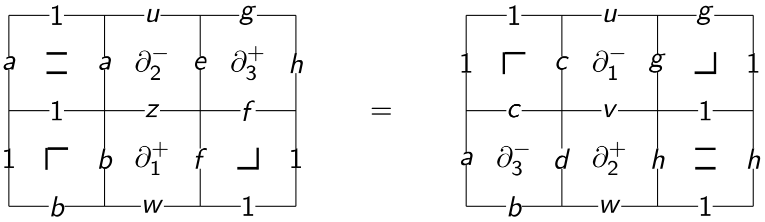

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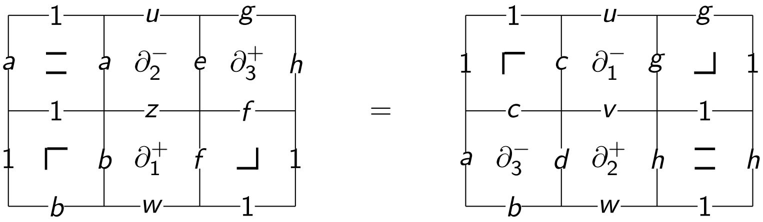

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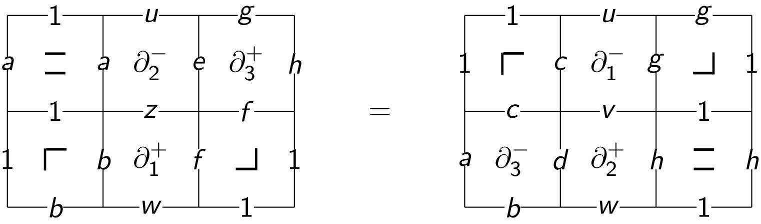

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$$\max, \min : [0,1] \times [0,1] \to [0,1]$$

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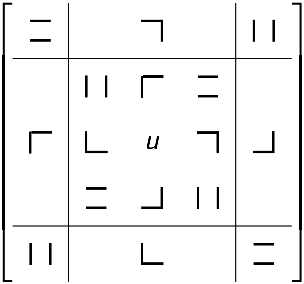
Now we prove $\tau\sigma(u) = u$ using 2-dimensional rewriting: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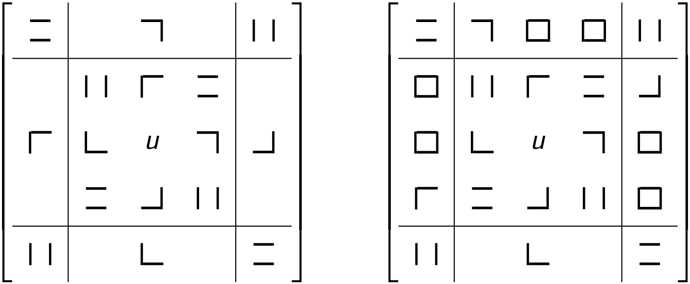
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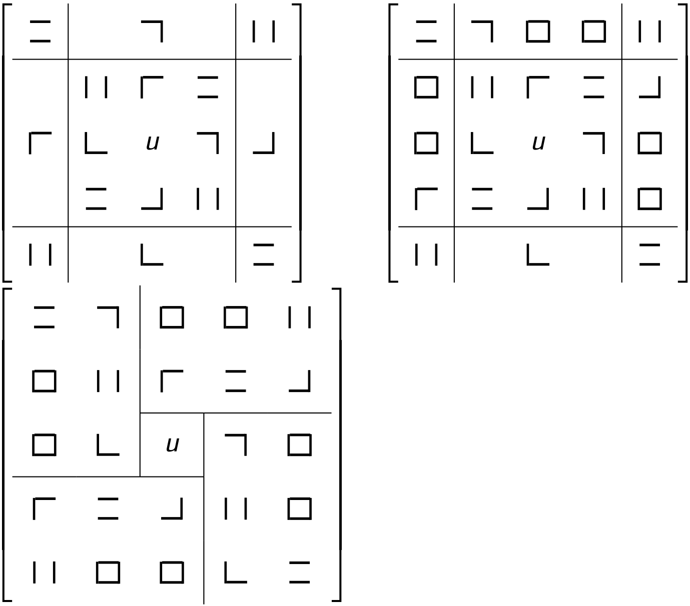
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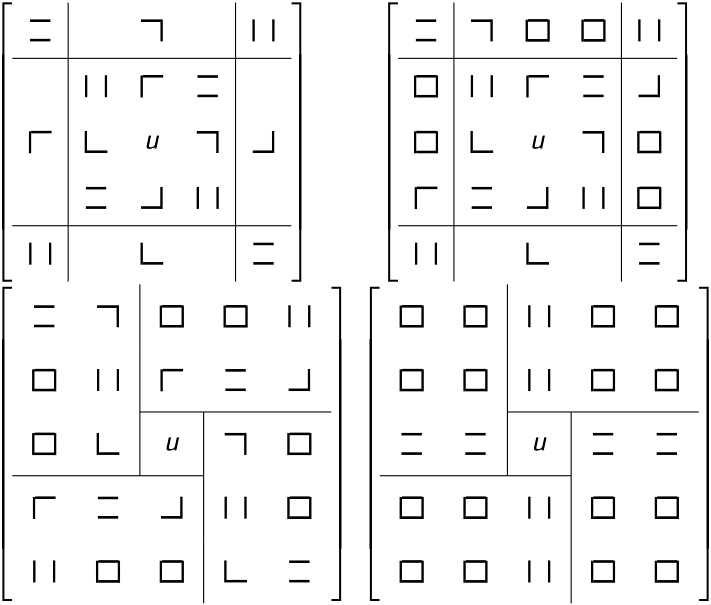

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46 years after Čech introduced higher homotopy groups.

