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GROUPOIDS AND THE MAYER-VIETORIS SEQUENCE

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1. Introduction

It was shown in [4] that a fibration $p:E\to B$ of groupoids gives rise to a family of exact sequences

(1.1) $$1\to F_{px}\{x\}\to E\{x\}\to B\{px\}\xrightarrow{\partial_x}\pi_0F_{px}\to\pi_0E\to\pi_0B$$

of groups and pointed sets, where $x\in \mathrm{Ob}(E)$, F_{px} denotes the fibre $p^{-1}p(x)$ over $p(x)$, π_0 is the set of components and for example $E\{x\}$ is the vertex group of E at x.

The advantage of using fibrations of groupoids is to give a simple and clear derivation of the operation of B on the family of sets $\{\pi_0F_y\}_{y\in \mathrm{Ob}(B)}$ and of extra properties of exactness at π_0F_{px}. The operation includes many operations in homotopy theory, involving change of base points. Similarly, many of the exact sequences in homotopy theory are easily derived from (1.1) (see [3], [4], [6], [19]-[22], [13]). Algebraic applications to exact sequences in non-abelian cohomology are given in [4], [5] and, more recently, to exact orbit sequences in group theory in [17]. These latter sequences also find applications in the Nielsen fixed point theory for fibre spaces [14].

The object of this paper, which goes back to an earlier version by the first author (1972), is to generalize the sequence (1.1) and the operations to a situation of a Mayer-Vietoris type. The prototype of such a sequence in a non-abelian situation was given somewhat obscurely in [1], a paper which was in effect a footnote to a paper [23] of Olum, which itself showed how non-abelian cohomology could be used to prove the van Kampen theorem on the fundamental group of a union of spaces.

Here we will show how a sequence of Mayer–Vietoris type arises from a pullback square in which two (opposite) maps are fibrations of groupoids. We also establish a weak form of operation and a type of 5-lemma result.

We give some applications to homotopy theory, and in the last section relate the methods of non-abelian cohomology to the groupoid version of the van Kampen theorem given in [2]. It is hoped that the relative forms of non-abelian cohomology given here will prove a pointer for higher dimensional analogues of those methods.

2. The Mayer–Vietoris sequence

Let $p: D \to C$ be a morphism of groupoids. We say p is a *fibration*, if p is star surjective; that is for each object $d \in \mathrm{Ob}(D)$ each element in C with initial point $p(d)$ can be 'lifted' to an element in D with initial point d (see [4; 1, 2.1]).

We adopt two conventions. (i) The component of an object x of a groupoid is written $\tilde{x}$ or $x^{\sim}$ – it will be clear from the context in which groupoid this component is taken. (ii) The maps induced by a morphism f will usually be written f, as the interpretation will be clear from the context.

Throughout this section we consider a pullback square of groupoids

(2.1)
$$\begin{array}{ccc} B & \xrightarrow{\bar{f}} & D \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & C \end{array}$$

so that B is the subgroupoid of $A \times D$ whose elements are pairs (α, δ) such that $f(\alpha) = p(\delta)$, and $\bar{p}, \bar{f}$ are given by $(\alpha, \delta) \mapsto \alpha$, $(\alpha, \delta) \mapsto \delta$ respectively.

Let $b_0 = (a_0, d_0)$ be an object of B (so that $f(a_0) = p(d_0)$), and let $c_0 = f(a_0)$.

2.2. Theorem. *If p is a fibration, then there is a function $\Delta : C\{c_0\} \to \pi_0 B$ which fits into a diagram, called the Mayer–Vietoris sequence*

(2.3)
$$\begin{array}{ccccccccc} & D\{d_0\} & & & & & \pi_0 D & & \\ {\scriptstyle \bar{f}}\nearrow & & \searrow{\scriptstyle p} & & & {\scriptstyle \bar{f}}\nearrow & & \searrow{\scriptstyle p} & \\ B\{b_0\} & & & C\{c_0\} & \xrightarrow{\Delta} & \pi_0 B & & & \pi_0 C \\ {\scriptstyle \bar{p}}\searrow & & \nearrow{\scriptstyle f} & & & {\scriptstyle \bar{p}}\searrow & & \nearrow{\scriptstyle f} & \\ & A\{a_0\} & & & & & \pi_0 A & & \end{array}$$

with the following 'exactness' properties:

(i) *If* $\tilde{a}$ *in* $\pi_0 A$, $\tilde{d}$ *in* $\pi_0 D$ *satisfy* $f(\tilde{a})=p(\tilde{d})$, *then there is a* $\tilde{b}$ *in* $\pi_0 B$ *such that*

$$\bar{f}(\tilde{b})=\tilde{d}, \qquad \bar{p}(\tilde{b})=\tilde{a}.$$

(ii) $\mathrm{Im}(\Delta)$ *is the intersection of* $\bar{f}^{-1}(\tilde{d}_0)$ *and* $\bar{p}^{-1}(\tilde{a}_0)$.

(iii) $\Delta(1)=\tilde{b}_0$.

(iv) γ_1, γ_2 *in* $C\{c_0\}$ *satisfy* $\Delta(\gamma_1)=\Delta(\gamma_2)$ *if and only if there are elements* α *in* $A\{a_0\}$, δ *in* $D\{d_0\}$ *such that*

$$\gamma_1 = f(\alpha)\gamma_2 p(\delta).$$

(v) $B\{b_0\}$ *is the pullback of* $A\{a_0\} \xrightarrow{f} C\{c_0\} \xleftarrow{p} D\{d_0\}$.

Proof. We first prove (i), which does not involve Δ. Since $f(\tilde{a})=p(\tilde{d})$, there is an element γ in $C(p(d), f(a))$. Since p is a fibration, γ lifts to an element δ in $D(d, d_1)$, say, where $p(d_1)=f(a)$ so that the component of (a, d_1) in B maps to $\tilde{a}, \tilde{d}$ by $\bar{p}, \bar{f}$ respectively. This proves (i).

We now define Δ.

Let $\gamma \in C\{c_0\}$. Since p is a fibration, γ lifts to an element δ in $D(d_0, d)$, say, and we define $\Delta(\gamma)$ to be the component of (a_0, d) in B. If δ' in $D(d_0, d')$ also lifts γ, and 1 is the identity at a_0, then $f(1)=p(\delta'\delta^{-1})$ and so $(1, \delta'\delta^{-1})$ joins (a_0, d) to (a_0, d') in B. Thus $\Delta(\gamma)$ is well defined. Further $\bar{p}(a_0, d)=a_0$, $\bar{f}(a_0, d)=d$ and $\tilde{d}=\tilde{d}_0$. This proves that $\bar{p}(\mathrm{Im}(\Delta))=\{\tilde{a}_0\}$, $\bar{f}(\mathrm{Im}(\Delta))=\{\tilde{d}_0\}$.

To prove the remaining part of (ii), suppose $\tilde{b}$ in $\pi_0 B$ satisfies $\bar{p}(\tilde{b})=\tilde{a}_0$, $\bar{f}(\tilde{b})=\tilde{d}_0$. Let $b=(a,d)$; then there are elements α, δ, say, in $A(a, a_0)$, $D(d_0, d)$ respectively. The element $\gamma=f(\alpha)p(\delta)$ is defined and belongs to $C\{c_0\}$. We prove that $\Delta(\gamma)=\tilde{b}$.

