THE FUNDAMENTAL GROUPOID AS A TOPOLOGICAL GROUPOID

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Introduction

Let X be a topological space. Then we may define the fundamental groupoid πX and also the quotient groupoid $(\pi X)/N$ for N any wide, totally disconnected, normal subgroupoid N of πX (**1**). The purpose of this note is to show that if X is locally path-connected and semi-locally 1-connected, then the topology of X determines a " lifted topology " on $(\pi X)/N$ so that it becomes a topological groupoid over X. With this topology the subspace $\mathrm{St}_{(\pi X)/N}x$ which is the fibre of the initial point map $\partial' : (\pi X)/N \to X$ over x in X, is the usual covering space $\tilde{X}_{N,x}$ of X determined by the normal subgroup $N\{x\}$ of the fundamental group $\pi(X, x)$.

One advantage of considering $(\pi X)/N$ rather than the collection of spaces $\tilde{X}_{N,x}$, $x \in X$, is that, in a suitable sense, $(\pi X)/N$ is functorial. For example, if G is a topological group acting on X, and N is G-invariant, then G acts also on $(\pi X)/N$. This gives us in Section 3 a formulation of the theory of the fundamental group of a transformation group due to F. Rhodes (**8, 9**) simply as an object group of the semi-direct product groupoid $G \ltimes (\pi X)/N$.

In Section 2 the topology on $(\pi X)/N$ is related to Ehresmann's work on locally trivial groupoids (**5**).

A final algebraic section shows that any extension of a group G arises as the exact sequence of the projection $G \ltimes A \to G$ of a semi-direct product of G with some groupoid A.

The genesis of this paper should be described. The main ideas of producing a lifted topology on $(\pi X)/N$, of describing its fundamental group, and of exploring the relationship with the work of (**5, 8, 9**) are to be found in (**4**). The present formulation of these results was worked out while the second author visited Bangor in 1972.†

1. The lifted topology on $(\pi X)/N$

A *topological groupoid* Γ *over* X is a groupoid such that $\mathrm{Ob}(\Gamma) = X$, and with topologies on both X and on the set Γ of arrows such that all the structure functions are continuous; these structure functions are: the initial and final maps $\partial', \partial : \Gamma \to X$; the unit map $u : X \to \Gamma$ which sends $x \mapsto 1_x$; the composition $\theta : \Gamma \ltimes \Gamma \to \Gamma$, $(a, b) \mapsto ba$, whose domain is the set of (a, b) such that $\partial a = \partial' b$; and the inverse map $\Gamma \to \Gamma$, $a \mapsto a^{-1}$. Thus a topological groupoid is a natural generalisation of a topological group.

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The topological groupoid Γ is *locally trivial* (5) if each x in $\mathrm{Ob}(\Gamma)$ has a neighbourhood U such that there is a continuous function $\lambda : U \to \Gamma$ such that $\lambda(y) \in \Gamma(x, y)$ for all $y \in U$.

Now let X be a topological space and N a totally disconnected, wide, normal subgroupoid of πX, the fundamental groupoid of X. Thus N assigns to each x in X a subgroup $N\{x\}$ of $\pi(X, x)$ so that if $a \in \pi X(x, y)$, then $aN\{x\}a^{-1} = N\{y\}$. The quotient groupoid $\pi/N = (\pi X)/N$ can then be defined to have object set X and arrows $x \to y$ the cosets $\alpha N\{x\}$ for $\alpha \in \pi X(x, y)$. In particular the object group of π/N at x is the quotient group $\pi(X, x)/N\{x\}$.

Let $p : \pi X \to \pi/N$ be the projection.

Theorem 1. *Let X be locally-path connected and semi-locally* 1*-connected.*

(*a*) *There is a topology on π/N so that it becomes a locally trivial topological groupoid over X with (topologically) discrete object groups.*

(*b*) *For each x in X, the subspace $\mathrm{St}_{\pi/N}x$ of π/N is the regular covering space of X based at x and determined by the subgroup $N\{x\}$ of $\pi(X, x)$.*

(*c*) *The fundamental group of π/N at 1_x is isomorphic to the subgroup of $\pi(X, x) \times \pi(X, x)$ of pairs (a, b) such that $aN\{x\} = bN\{x\}$.*