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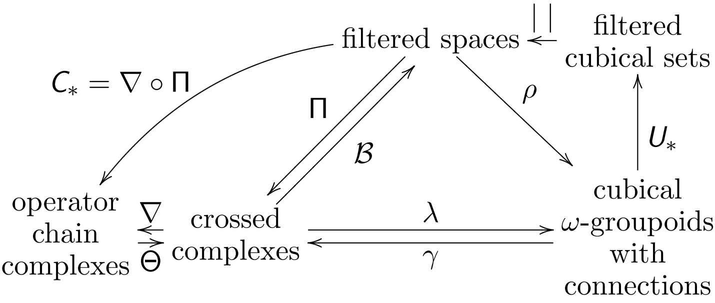

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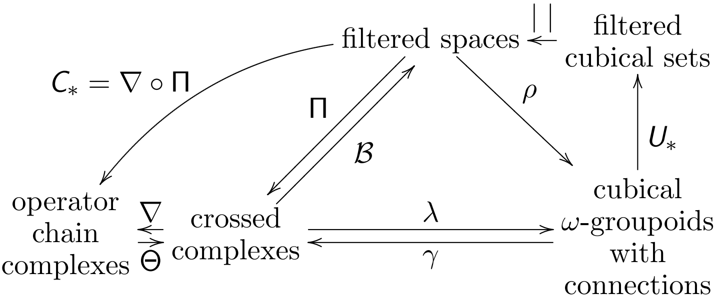

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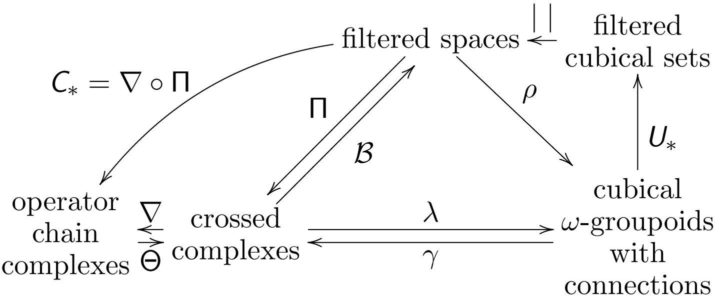

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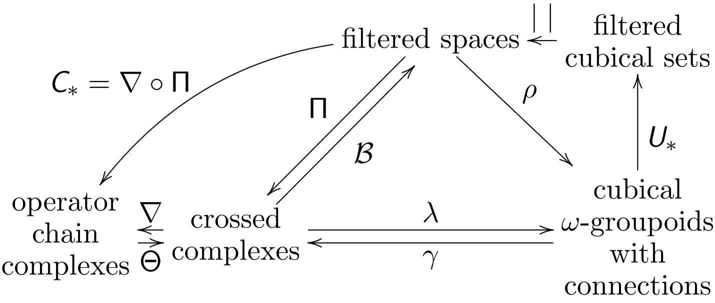

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- if $B = U \circ \mathcal{B}$: (crossed complexes) $\to$ (spaces) then there is a homotopy classification theorem

$$[X, BC] \cong [\Pi X_*, C]$$

for CW X and crossed complex C.

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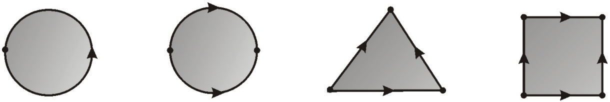

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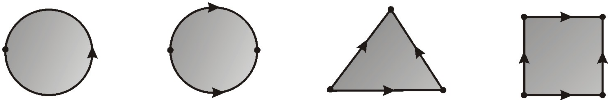

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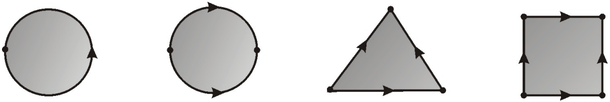

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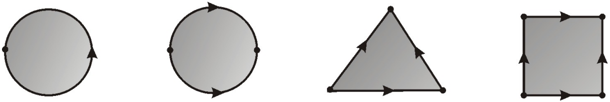

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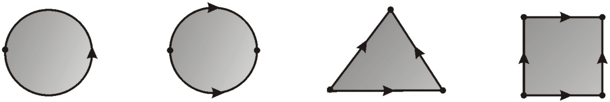

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- Free crossed resolutions for algebras?

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Link with analysis and geometry may need algebras with many objects. Higher dimensional versions of these have been investigated by Ghaffar Mosa in his 1987 Bangor thesis (now scanned to internet).

Some Context for Higher Dimensional Group Theory