Let δ_1 in $D(d_0, d_1)$, say, be a lift of γ. Then

$$p(\delta_1)=\gamma=f(\alpha)p(\delta)$$

and it follows that $(\alpha, \delta_1\delta^{-1})$ is an element of $B((a,d),(a_0,d_1))$. Hence $\Delta(\gamma)= (a_0, d_1)^{\sim}=(a,d)^{\sim}=\tilde{b}$, which proves (ii).

The proof of (iii) is simple; the identity at d_0 lifts the identity of $C\{c_0\}$, so that $\Delta(1)=\tilde{b}_0$.

To prove (iv), suppose for $\varepsilon=1,2$ that γ_ε belongs to $C\{c_0\}$ and that δ_ε in $D(d_0, d_\varepsilon)$ lifts γ_ε, so that $\Delta(\gamma_\varepsilon)$ is the component of (a_0, d_ε) in B.

Suppose there are elements α in $A\{a_0\}$, δ in $D\{d_0\}$ such that

$$\gamma_1 = f(\alpha)\gamma_2 p(\delta).$$

Then

$$p(\delta_1)=\gamma_1=f(\alpha)p(\delta_2)p(\delta).$$

Thus if $\kappa=\delta_1\delta^{-1}\delta_2^{-1}$ in $D(d_2, d_1)$, then $p(\kappa)=f(\alpha)$ and (α, κ) joins (a_0, d_2) to (a_0, d_1) in B. Hence $\Delta(\gamma_1)=\Delta(\gamma_2)$.

Suppose conversely that $\Delta(\gamma_1)=\Delta(\gamma_2)$. Then we can find an element (α, κ) in B

joining (a_0, d_2) to (a_0, d_1). Then $\alpha \in A\{a_0\}$, $\kappa \in D(d_2, d_1)$ and $f(\alpha) = p(\kappa)$. Let $\delta^{-1} = \delta_1^{-1}\kappa\delta_2$ in $D\{d_0\}$. Then $\delta_1 = \kappa\delta_2\delta$ and

$$\gamma_1 = p(\delta_1) = f(\alpha)\gamma_2 p(\delta).$$

Finally, the proof of (v) is easy, using the definition of B. □

We remark that if A is a trivial groupoid then 2.2 specializes to the exact sequence of a fibration of groupoids [4; 4.3(b)] except for the operation of $C\{c_0\}$ on the set of components of the fibre. We consider operations in Section 3.

The special case of 2.2 where A is *connected*, that is $\pi_0 A = 1$, has already been used in [16], dealing with problems of duality in homotopy theory.

Theorem 2.2 is used in [25] to deduce a Mayer–Vietoris sequence for homotopy pullbacks of groupoids. Analogous results have also been stated in [18].

Note that part of Theorem 2.2, namely (ii)–(v) could have been obtained from [4; (4.2)] by an alternative approach using the methods of [9] for groupoids instead of pointed topological spaces.

From Theorem 2.2 we can deduce a sequence of a more familiar form.

Let $\pi_0 A \sqcap \pi_0 D$ be the pullback of

$$\begin{array}{ccc} & & \pi_0 D \\ & & \big\downarrow{\scriptstyle p} \\ \pi_0 A & \xrightarrow[f]{} & \pi_0 C \end{array}$$

and let $\phi = (\bar{p}, \bar{f}) : \pi_0 B \to \pi_0 A \sqcap \pi_0 D$.

2.4. Corollary. *The following sequence is exact*

$$1 \to B\{b_0\} \xrightarrow{(\bar{p}, \bar{f})} A\{a_0\} \times D\{d_0\} \xrightarrow{f^{-1}p} C\{c_0\} \xrightarrow{\Delta} \pi_0 B \xrightarrow{\phi} \pi_0 A \sqcap \pi_0 D \to 1,$$

where $(f^{-1}p)(\alpha, \delta) = f(\alpha^{-1})p(\delta)$. □

The sequence in 2.4 contains less information than (2.3) as can be seen by comparing the exactness at $C\{c_0\}$ of 2.4 with 2.2(iv).

One part of Corollary 2.4 is that ϕ is surjective. We are interested in conditions for ϕ to be bijective, i.e. for π_0 to preserve the pullback (2.1).

Recall that a groupoid C is *simply connected* if $C\{c_0\}$ consists of a single element for all c_0 in $\mathrm{Ob}(C)$; C is *1-connected*, if C is connected and simply connected.

2.5. Corollary. *The function* ϕ *of* 2.4 *is a bijection if and only if for all choices of* $(a_0, d_0) \in \mathrm{Ob}(B)$, *the group* $C\{c_0\}$ (*where* $c_0 = f(a_0) = p(d_0)$) *is the product set*

$$(fA\{a_0\})(pD\{d_0\}).$$

In particular, ϕ is bijective if C is simply connected, or if $f:A\{a_0\}\to C\{c_0\}$ or $p:D\{d_0\}\to C\{c_0\}$ is surjective.

Proof. Suppose that ϕ is bijective. Then for any $(a_0,d_0)\in \mathrm{Ob}(B)$ there is a sequence (2.3) with base points a_0, d_0 etc. For this sequence $\mathrm{Ker}\,\phi=1$ by hypothesis, and so $f^{-1}p$ is surjective in 2.4.

Conversely, let $b_0=(a_0,d_0)$, $b=(a,d)\in\mathrm{Ob}(B)$ and suppose $\phi(\tilde b_0)=\phi(\tilde b)$. Form (2.3) with the base points a_0, d_0 etc. Then $\tilde b\in \mathrm{Ker}\,\phi=\mathrm{Im}\,\Delta$ by 2.4. So there is a $\gamma\in C\{c_0\}$ such that $\Delta(\gamma)=\tilde b$. Then $\gamma=f(\alpha)p(\delta)$ for some $\alpha\in A\{a_0\}$, $\delta\in D\{d_0\}$ by hypothesis, so $\tilde b=\Delta(\gamma)=\Delta(1)=\tilde b_0$ by 2.2(iv). □

It will be seen in Section 5 that Corollary 2.5 is the essence of the method of Olum in [23]. The extension of Olum's method discussed in [1] comes from the following simple type of corollary.

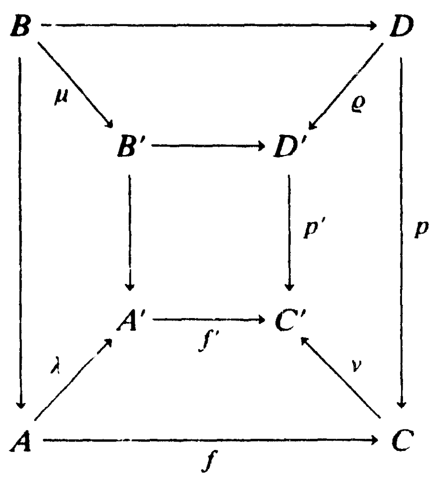
2.6. Corollary. *If both A and D are 1-connected, then $\Delta: C\{c_0\}\to\pi_0 B$ is bijective.* □

We have a complete description of $\pi_0 B$ which generalizes 4.4 of [4]. The simple proof is left to the reader.