For the proof of Theorem 1, we consider the groupoid Q such that $\mathrm{Ob}(Q)$ consists of the arrows of π/N, and if $\alpha, \beta \in \pi/N$ then $Q(\alpha, \beta)$ consists of triples (a, α, b) such that a, b are arrows of πX and $p(b)\alpha p(a^{-1})$ is defined and equal to β. Composition in Q is given by $(a', \alpha', b')(a, \alpha, b) = (a'a, \alpha, b'b)$. (Thus Q is isomorphic to the comma category $(p\downarrow p)$, (7) p. 47.) There is a projection $q : Q \to \pi X \times \pi X$ given by $(a, \alpha, b) \mapsto (a, b)$; it is easy to check that q is a covering morphism of groupoids. But $\pi X \times \pi X = \pi(X \times X)$ ((1) p. 189). By (1) p. 309 the topology of $X \times X$ lifts uniquely to a topology on $\mathrm{Ob}(Q) = \pi/N$ such that $\mathrm{Ob}(q) : \pi/N \to X \times X$ is a covering map and there is an isomorphism $r : \pi(\pi/N) \to Q$ making the following diagram commutative

$$\begin{tikzcd} \pi(\pi/N) \arrow[rr, "r"] \arrow[dr] & & Q \arrow[dl, "q"] \\ & \pi(X \times X) & \end{tikzcd}$$

It follows that $\pi(\pi/N, 1_x)$ is isomorphic to the object group $Q\{1_x\}$, which consists of triples $(a, 1_x, b)$ such that $p(a)1_x p(b^{-1})$ is defined and equal to 1_x. This verifies (*c*) of Theorem 1.

In order to prove (*a*) and (*b*) of Theorem 1, we need to describe in detail the lifted topology on Q.

Let $\mathscr{U}$ be the open cover of X consisting of all open, path-connected subsets U of X such that the inclusion $i : U \to X$ maps each fundamental group of U to the trivial group. The elements of U will be called *canonical neighbourhoods*. For each U in $\mathscr{U}$ and x in U, define $\lambda_x : U \to \pi X$ by choosing for each x' in U a path in U from x to x' and letting $\lambda_x(x')$ be the class in $\pi X(x, x')$ of this path—

the conditions on U imply that $\lambda_x(x')$ is independent of the choice of path in U from x to x'. Let $\mu_x = p\lambda_x : U \to \pi/N$, and let $\tilde{U}_x = \mu_x(U)$.

Lemma 2. *Let $\alpha \in \pi/N(x, y)$. Then the sets $\tilde{V}_y \alpha \tilde{U}_x^{-1}$ for all U, V in $\mathscr{U}$ such that $x \in U$, $y \in V$, form a set of basic neighbourhoods for the lifted topology on π/N.*

The proof follows easily from the description of the lifted topology in (**1**) pp. 307-8.

We can now prove that π/N is a topological groupoid. In fact, the only non-trivial part of the proof is continuity of composition $\theta : \pi/N \tilde{\times} \pi/N \to \pi/N$. Let $\theta(\beta, \alpha) = \beta\alpha$, where $\alpha \in \pi/N(x, y)$, $\beta \in \pi/N(y, z)$. Let $\tilde{W}_z \beta\alpha \tilde{U}_x^{-1}$ be a basic neighbourhood of $\beta\alpha$. Then for any V in $\mathscr{U}$ such that $y \in V$,

$$\theta(\tilde{W}_z \beta \tilde{V}_y^{-1} \tilde{\times} \tilde{V}_y \alpha \tilde{U}_x^{-1}) = \tilde{W}_z \beta\alpha \tilde{U}_x^{-1}$$

and so θ is continuous.

The mappings μ_x give the local trivialisations of π/N.

If $\alpha \in \pi/N\{x\}$, then the intersection of $\pi/N\{x\}$ with a basic neighbourhood of α is $\{\alpha\}$—so $\pi/N\{x\}$ has the discrete topology.

If $\alpha \in \pi/N(x, y)$, then a basic neighbourhood of α in $\mathrm{St}_{\pi/N}x$ is $\tilde{U}_y\alpha$, and these are the usual basic neighbourhoods in the theory of covering spaces. This completes the proof of Theorem 1.

The topological groupoid $(\pi X)/N$ is functorial in the following sense.

Let $f: Y \to X$ be a map of locally path-connected and semi-locally 1-connected spaces. Let M, N be wide, totally disconnected, normal subgroupoids of πY, πX respectively such that $(\pi f)(M) \subseteq N$.