2.7. Corollary. *Let $X\subset \mathrm{Ob}(B)$ be a complete set of representatives of the fibres of $\phi:\pi_0 B\to\pi_0 A\sqcap\pi_0 D$. There is a bijection of $\pi_0 B$ with the disjoint union over X of double cosets of $fA\{a_0\}, pD\{d_0\}$ in $C\{fa_0\}$ for all $(a_0,d_0)\in X$.* □

The next proposition deals with the naturality of the function Δ. Suppose given a commutative diagram of morphisms of groupoids

(2.8)



in which the inner and outer squares are pullbacks and p and p' are fibrations. Let $b_0=(a_0,d_0)$ be an object of B, let $c_0=f(a_0)$, and let

$$a_0'=\lambda(a_0),\qquad d_0'=\varrho(d_0),\qquad b_0'=\mu(b_0),\qquad c_0'=\nu(c_0).$$

2.9. Proposition. *The induced diagram*

$$\begin{array}{ccc} C\{c_0\} & \xrightarrow{\Delta} & \pi_0 B \\ \Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \mu} \\ C'\{c_0'\} & \xrightarrow[\Delta']{} & \pi_0 B' \end{array}$$

is commutative. □

The amount of structure discussed so far enables us to prove a 4-lemma type result, in fact that part of the 5-lemma which deals with injectivity.

We suppose given the situation of (2.8) and the induced diagram

$$\begin{array}{ccc} \pi_0 B & \xrightarrow{\phi} & \pi_0 A \sqcap \pi_0 D \\ {\scriptstyle \mu}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda \sqcap \varrho} \\ \pi_0 B' & \xrightarrow[\phi']{} & \pi_0 A' \sqcap \pi_0 D' \end{array} \tag{2.10}$$

2.11. Theorem. *If* $\lambda : A\{a_0\} \to A'\{a_0'\}$ *and* $\varrho : D\{d_0\} \to D'\{d_0'\}$ *are surjective and* $\nu : C\{c_0\} \to C'\{c_0'\}$ *is injective, then* μ *is injective on* Ker $\phi = \phi^{-1}(\tilde{a}_0, \tilde{d}_0)$.

Proof. Let $\tilde{b}_1, \tilde{b}_2 \in \pi_0 B$ satisfy $\mu(\tilde{b}_1) = \mu(\tilde{b}_2)$, and $\phi(\tilde{b}_1) = \phi(\tilde{b}_2) = (\tilde{a}_0, \tilde{d}_0) = \phi(\tilde{b}_0)$. Then by 2.2(ii) we have $\tilde{b}_1 = \Delta(\gamma_1)$, $\tilde{b}_2 = \Delta(\gamma_2)$ for some $\gamma_1, \gamma_2 \in C\{c_0\}$, and so

$$\Delta'\nu(\gamma_1) = \mu\Delta(\gamma_1) = \mu(\tilde{b}_1) = \mu(\tilde{b}_2) = \mu\Delta(\gamma_2) = \Delta'\nu(\gamma_2).$$

Hence by 2.2(iv) there are $\alpha' \in A'\{a_0'\}$, $\delta' \in D'\{d_0'\}$ such that $\nu(\gamma_1) = f'(\alpha')\nu(\gamma_2)p'(\delta')$. Since $\lambda : A\{a_0\} \to A'\{a_0'\}$, $\varrho : D\{d_0\} \to D'\{d_0'\}$ are assumed to be surjective, we have $\alpha' = \lambda(\alpha)$, $\delta' = \varrho(\delta)$ for some $\alpha \in A\{a_0\}$, $\delta \in D\{d_0\}$. So

$$\nu(\gamma_1) = f'\lambda(\alpha)\nu(\gamma_2)p'\varrho(\delta) = \nu f(\alpha)\nu(\gamma_2)\nu p(\delta) = \nu(f(\alpha)\gamma_2 p(\delta)).$$

Since $\nu : C\{c_0\} \to C'\{c_0'\}$ is injective, it follows that

$$\gamma_1 = f(\alpha)\gamma_2 p(\delta)$$

and hence $\tilde{b}_1 = \Delta(\gamma_1) = \Delta(\gamma_2) = \tilde{b}_2$ by 2.2(iv). □

For the following corollary we suppose given the situation of diagram (2.8).

2.12. Corollary. *Suppose* $\lambda \sqcap \varrho : \pi_0 A \sqcap \pi_0 D \to \pi_0 A' \sqcap \pi_0 D'$ *is injective and for all objects* (a_0, d_0) *of* B, $\lambda : A\{a_0\} \to A'\{\lambda a_0\}$, $\varrho : D\{d_0\} \to D'\{\varrho d_0\}$ *are surjective and* $\nu : C\{c_0\} \to C'\{\nu c_0\}$ *is injective, where* $c_0 = f(a_0)$. *Then* $\mu : \pi_0 B \to \pi_0 B'$ *is injective.*

Proof. Let $\tilde{b}_0, \tilde{b} \in \pi_0 B$, and suppose $\mu(\tilde{b}_0) = \mu(\tilde{b})$. Then $\phi(\tilde{b}_0) = \phi(\tilde{b})$ by commutativity of (2.10) and injectivity of $\lambda \sqcap \varrho$. Suppose that $b_0 = (a_0, d_0)$. Then Theorem 2.11 applies to show that $\tilde{b} = \tilde{b}_0$. □

2.13. Remark. It seems that an extra structure of some kind of operation is needed to give 5-lemma type conditions for $\mu : \pi_0 B \to \pi_0 B'$ to be surjective. This question is discussed in the next section.

3. An operation

The boundary mapping $\partial_x : B\{px\} \to \pi_0 F_{px}$ of the sequence (1.1) can be described as the restriction of an operation of $B\{px\}$ on $\pi_0 F_{px}$ to the base point of $\pi_0 F_{px}$ and so does not depend on the choice of x in its component of F_{px}. However, the following example shows that the boundary $\Delta : C\{c_0\} \to \pi_0 B$ of Theorem 2.2 does in general depend on the choice of $b_0 = (a_0, d_0)$ in its component in B.

3.1. Example. *In Theorem 2.2, let C be a group and let f be the inclusion of a subgroup A of C. Suppose that D is simply connected, and that Δ is independent of the choice of $b_0 = (a_0, d_0)$ in its component. Then A is normal in C.*

Proof. Let $\alpha \in A$ and let $\gamma \in C$. Let α, γ lift to elements $\beta \in D(d_0, d_0')$, $\delta \in D(d_0, d)$. Let also γ lift to $\delta' \in D(d_0', d')$. By the assumption on Δ, there is an element $(\xi, \eta) \in B((a_0, d), (a_0, d'))$. Since D is simply connected $\eta\delta = \delta'\beta$. Hence $p(\eta)\gamma = \gamma\alpha$. But $p(\eta) = \xi \in A$, and we have shown $\gamma\alpha\gamma^{-1} \in A$. □

Thus Δ cannot in general be described as the restriction of an operation of $C\{c_0\}$ on $\pi_0 B$. However, we can give an operation of a subgroup of $C\{c_0\}$ on the subset $\Psi(a_0)$ of $\pi_0 B$ which lies over the component of a_0.

In detail, let (2.1) be a pullback of groupoids where p is a fibration; let $b_0 = (a_0, d_0)$ be an object of B; define $c_0 = f(a_0)$; and let

$$\Psi(a_0) = \bar{p}^{-1}(\tilde{a}_0),$$

where $\bar{p} : \pi_0 B \to \pi_0 A$.

3.2. Remarks. (i) Let $j : F \to B$ denote the inclusion of the fibre F of $\bar{p} : B \to A$ over a_0. Since p is a fibration and (2.1) is a pullback, $\bar{p}$ is a fibration. Thus by exactness of the sequence $\pi_0 F \xrightarrow{j} \pi_0 B \xrightarrow{\bar{p}} \pi_0 A$ we have

$$\Psi(a_0) = \bar{p}^{-1}(\tilde{a}_0) = \operatorname{Im} j.$$

Thus $\Psi(a_0)$ is the subset of $\pi_0 B$ of those elements $\tilde{b}$ which are of the form $\tilde{b} = (a_0, d)^{\sim}$, where $(a_0, d) \in \operatorname{Ob}(B)$.

(ii) If A is connected, then $\Psi(a_0) = \pi_0 B$.

Let $N(a_0)$ denote the *normalizer* of $fA\{a_0\}$ in $C\{c_0\}$, i.e.

$$N(a_0)=\{\gamma\in C\{c_0\}\mid \gamma\cdot fA\{a_0\}\cdot\gamma^{-1}=fA\{a_0\}\}.$$

We say f is *normal at* a_0 if $fA\{a_0\}$ is normal in $C\{c_0\}$ (i.e. if $N(a_0)=C\{c_0\}$) and we say f is *normal* if it is normal at all $a_0\in \mathrm{Ob}(A)$.

3.3. Theorem. *There is an operation* $\cdot$ *of* $N(a_0)$ *on* $\Psi(a_0)$ *such that* $\Delta(\gamma)=\gamma\cdot\tilde b_0$ *for each* $\gamma\in N(a_0)$.

Let ϕ *be the canonical map* $\pi_0B\to\pi_0A\sqcap\pi_0D$. *Then for* $\tilde b,\tilde b'\in\Psi(a_0)$ *we have* $\phi(\tilde b)=\phi(\tilde b')$ *if there is a* $\gamma\in N(a_0)$ *such that* $\gamma\cdot\tilde b=\tilde b'$. *The converse is true if* f *is normal at* a_0.

Proof. Let $\gamma\in N(a_0)$, $\tilde b=(a_0,d)^\sim\in\Psi(a_0)$. Then we define $\gamma\cdot\tilde b$ to be $(a_0,d_1)^\sim$ where d_1 is the end point of a lift $\theta\in D(d,d_1)$ of γ. For other choices $(a_0,d')\in\tilde b$ and $\theta'\in D(d',d_1')$ lifting γ, there is a morphism $(\alpha,\delta):(a_0,d)\to(a_0,d')$ in B. Let $\delta'=\theta'\delta\theta^{-1}:d_1\to d_1'$. Then $p(\delta')=\gamma\cdot f(\alpha)\cdot\gamma^{-1}$ which belongs to $fA\{a_0\}$ because $\gamma\in N(a_0)$. Thus $p(\delta')=f(\alpha')$ for some $\alpha'\in A\{a_0\}$ and (α',δ') is a morphism $(a_0,d_1)\to(a_0,d_1')$ in B. Hence $\gamma\cdot\tilde b$ is well defined.

Note also that $\gamma\cdot\tilde b_0=\Delta(\gamma)$ for $\gamma\in N(a_0)$.

The axioms for an operation are easily verified, as is the fact that $\phi(\gamma\cdot\tilde b)=\phi(\tilde b)$ when $\gamma\cdot\tilde b$ is defined. Suppose conversely that $\tilde b,\tilde b'\in\Psi(a_0)$ satisfy $\phi(\tilde b)=\phi(\tilde b')$, and that $\tilde b=(a_0,d)^\sim$, $\tilde b'=(a_0,d')^\sim$. Then $(\tilde a_0,\tilde d)=(\tilde a_0,\tilde d')$ and so $\tilde d=\tilde d'$. Let $\theta:d\to d'$ in D. Define $\gamma=p(\theta)$ in $C\{c_0\}$. Then $\gamma\in N(a_0)$ (by assumption) and $\gamma\cdot\tilde b=\tilde b'$ by definition of the operation. □

The operation of 3.3 is natural. In the situation of diagram (2.8) with given base points $b_0=(a_0,d_0)$ etc. we have in analogy with 2.9:

3.4. Proposition. *If* $\tilde b\in\Psi(a_0)$, *then* $\mu(\tilde b)\in\Psi(a_0')$.

If $\gamma\in N(a_0)$ *and* $\nu(\gamma)\in N(a_0')$, *then* $\mu(\gamma\cdot\tilde b)=\nu(\gamma)\cdot\mu(\tilde b)$. □

This naturality of the operations enables us to prove a 4-lemma type result, namely that part of the 5-lemma dealing with surjectivity. We suppose given the situation of the previous proposition, and consider the diagram

$$(3.5)\qquad \begin{array}{ccccccc}
C\{c_0\} & \xrightarrow{\Delta} & \pi_0B & \xrightarrow{\phi} & \pi_0A\sqcap\pi_0D & \xrightarrow{\tau} & \pi_0C\\
\downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\lambda\sqcap\varrho} & & \downarrow{\scriptstyle\nu}\\
C'\{c_0'\} & \xrightarrow{\Delta'} & \pi_0B' & \xrightarrow{\phi'} & \pi_0A'\sqcap\pi_0D' & \xrightarrow{\tau'} & \pi_0C'
\end{array}$$

where τ is defined by the maps $\pi_0A\to\pi_0C$, $\pi_0D\to\pi_0C$.

3.6. Theorem. *Let f be normal at a_0, f' be normal at a_0'. Suppose $\nu : \pi_0 C \to \pi_0 C'$ is injective, $\nu : C\{c_0\} \to C'\{c_0'\}$ and $\varrho : \pi_0 D \to \pi_0 D'$ are surjective. Then*

$$\mu : \Psi(a_0) \to \Psi(a_0')$$

is surjective.

Proof. Let $\tilde{b}' = (a_0', d')^{\sim} \in \Psi(a_0')$, and consider $\phi'(\tilde{b}') = (\tilde{a}_0', \tilde{d}')$. Since $\varrho : \pi_0 D \to \pi_0 D'$ is surjective, we can find $\tilde{d} \in \pi_0 D$ with $\varrho(\tilde{d}) = \tilde{d}'$. We have

$$\nu f(\tilde{a}_0) = \tilde{c}_0' = \nu p(\tilde{d}).$$

Since ν is injective it follows that $f(\tilde{a}_0) = p(\tilde{d})$, i.e. $(\tilde{a}_0, \tilde{d}) \in \pi_0 A \sqcap \pi_0 D$.