Proposition 3. *The morphism $g : (\pi Y)/M \to (\pi X)/N$ induced by πf is continuous.*

Proof. Let $\alpha : y \to y'$ be an arrow of $(\pi Y)/M$ and let $\beta = g(\alpha) : x \to x'$. Let U, U' be canonical neighbourhoods of x, x' respectively. Then there are canonical neighbourhoods V, V' of y, y' respectively such that $f(V) \subseteq U$, $f(V') \subseteq U'$. Clearly

$$g(\tilde{V}'_{y'} \alpha \tilde{V}_y^{-1}) \subseteq \tilde{U}'_{x'} \beta \tilde{U}_x^{-1}$$

and so g is continuous.

We now consider *topological covering morphisms* $p: G \to H$ of topological groupoids. A definition of this given in (**4**), that each induced map $\mathrm{St}_G x \to \mathrm{St}_H p(x)$ is a homeomorphism, turns out not to be strong enough; the definition which follows is due to R. Brown and J. P. L. Hardy (**6**), (**10**).

First of all let $q: G \to H$ be a morphism of topological groupoids and consider the pull-back diagram

$$\begin{tikzcd} H \tilde{\times} \mathrm{Ob}(G) \arrow[r] \arrow[d] & \mathrm{Ob}(G) \arrow[d, "q"] \\ H \arrow[r, "\partial'"'] & \mathrm{Ob}(H) \end{tikzcd}$$

Then $q: G \to H$ and $\partial': G \to \mathrm{Ob}(G)$ determine a continuous function

$$\tilde{q}: G \to H \tilde{\times} \mathrm{Ob}(G).$$

Definition. The morphism $q: G \to H$ is a *topological covering morphism* if $\tilde{q}: G \to H \tilde{\times} \mathrm{Ob}(G)$ is a homeomorphism.

Let X, Y, M, N satisfy the conditions of Proposition 3.

Theorem 4. *If $f: Y \to X$ is a covering map and $(\pi f)(M) = N$ then the morphism $g: (\pi Y)/M \to (\pi X)/N$ induced by f is a topological covering morphism of topological groupoids.*

Proof. By 9.2.1 of (1), $\pi f: \pi Y \to \pi X$ is (abstractly) a covering morphism, and it is easy to prove that so also is $g: (\pi Y)/M \to (\pi X)/N$. Certainly

$$(g, \partial'): (\pi Y)/M \to (\pi X)/N \tilde{\times} Y$$

is continuous and bijective. We prove (g, ∂') is an open mapping.

Let $\alpha: y \to y'$ be an arrow of $(\pi Y)/M$. Then we can choose canonical neighbourhoods V, V' of y, y' respectively such that $U = f(V)$, $U' = f(V')$ are canonical neighbourhoods of $x = f(y)$, $x' = f(y')$ respectively. If

$$W = \tilde{V}_y \alpha \tilde{V}_{y'}'^{-1},$$

then $g(W)$ is a basic neighbourhood of $g(\alpha)$, while $(g, \partial')(W) = g(W) \tilde{\times} V$, which is open in $(\pi X)/N \tilde{\times} Y$. This completes the proof.

We now examine products. Let X, Y be locally path-connected and semi-locally 1-connected. Let N, M be totally disconnected, wide, normal subgroupoids of πX, πY respectively.

Proposition 5. *The lifted topology on $\pi(X \times Y)/N \times M$ makes it isomorphic as topological groupoid to $(\pi X)/N \times (\pi Y)/M$.*

Proof. This follows easily from an examination of basic neighbourhoods, those in $\pi(X \times Y)/N \times M$ being of the form $(\tilde{U}_{x'} \times \tilde{V}_{y'})(\alpha, \beta)(\tilde{U}_x^{-1} \times \tilde{V}_y^{-1})$ for $\alpha: x \to x'$ in $(\pi X)/N$, $\beta: y \to y'$ in $(\pi Y)/M$.

2. The Ehresmann theory of locally trivial groupoids

Throughout this section, G will be a locally trivial, transitive†, topological groupoid over X. The study of such groupoids was initiated by Ehresmann, who has proved (5):

Theorem 5. (i) *The initial point map $\partial': G \to X$ is the projection of a fibre bundle with fibre $\mathrm{St}_G x$ over x and group $G\{x\}$.*

(ii) *If $x \in X$, the final point map $\partial: \mathrm{St}_G x \to X$ is the projection of a principal fibre bundle with group $G\{x\}$.*

It is also true that $(\partial', \partial): G \to X \times X$ is a fibre bundle with fibre $G\{x\}$ and group $G\{x\} \times G\{x\}$ (5), but this result will not concern us here.