Since ϕ is surjective, there is a $\tilde{b}$ in $\pi_0 B$ such that $\phi(\tilde{b}) = (\tilde{a}_0, \tilde{d})$. We see $\tilde{b} \in \bar{p}^{-1}(\tilde{a}_0) = \Psi(a_0)$ whence $\mu(\tilde{b}) \in \Psi(a_0')$.

Now $\phi' \mu(\tilde{b}) = \phi'(\tilde{b}')$. Hence by 3.3 there is a γ' in $C'\{c_0'\}$ such that $\gamma' \cdot \mu(\tilde{b}) = \tilde{b}'$. Since $\nu : C\{c_0\} \to C'\{c_0'\}$ is surjective, there is a γ in $C\{c_0\}$ such that $\nu(\gamma) = \gamma'$. Then by 3.4

$$\mu(\gamma \cdot \tilde{b}) = \gamma' \cdot \mu(\tilde{b}) = \tilde{b}'.$$

This proves $\mu : \Psi(a_0) \to \Psi(a_0')$ is surjective. □

For the following corollary let (2.8) be as before a commutative diagram in which the inner and outer squares are pullbacks and p and p' are fibrations.

3.7. Corollary. *Suppose $\nu : \pi_0 C \to \pi_0 C'$ is injective, $\lambda : \pi_0 A \to \pi_0 A'$ and $\varrho : \pi_0 D \to \pi_0 D'$ are surjective. Assume, f, f' are normal and $\nu : C\{f(a_0)\} \to C'\{\nu f(a_0)\}$ is surjective. Then $\mu : \pi_0 B \to \pi_0 B'$ is surjective.*

Proof. The method is to start with an element of $\pi_0 B'$, and then choose appropriate $a_0' \in \mathrm{Ob}(A')$, $a_0 \in \mathrm{Ob}(A)$ so that 3.6 can be applied. □

4. Applications to homotopy theory

If X, E are pointed topological spaces, then let πE^X denote the pointed track groupoid whose objects are the pointed maps $X \to E$ and whose morphisms are the homotopy classes of pointed homotopies rel end maps. Then the set of components $\pi_0(\pi E^X)$ may be identified with $[X, E]$, the set of pointed homotopy classes of pointed maps, and the vertex group $\pi E^X\{\cdot\}$, where $\cdot$ is the constant map, may be identified with $[\Sigma X, E]$, where ΣX is the reduced suspension of X (cf. [3]).

4.1. Proposition. *Suppose we are given a pullback of pointed spaces*

$$\begin{array}{ccc} X \sqcap E & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \bar{p}} \downarrow & & \downarrow {\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

in which p is an h-fibration in the category of pointed topological spaces. Then for any pointed space Z the canonical map $\psi : \pi((X \sqcap E)^Z) \to \pi(X^Z) \sqcap \pi(E^Z)$ induces a bijection $\pi_0 \psi : [Z, X \sqcap E] \to \pi_0(\pi(X^Z) \sqcap \pi(E^Z))$. □

The proof of 4.1 is a pointed version of [15; 2.3], and is left to the reader.

In the following corollary $\pi_0 Y$ denotes the pointed set of path components of the pointed space Y and, for $n \geq 1$, $\pi_n Y$ denotes the nth homotopy group of Y.

4.2. Corollary. *In the situation of* 4.1 *there is an infinite sequence*

$$\begin{array}{ccccccccccc}
\pi_2 E & & & & \pi_1 E & & & & \pi_0 E & & \\
& \searrow^{p_*} & & \nearrow^{\bar{f}_*} & & \searrow^{p_*} & & \nearrow^{\bar{f}_*} & & \searrow^{p_*} & \\
\cdots & & \pi_2 B \xrightarrow{\Delta} \pi_1(X \sqcap E) & & & & \pi_1 B \xrightarrow{\Delta} \pi_0(X \sqcap E) & & & & \pi_0 B. \\
& \nearrow_{f_*} & & \searrow_{\bar{p}_*} & & \nearrow_{f_*} & & \searrow_{\bar{p}_*} & & \nearrow_{f_*} & \\
\pi_2 X & & & & \pi_1 X & & & & \pi_0 X & &
\end{array}$$

The sequence is exact in the sense of 2.2. *Furthermore there is an operation of the normalizer of $f_*(\pi_1 X)$ in $\pi_1 B$ on the subset of elements of $\pi_0(X \sqcap E)$ which project to zero in $\pi_0 X$.*

Proof. The seven sets/groups to the right come from a direct application of 2.2, together with the information given before 4.1, in the case Z is the 0-sphere S^0. The infinite continuation is given inductively by putting $Z_1 = S^0$, $Z_{n+1} = \Sigma Z_n$ in a manner similar to [4; §6]. Finally the operation is a straightforward interpretation of Theorem 3.4. □

Clearly there are results in the vein of Corollary 4.2 corresponding to Corollaries 2.4 to 2.7. By replacing $f : X \to B$ by its associated mapping track fibration the result corresponding to 2.4 allows one to deduce the results of [9; §1].

In the following example, we give an application to Eilenberg-MacLane spaces generalizing [12; §17, Exercise 10].

4.3. Example. *Let $X \xrightarrow{f} B \xleftarrow{p} E$ be a diagram of Eilenberg-MacLane spaces of types (G, n), (L, n), (H, n), respectively $(n \geq 1)$. Assume that p is a fibration and*

$k:G\times H\to L$, $(g,h)\mapsto f_*(g)\cdot p_*(h)$, is surjective. Then the pullback of f and p is a $K(G\sqcap H,n)$, where $G\sqcap H$ is the pullback of $G\xrightarrow{f_*}L\xleftarrow{p_*}H$. In particular, this holds if f_* or p_* is surjective.

Proof. The result follows from 2.4 and 4.1 which gives an exact sequence

$$\pi_{i+1}(B)\to\pi_i(X\sqcap E)\twoheadrightarrow\pi_i(X)\sqcap\pi_i(E)\to 1.$$

The assumption on k is needed to show $\pi_{n-1}(X\sqcap E)=0$. □

General remarks. There are also free versions of the above results, as well as dual results, both pointed and free. All four cases can be obtained by working more generally in a suitable abstract category with homotopy system as indicated in [15]. That is, the arguments go over formally to such general categories and so can be applied there.

Some of the results in the dual, pointed case can be found in [24] under stronger assumptions than given here. An example of the formal argument in the category of topological pairs is given in [10].

5. Non-abelian cohomology and van Kampen's theorem

Let X be a space with base point x_0 and let G be a group, not necessarily abelian. Olum in [23] defined a cohomology set $H^1(X,x_0;G)$, and he showed that if $X=U\cup V$; $W=U\cap V$; $x_0\in W$; U,V,W are path-connected, and suitable local conditions hold (for example U,V are open), then the square induced by inclusions

$$(5.1)\qquad \begin{array}{ccc} H^1(X,x_0;G) & \longrightarrow & H^1(U,x_0;G)\\ \downarrow & & \downarrow\\ H^1(V,x_0;G) & \longrightarrow & H^1(W,x_0;G)\end{array}$$

is a pullback. Olum also proved that there is a natural bijection

$$(5.2)\qquad H^1(Y,y_0;G)\xrightarrow{\cong}\mathrm{Hom}(\pi_1(Y,y_0),G)$$

for each path-connected space Y and $y_0\in Y$. The two results easily imply the van Kampen theorem, for path-connected $U,V,U\cap V$.

Another useful consequence of (5.2) is the 1-dimensional Hurewicz theorem. Suppose that G is abelian. Then $H^1(X,x_0;G)$ is the usual 1-dimensional cohomology with coefficients in G. Also there is a natural surjection

$$(5.3)\qquad H^1(X,x_0;G)\to\mathrm{Hom}(H_1(X),G)$$

which is bijective if X is path-connected. From (5.2) and (5.3) we deduce easily that if X is path-connected, then $H_1(X)$ is $\pi_1(X,x_0)$ made abelian.

The pullback square (5.1) was generalized in [1] to the non-connected case, by establishing a sequence of the form given in Theorem 2.2 above. This gave for example a determination of $\pi_1(X, x_0)$ when U, V are open and 1-connected and $W = U \cap V$ has $n+1$ path-components. In particular, it gave a new proof that $\pi_1(S^1, 1) \cong \mathbb{Z}$. It seemed unlikely, though, that such a Mayer–Vietoris sequence would determine $\pi_1(X, x_0)$ completely in the general case.

Let X_0 be a subset of X. In [2] the fundamental groupoid[1] $\pi X X_0$ of the pair (X, X_0) was defined as the set of homotopy classes rel $\dot{I}$ of maps $(I, \dot{I}) \to (X, X_0)$, with groupoid operation induced by composition of paths. The van Kampen theorem was then generalized to a pushout theorem for $\pi X X_0$, the conditions of connectivity of U, V, W being replaced by the condition that X_0 meets each path-component of U, V and W. Remarkably, this result does determine $\pi_1(U \cup V, x_0)$ completely even when $U \cap V$ is not path-connected. It is not clear how such a result could be obtained by standard cohomological methods, which usually relate invariants in different dimensions by an exact sequence. Also, the pushout theorem looks like a relative result (on the fundamental groupoid of a pair (X, X_0)), but does not fit into standard notions of relative results.

The purpose of this section is to show that by combining the non-abelian cohomological methods of [23] with the groupoid methods of [2] and [5] we are able to resolve the above anomalies. Thus we consider ***relative non abelian cohomology with coefficients in a groupoid*** to establish simplicial analogues of the main results on Mayer–Vietoris sequences and van Kampen's theorem in [23], [1], [2], and [3]. The section closes with an indication of how the topological and simplical theories are related.

In the following let K be a simplicial set with face maps $\partial_i : K_n \to K_{n-1}$, $i = 0, \dots, n$, and degeneracies $s_j : K_{n-1} \to K_n$, $j = 0, \dots, n-1$; let G be a groupoid.

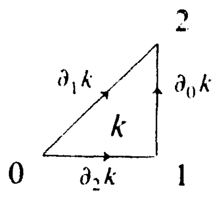
5.4. Definition. A *1-cocycle z of K with coefficients in G* consists of two functions

$$z_0 : K_0 \to \mathrm{Ob}(G), \qquad z_1 : K_1 \to G$$

satisfying

(i) $\quad z_1(k) \in G(z_0(\partial_1 k), z_0(\partial_0 k)) \quad$ for $k \in K_1$,

(ii) $\quad z_1(\partial_1 k) = z_1(\partial_0 k) z_1(\partial_2 k) \quad$ for $k \in K_2$.



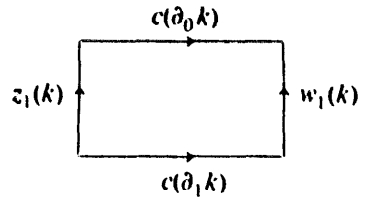
Note that the composition on the right hand side of (ii) is defined and that (ii) implies

[1] The notation is borrowed from [3] and differs from [2]

(iii) $z_1(s_0 a) = 1_{z_0(a)}$ for $a \in K_0$.

5.5. Definition. Let z, w be 1-cocycles of K with coefficients in G. A *homotopy* $c: z \simeq w$ is a function $c: K_0 \to G$ satisfying

(i) $c(a) \in G(z_0(a), w_0(a))$ for $a \in K_0$,
(ii) $c(\partial_0 k) z_1(k) = w_1(k) c(\partial_1 k)$ for $k \in K_1$.



By $Z^1(K; G)$ we denote the groupoid which has as objects the 1-cocycles of K with coefficients in G, and as morphisms the homotopies of 1-cocycles, with the obvious composition.

5.6. Definition. The *1-dimensional cohomology of K with coefficients in G* is defined as

$$H^1(K; G) = \pi_0 Z^1(K; G).$$

A map $f: L \to K$ of simplicial sets induces a morphism of groupoids $f^*: Z^1(K; G) \to Z^1(L; G)$ in an obvious way. Thus $Z^1(K; G)$ becomes a functor of K.

5.7. Proposition. *Let $f: L \to K$ be a map of simplicial sets such that $f_0: L_0 \to K_0$ is injective. Then*

$$f^*: Z^1(K; G) \to Z^1(L; G)$$

is a fibration of groupoids.

Proof. Let z be an object of $Z^1(K; G)$, and let $d: f^*(z) \simeq w$ be a homotopy in $Z^1(L; G)$. Then a homotopy c of z satisfying $f^*(c) = d$ is given by

$$c(k) = \begin{cases} d(f^{-1}k) & \text{if } k \in f(L_0), \\ z_1(s_0 k) & \text{if } k \notin f(L_0). \end{cases} \qquad \square$$

Next we define relative cohomology. We cannot define a relative theory by considering cocycles vanishing on simplicial subsets since for groupoids as coefficients there is no standard meaning to be given to 'vanishing'. Instead, we have relative cocycles for each choice of cocycle on a simplicial subset.

More generally, consider a map $i: K' \to K$ of two simplicial sets. Let ϕ' be a 1-cocycle of K' with coefficients in the groupoid G. Define $Z^1(K, \phi'; G)$ to be the fibre of $i^*: Z^1(K; G) \to Z^1(K'; G)$ over ϕ'.

The groupoid $Z^1(K,\phi';G)$ has a set of components and for each of its objects ϕ (so that ϕ is a 1-cocycle relative to ϕ') has a vertex group at ϕ. The set π_0 of components is written $H^1(K,\phi';G)$ and called the *1-dimensional relative cohomology set*. The vertex group at ϕ is written $H^0_\phi(K,\phi';G)$ and is called the *0-dimensional relative cohomology group at* ϕ.

We need a relative version of 5.7.

5.8. Proposition. *Let*

$$\begin{array}{ccc} L' & \xrightarrow{f'} & K' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ L & \xrightarrow[f]{} & K \end{array}$$

be a commutative diagram of simplicial maps such that the corresponding diagram in dimension 0 *is a diagram of inclusions. Suppose* $L_0' = L_0 \cap K_0'$.

Then, if ϕ' *is a* 1*-cocycle in* $Z^1(K';G)$ *and* $\psi' = f'^*(\phi)$, *the induced morphism*

$$f^*: Z^1(K,\phi';G) \to Z^1(L,\psi';G)$$

is a fibration of groupoids. □

The proof if given by the same formula as in the proof of 5.7.