† It is convenient, for topological groupoids to replace the term "connected groupoid" of (1) by "transitive groupoid".

As a consequence of Theorem 5 (i) we have a transgression function $\Delta : \pi(X, x) \to G\{x\}$.

Corollary 6. *Let X be path-connected and $x \in X$. The following conditions are equivalent.*

(i) $\mathrm{St}_G x$ *is path-connected.*
(ii) $\Delta : \pi(X, x) \to G\{x\}$ *is surjective.*
(iii) $\partial'_* : \pi(G, 1_x) \to \pi(X, x)$ *is surjective.*

Proof. These follow immediately from the exact homotopy sequences of the fibrations described in Theorem 5.

There is an inverse to the operator taking G to the principal bundle $\mathrm{St}_G x$. Let A be a topological group, and $p : E \to X$ a principle A-bundle. Then the groupoid $\mathscr{G}(E)$ of admissible maps of E is defined—if $x, y \in X$, then $\mathscr{G}(E)(x, y)$ is the set of admissible maps $E_x \to E_y$. The local triviality of $p : E \to X$ enables one to define a topology on $\mathscr{G}(E)$ so that it becomes a locally trivial, transitive topological groupoid over X.

The further results of Ehresmann are:

Theorem 7. *If $p : E \to X$ is a principal A-bundle, and $x \in X$, then there is an isomorphism of A-bundles, $E \cong \mathrm{St}_{\mathscr{G}(E)} x$.*

Theorem 8. *If $\mathrm{St}_G x$ is given its structure as a principal $G\{x\}$-bundle, then there is an isomorphism of topological groupoids $G \cong \mathscr{G}(\mathrm{St}_G x)$.*

We can now state a classification theorem for certain topological groupoids over a path-connected space X.

Theorem 9. *Suppose G is transitive, locally trivial, and has topologically discrete vertex groups, that X is path-connected, and $x \in X$. Then the following conditions are equivalent:*

(i) $\mathrm{St}_G x$ *is path connected,*
(ii) *G is isomorphic to $(\pi X)/N$ for some wide, normal totally disconnected subgroupoid N of πX.*

Proof. That (ii)$\Rightarrow$(i) is clear, since $\mathrm{St}_G x$ is then a connected covering space of X; so we prove (i)$\Rightarrow$(ii).

The given conditions imply that $\mathrm{St}_G x$ is a path-connected regular covering space over X, corresponding to a normal subgroup $N\{x\}$ of $\pi(X, x)$. Since X is path-connected, we can define $N\{y\}$ for any $y \in X$ by $N\{y\} = \alpha N\{x\}\alpha^{-1}$ for any $\alpha \in \pi X(x, y)$. Then $N\{y\}$ is independent of choice of α, by normality, and the collection $N\{y\}$, $y \in X$, is a wide, normal, totally disconnected normal subgroupoid of πX.

So we have isomorphisms of principal $G\{x\}/N\{x\}$ bundles over X

$$\mathrm{St}_G x \cong \tilde{X}_{N,x} \cong \mathrm{St}_{(\pi X)/N} x.$$

Hence $G \cong \mathscr{G}(\mathrm{St}_G x) \cong \mathscr{G}(\mathrm{St}_{(\pi X)/N} x) \cong (\pi X)/N$. This completes the proof.

The isomorphism of groupoids $(\pi X)/N = \mathscr{G}(\tilde{X}_{N,x})$ and Ehresmann's topology on the latter groupoid gives an alternative method of topologising the groupoid $(\pi X)/N$. The disadvantages of this method are the dependence on the base point, and that further work is needed to describe the fundamental group of the space $(\pi X)/N$.

3. Transformation groups

Let G be a topological transformation group of the space X, and assume X has the local conditions of Theorem 1. Let π/N be as in Section 1, and assume further that N is invariant under G in that $gN\{x\} = N\{^g x\}$ for all $g \in G$, $x \in X$. Then G acts also on the groupoid π/N and it is easy to see that if π/N has the lifted topology then this action is continuous.

We now form the groupoid $G \ltimes \pi/N$ considered in (3). (It is preferable to call this the *semi-direct product* rather than split-extension as in (3).) The object set of $G \ltimes \pi/N$ is X, the elements in $G \ltimes \pi/N$ from x to y are pairs (g, a) with $g \in G$, $a \in \pi/N(^g x, y)$, and composition is

$$(h, b)(g, a) = (hg, b\,{}^g a).$$

Let X have its given topology, and let $G \ltimes \pi/N$ have its topology as a subset of the product. Then it is easy to check that $G \ltimes \pi/N$ is a topological groupoid.