Note that the assumption on L_0' cannot be dropped. If $K = L = K' = \{*\}$, $L' = \emptyset$, and G is a group, then the induced morphism of fibres is the group homomorphism $\{1\} \to G$ which is not a fibration if G is non-trivial.

For the applications we are interested in an interpretation of cohomology in the case of Kan complexes.

Recall that if K is a Kan complex then the fundamental groupoid πK is defined [11; IV.5.2]. Recall also that if H, G are groupoids, then there is a groupoid (HG) whose objects are the morphisms $H \to G$ and whose arrows are the homotopies of morphisms [4; §1]. We write

$$[H,G] = \pi_0(HG).$$

5.9. Proposition. *If* K *is a Kan complex, then there is a natural isomorphism of groupoids*

$$\Gamma: Z^1(K;G) \to ((\pi K)G).$$

Proof. Because K is a Kan complex, each element α of $\pi K(x,y)$ is represented by a 1-simplex $k \in K_1$, and if k, l are two representatives of α, then there is a 2-simplex k_2 such that $\partial_1 k_2 = k$, $\partial_2 k_2 = l$, $\partial_0 k_2 = s_0 y$. So 5.4 implies that if z is a 1-cocycle, then the function $\Gamma(z): \pi K \to G$, $\operatorname{cls} k \mapsto z_1(k)$, is well defined. The definition of composition in πK, and the cocycle condition, together imply that $\Gamma(z)$ is a morphism of groupoids.

Let $c:z\simeq w$ be a homotopy in $Z^1(K;G)$. Then c determines also a homotopy $\Gamma(z)\simeq\Gamma(w)$. Thus Γ is a function $Z^1(K;G)-((\pi K)G)$ which is clearly a morphism.

An inverse Θ to Γ is defined on objects by

$$\Theta(f):k\mapsto f(\text{cls } k)$$

for any morphism $f:\pi K\to G$; and Θ is defined on homotopies as the inverse of Γ. □

5.10. Corollary. *If K is a Kan complex, there is a natural bijection*

$$\Gamma:H^1(K;G)\to[\pi K,G]. \qquad \square$$

In the relative case, let K and K' be Kan complexes, and let $i:K'\to K$ be a map. A cocycle ϕ' in $Z^1(K';G)$ defines a morphism $\phi':\pi K'\to G$. Let $((\pi K)G)_{\phi'}$ be the fibre of $i^*:((\pi K)G)\to((\pi K')G)$ over ϕ', and let

$$[\pi K,G]_{\phi'}=\pi_0((\pi K)G)_{\phi'}.$$

Clearly the isomorphisms $Z^1(K;G)\cong((\pi K)G)$ and $Z^1(K';G)\cong((\pi K')G)$ induce an isomorphism $Z^1(K,\phi';\ i)\cong((\pi K)G)_{\phi'}$ and hence a bijection

$$H^1(K,\phi';G)\cong[\pi K,G]_{\phi'}. \tag{5.11}$$

It is the latter set for which we wish to give an explicit description in some useful cases.

We assume that $i:K_0'\to K_0$ is an inclusion mapping. Then the groupoid $\pi KK_0'$ is defined as the full subgroupoid of πK on K_0'. Let $\text{Hom}(\pi KK_0',G)_{\phi'}$ denote the set of morphisms from $\pi KK_0'$ to G which give ϕ' on composition with $i:\pi K'\to\pi KK_0'$.

5.12. Proposition. *Suppose in addition to the above assumption that K_0' meets each component of K. Then there is a bijection*

$$\alpha:[\pi K,G]_{\phi'}\cong\text{Hom}(\pi KK_0',G)_{\phi'}.$$

Proof. Since K_0' meets each component of K, the inclusion $i:\pi KK_0'\to\pi K$ is a homotopy equivalence and in fact makes $\pi KK_0'$ a deformation retract of πK. Let $r:\pi K\to\pi KK_0'$ be a retraction.

Define α by $\alpha(\bar\xi)=\xi i$. Then α is well defined because any homotopy $c:\xi\simeq\eta$ in $((\pi K)G)_\phi$ is, by definition of this groupoid, constant on K_0'.

Define an inverse β to α by $\beta(\xi')=(\xi' r)^-$. Then $\alpha\beta(\xi')=\xi' ri=\xi'$, and $\beta\alpha(\bar\xi)=(\xi ir)^-=\bar\xi$ since $ir\simeq 1$ rel $\pi KK_0'$. □

A particular case of this relative theory is when K' consists of a single point k_0 of K, and $i:K'\to K$ is the inclusion. Suppose also G is a group. Then there is a unique choice of the cocycle ϕ' on K', namely the constant function $K_1' \to G$ with value the identity of G. We denote the set $H^1(K, \phi'; G)$ by $H^1(K, k_0; G)$. As an immediate consequence of the bijections (5.11) and of 5.12 we have:

5.13. Corollary. *If G is a group and K is a connected Kan complex with $k_0 \in K_0$, then there is a bijection*

$$\alpha : H^1(K, k_0; G) \to \mathrm{Hom}(\pi_1(K, k_0), G). \qquad \square$$

Suppose we are given a pushout square of simplicial sets

$$\text{(5.14)} \qquad \begin{array}{ccc} L & \xrightarrow{f} & P \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ K & \xrightarrow[\bar{f}]{} & Q \end{array}$$

5.15. Proposition. *For any groupoid G, the induced square*

$$\begin{array}{ccc} Z^1(Q; G) & \xrightarrow{\bar{i}^*} & Z^1(F; G) \\ {\scriptstyle \bar{f}^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Z^1(K; G) & \xrightarrow[i^*]{} & Z^1(L; G) \end{array}$$

is a pullback of groupoids. $\square$

The proof is straightforward.

We need a relative version of 5.15. Consider then a pushout square

$$\text{(5.16)} \qquad \begin{array}{ccc} L' & \xrightarrow{f'} & P' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \bar{i}'} \\ K' & \xrightarrow[\bar{f}']{} & Q' \end{array}$$

of simplicial subsets of L, K, P, Q, the maps being obtained by restriction of those of (5.14). Let ξ, η be 1-cocycles of K', P' respectively, with coefficients in G, such that $i'^*(\xi) = f'^*(\eta) = \phi$, say, and let ψ be the 1-cocycle on Q' determined by ξ, η.

5.17. Proposition. *With the above as data, the induced square*

$$
\begin{array}{ccc}
Z^1(Q,\psi;G) & \xrightarrow{\ \bar{i}^*\ } & Z^1(P,\eta;G) \\
{\scriptstyle \bar{j}^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
Z^1(K,\xi;G) & \xrightarrow[\ i^*\ ]{} & Z^1(L,\phi;G)
\end{array}
$$

is a pullback of groupoids.

Proof. This follows from Proposition 5.15 and the easily verified fact that if κ is a map $S \to S'$ of pullback squares, and $*$ is a square of base points in S', then the fibre $\kappa^{-1}(*)$ is also a pullback square. □

We now fulfill our main aim by applying the Mayer–Vietoris sequence (2.3) to two special cases of Proposition 5.17, and so show how the simplicial versions of the main results of [1], [2], [23] follow.