The object groups of $G \ltimes \pi/N$ have been considered by F. Rhodes (9). He defined a group $\sigma_\rho(X, x_0, G)$ for the case ρ is a G-invariant normal subgroup of $\pi(X, x_0)$, the elements of this group being equivalence classes $[f;\ g]_\rho$ of pairs $(f;\ g)$ such that $g \in G$, f is a path from x_0 to $^g x_0$, and two pairs $(f;\ g)$, $(f';\ g')$ are equivalent if and only if $g = g'$ and the class $[f' - f] \in \rho$.

Proposition 10. *If $x_0 \in X$ and $\rho = N\{x_0\}$, the map*

$$(G \ltimes \pi/N)\{x_0\} \to \sigma_\rho(X, x_0, G)$$

given by $(g, [f]) \mapsto [-f;\ g]$, is an isomorphism.

The proof is obvious. If $\rho = 1$, then $\sigma_\rho(X, x_0, G)$ is written $\sigma(X, x_0, G)$.

The exact sequences given in (9) p. 906 can be derived as follows. The projection $G \ltimes \pi/N \to G$ is a fibration of groupoids (Proposition 2.4 of (3)). By Theorem 4.3 of (2), and Proposition 4 there is an exact sequence

$$1 \to \pi(X, x_0)/\rho \to \sigma_\rho(X, x_0, G) \to G \to \pi_0 X \to \pi_0(G \ltimes \pi/N) \to 1.$$

Further the quotient mappings $\pi X \to \pi/N$ is a fibration of groupoids and so determines a fibration $G \ltimes \pi X \to G \ltimes \pi/N$ (Proposition 2.7 of (3)). The exact sequence of this fibration at x_0 is

$$1 \to \rho \to \sigma(X, x_0, G) \to \sigma_\rho(X, x_0, G) \to 1.$$

Results about actions of groups on covering spaces also fall within the present framework. The topological groupoid $G \ltimes \pi/N$ acts on the space π/N via $\partial : \pi/N \to X$ by the rule

$$(g, b).a = b\,{}^g a.$$

In particular, if $x_0 \in X$, then the group $\sigma_\rho(X, x_0, G) = (G \ltimes \pi/N)\{x_0\}$ acts on $\partial^{-1}(x_0) = \mathrm{Cost}_{\pi/N} x_0$. But $\mathrm{Cost}_{\pi/N} x_0$ is, like $\mathrm{St}_{\pi/N} x_0$, the covering space of X determined by the subgroup $\rho = N\{x_0\}$ of $\pi(X, x_0)$. More generally, if N' is another normal, G-invariant subgroupoid of πX such that $N' \subseteq N$, then $G \ltimes \pi/N'$ acts on π/N by the above rule, and so σ_ρ (X, x_0, G) acts on the covering space $\mathrm{Cost}_{\pi/N} x_0$ of X—this is Proposition 2 of (9).

4. An algebraic remark

Let G be a group and A a transitive groupoid which is a G-module. If $x \in \mathrm{Ob}\,(A)$ and $B = A\{x\}$ then the fibration $G \ltimes A \to G$ determines an exact sequence of groups

$$1 \to B \to (G \ltimes A)\{x\} \to G \to 1, \tag{4.1}$$

i.e. an extension of G by B.

Conversely, suppose given an extension

$$1 \to B \to E \xrightarrow{p} G \to 1 \tag{4.2}$$

we show that this extension is isomorphic to (4.1) for some groupoid A—thus *every extension of groups derives from a semi-direct product of groupoids.*

For the proof, we take A to be the covering groupoid $A = E \ltimes G$ determined by the action of E on G on the right. Thus $\mathrm{Ob}\,(A) = G$, and

$$A(g, g') = \{(e, g) \in E \times G : g = g'e\}$$

with composition $(e', g')(e, g) = (e'e, g)$. An action of G on A is then defined by left multiplication of objects, and on arrows by $h.(e, g) = (e, hg)$.

Now $(G \ltimes A)\{1\}$ consists of triples (h, e, g) such that $(e, g) \in A(h, 1)$, whence $h = g = 1.e$. So we have an isomorphism $(G \ltimes A)\{1\} \to E$, $(p(e), e, p(e)) \mapsto e$, which maps the subgroup $A\{1\}$ isomorphically to B.

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