For the first application we take L', K', P', Q' to be base points in L, K, P, Q respectively, and we take G to be a group. We assume that $i: L_0 \to K_0$ is injective. As base cocycles we take constant functions on 1-simplices with value 1. We abbreviate cohomologies such as $H^1(K, k_0; G)$ to $H^1(K)$. In this situation we also have a natural choice of cocycles extending those on simplicial subsets namely the constant cocycle with value 1, and this gives a 0-dimensional cohomology group $H^0(K, k_0; G)$ which we abbreviate to $H^0(K)$.

Under all these assumptions we have by 5.8, 5.17 and 2.2:

5.18. Proposition. *There is a Mayer–Vietoris sequence*

$$
\begin{array}{ccccccccc}
 & & H^0(K) & & & & H^1(K) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
H^0(Q) & & & & H^0(L) \xrightarrow{\ \Delta\ } H^1(Q) & & & & H^1(L) \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & H^0(P) & & & & H^1(P) & &
\end{array}
$$

which is exact in the sense of Theorem 2.2. □

For the second application, let $i: L \to K$ be an inclusion. We take K', P' to be discrete simplicial subsets of K and P, $L' = L \cap K'$ and assume that L', K', P' and hence Q' meet each component of L, K, P, Q respectively.

Then the group $H^0_\phi(L, \phi'; G)$ is trivial, and so we obtain by 5.8, 5.17 and 2.5 from the Mayer–Vietoris sequence:

5.19. Proposition. *The square*

$$\begin{array}{ccccc} & & H^1(P,\eta;G) & & \\ & \nearrow^{\bar{i}^*} & & \searrow^{f^*} & \\ H^1(Q,\psi;G) & & & & H^1(L,\phi;G) \\ & \searrow_{\bar{j}^*} & & \nearrow_{i^*} & \\ & & H^1(K,\xi;G) & & \end{array}$$

is a pullback of sets. □

In particular, if all the simplicial sets are Kan complexes, this result can be translated to:

5.20. Proposition. *The square*

$$\begin{array}{ccccc} & & \mathrm{Hom}(\pi PP',G)_\eta & & \\ & \nearrow & & \searrow & \\ \mathrm{Hom}(\pi QQ',G)_\psi & & & & \mathrm{Hom}(\pi LL',G)_\phi \\ & \searrow & & \nearrow & \\ & & \mathrm{Hom}(\pi KK',G)_\xi & & \end{array}$$

is a pullback of sets. □

Since this result is true for any G and compatible ψ,ξ,η,ϕ, we have as a consequence:

5.21. Proposition. *The square*

$$\begin{array}{ccc} \pi LL' & \longrightarrow & \pi KK' \\ \downarrow & & \downarrow \\ \pi PP' & \longrightarrow & \pi QQ' \end{array}$$

is a pushout of groupoids. □

Thus, from this result, the groupoid $\pi QQ'$ is determined completely.

Our simplicial results contain in essence the corresponding topological results of [23], [1], [2], [3]. To see this, as in [23], we have to relate the simplicial and the topological theory.

If X is a topological space with base point x_0 and G is a group then the cohomology set $H^1(X,x_0;G)$ and the cohomology group $H^0(X,x_0;G)$ defined by

Olum in [23] are exactly the relative cohomology set $H^1(K, x_0; G)$ and, respectively, the relative cohomology group $H^0(K, x_0; G)$ where $K = SX$ is the singular complex of X.

If X is a topological space and A is a subset of X, then the fundamental groupoid πXA of the pair (X, A) may easily be identified (compare [11; Ch. IV, §5]) with the groupoid $\pi KK'_0$ where $K = SX$ is the singular complex of X and $K' = SA$.

Let $\mathscr{F}$ be any family of subsets of X. We may define $S_{\mathscr{F}}X$ to be the complex of singular simplices σ of X such that there is a set F of $\mathscr{F}$ with $\operatorname{Im}\sigma \subset F$. We note that if $\mathscr{F}$ consists of two sets X_1 and X_2 with $X_1 \cap X_2 = X_0$, then the diagram of inclusions

(5.22)
$$\begin{array}{ccc} SX_0 & \longrightarrow & SX_1 \\ \downarrow & & \downarrow \\ SX_2 & \longrightarrow & S_{\mathscr{F}}X \end{array}$$

is a pushout of simplicial sets.

Let $i : S_{\mathscr{F}}X \to SX$ be the inclusion, and let G be a groupoid. The proof on p. 662 of [23] can be generalized and sharpened to show that if the interiors of the sets of $\mathscr{F}$ cover X, then the inclusion i induces an isomorphism of groupoids

(5.23) $$i^* : Z^1(SX; G) \to Z^1(S_{\mathscr{F}}X; G).$$

Now let K' be any subcomplex of SX, and let $K'_{\mathscr{F}} = K' \cap S_{\mathscr{F}}X$. Let ϕ' be an object of $Z^1(K'; G)$ and let $\phi'_{\mathscr{F}}$ be the restriction of ϕ' to $K'_{\mathscr{F}}$. Then we have a map of fibrations (see 5.7)

(5.24)
$$\begin{array}{ccc} Z^1(SX, \phi'; G) & \xrightarrow{\;i_3^*\;} & Z^1(S_{\mathscr{F}}X, \phi'_{\mathscr{F}}; G) \\ \downarrow & & \downarrow \\ Z^1(SX; G) & \xrightarrow{\;i_2^*\;} & Z^1(S_{\mathscr{F}}X; G) \\ \downarrow & & \downarrow \\ Z^1(K'; G) & \xrightarrow{\;i_1^*\;} & Z^1(K'_{\mathscr{F}}; G) \end{array}$$

We assume that the interiors of the sets of $\mathscr{F}$ cover X. Hence by (5.23) i_2^* is an isomorphism. So i_3^* is an isomorphism if i_1^* is an isomorphism. The latter will be true if, for example:

(1) A is a subspace of X and $K' = SA$;

(2) $K' \subset S_{\mathscr{F}}X$.

The first example is considered by Olum (with G a group and ϕ' the trivial cocycle).

Proposition 5.18 applied to the pushout (5.23) endowed with base points, together with the induced isomorphism of (5.24), gives the results of [23] and [1].

The second example allows us to discuss πXA if we take K' to consist of all constant singular simplices in A. Proposition 5.19 applied to the pushout (5.22), together with the induced isomorphism of (5.24), enables one to deduce the groupoid version of the van Kampen theorem (6.7.2 of [3]).

A generalization to all dimensions of the van Kampen theorem has been given in [7], which also discusses the case of arbitrary covers. The appropriate generalization of the fundamental groupoid of a pair is the homotopy crossed complex $\pi\mathbf{X}$ of a filtered space $\mathbf{X}$. If X is a CW-complex a notion of *cohomology of X with coefficients in a crossed complex C* is suggested in [8] as[2] $H^0(X; C) = [\pi\mathbf{X}, C]$, the set of homotopy classes of maps $\pi\mathbf{X} \to C$, where $\mathbf{X}$ is the skeletal filtration of X. This can be extended to any space Z by defining $H^0(Z; C) = H^0(|SZ|; C)$ where $|SZ|$ is the geometric realization of the singular complex of Z. It would be interesting to know if the colimit theorems of [7] have cohomological proofs.

References

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