ALGEBRAIC MODELS OF 3-TYPES AND AUTOMORPHISM STRUCTURES FOR CROSSED MODULES

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Introduction

In this paper we are led, from consideration of an automorphism structure for crossed modules, to the notion of *braided, regular crossed modules*. These are then shown to be closely related to simplicial groups: we prove that the category of braided, regular crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2. This equivalence establishes the rôle of braided, regular crossed modules as algebraic models of homotopy 3-types.

We now review our motivation from the theory of automorphisms. Associated to the automorphism group $\operatorname{Aut} G$ of a group G is the homomorphism $\chi\colon G \to \operatorname{Aut} G$ that sends $x \in G$ to the inner automorphism $g \mapsto g^x = x^{-1}gx$. The group $\operatorname{Aut} G$ acts on G, and χ satisfies the two properties:

(i) $\chi(g^f) = f^{-1}\chi(g)f$,

(ii) $g^{\chi(x)} = x^{-1}gx$,

for all $g, x \in G$ and $f \in \operatorname{Aut} G$. We see that $\operatorname{Aut} G$ is naturally considered as part of a *crossed module*: that is, a group homomorphism $\partial\colon M \to P$ together with an action of P on M satisfying

CM1: $\partial(m^p) = p^{-1}\,\partial(m)p$,

CM2: $m_0^{\partial(m)} = m^{-1}m_0m$,

for all $m_0, m \in M$ and $p \in P$.

Crossed modules were introduced by J. H. C. Whitehead **[16]** and among the standard examples are the inclusion $M \hookrightarrow P$ of a normal subgroup M of P, the zero homomorphism $M \to P$ when M is a P-module, and any surjection $M \twoheadrightarrow P$ with central kernel. There is also an important topological example: if $F \to E \to B$ is a fibration sequence of pointed spaces, then the induced homomorphism $\pi_1 F \to \pi_1 E$ of fundamental groups is naturally a crossed module.

Now groups are *algebraic models of* 1-*types*: that is, there is a *classifying space* functor

$$B\colon (\text{groups}) \to (CW\text{-complexes})$$

such that for any group G the space BG satisfies

$$\pi_1 BG \cong G \quad \text{and} \quad \pi_j BG = 0 \ \text{ for } j > 1,$$

and further any pointed, connected CW-complex X with $\pi_j X = 0$ for $j > 1$ is of the homotopy type of $B\pi_1 X$.

Crossed modules are *algebraic models of* 2-*types.* There is a classifying space functor

$$B\colon (\text{crossed modules}) \to (CW\text{-complexes})$$

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such that if $\partial\colon M \to P$ is a crossed module then $B(M \to P)$ has

$$\pi_1 B(M \to P) \cong \operatorname{coker} \partial, \quad \pi_2 B(M \to P) \cong \ker \partial,$$

and

$$\pi_j B(M \to P) = 0 \quad \text{for } j > 2.$$

Further, any connected CW-complex X with $\pi_j X = 0$ for $j > 2$ is of the homotopy type of $B(M \to P)$ for some crossed module $M \to P$ [**13**, **9**]. Note also that for the crossed module $\chi\colon G \to \operatorname{Aut} G$, the first and second homotopy groups of the classifying space are $\operatorname{Out} G$ and $Z(G)$.

We see that the automorphisms of an algebraic model of a 1-type are naturally considered as an algebraic model of a 2-type. The original motivation for the present work was to investigate whether, for crossed modules, an automorphism structure could be found that could be considered as an algebraic model of a 3-type.

Our derivation of such an automorphism structure employs a procedure of independent interest. Let $\mathbb{C}$ be a monoidal closed category with tensor product $-\otimes -$ and internal hom functor HOM. Thus for any objects A, B, and C of $\mathbb{C}$ there is a natural isomorphism $\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, \mathrm{HOM}(B, C))$. Then for any object C of $\mathbb{C}$ the object $\mathrm{END}(C) = \mathrm{HOM}(C, C)$, together with a canonical map $\mathrm{END}(C) \otimes \mathrm{END}(C) \to \mathrm{END}(C)$, is a *monoid* in $\mathbb{C}$ and in many cases there is a submonoid of $\mathrm{END}(C)$ which can reasonably be labelled $\mathrm{AUT}(C)$. It is this object, with its monoid structure in $\mathbb{C}$, which gives *automorphism structures* for the category $\mathbb{C}$.

In order to treat automorphism structures for crossed modules, we have to embed the category of crossed modules in a larger category which is monoidal closed: we regard a crossed module as a 2-truncated *crossed complex*. We review the necessary facts on crossed modules and crossed complexes over groupoids in § 1 and we indicate the main results on the monoidal closed structure on the category $\mathscr{C}_{\mathscr{R}\mathscr{S}}$ of crossed complexes as given in [**2**]. We also define the additional structural features which identify crossed modules over groupoids that are monoids in $\mathscr{C}_{\mathscr{R}\mathscr{S}}$. These are the *braided* and *semiregular* crossed modules: we borrow the term *braided* from [**7**]. So if C is a crossed module, then $\mathrm{END}(C)$ is braided and semiregular. The automorphism structure $\mathrm{AUT}(C)$ inherits a braiding from $\mathrm{END}(C)$ and a stronger internal symmetry making it braided and *regular*.

We establish the role of $\mathrm{AUT}(C)$ as an *algebraic model of a* 3-*type* in our main technical result.

THEOREM. *The category of braided, regular crossed modules is equivalent to the category of simplicial groups with Moore complex of length* 2.

This theorem occupies the bulk of § 2. The use of simplicial groups as algebraic models of homotopy types is of long standing (see [**5**]). D. Conduché has shown in [**4**] that the category of simplicial groups with Moore complex of length 2 is also equivalent to the category of 2-*crossed modules*. The essence of the resulting equivalence between braided, regular crossed modules and 2-crossed modules is that a braided, regular crossed module contains as a canonical substructure the Moore complex of its equivalent simplicial group. The Moore complex is a 2-crossed module, and determines the braided, regular crossed module up to isomorphism.

The equivalence stated in the theorem also sheds light on the algebraic models of 3-types developed in unpublished work of A. Joyal and M. Tierney. As mentioned in [7], Joyal and Tierney model simply-connected 3-types by *braided, categorical groups*. These are equivalent to braided crossed modules *of groups* in the sense of this paper. Any crossed module of groups is regular, and as a corollary to our theorem we find that the category of braided crossed modules of groups is equivalent to the category of *reduced* simplicial groups with Moore complex of length 2.

In § 3 we return to the investigation of automorphism structures for crossed modules of groups. We give a detailed description of AUT(C) in this case, calling on work of Whitehead [15] (see also [11]) and its extension by K. J. Norrie [14]. By regarding AUT(C) as a 2-crossed module, we compute the homotopy groups of the corresponding 3-type in some special cases. Further, we see that we may also consider the automorphism structure of a crossed module as a *crossed square,* as has been independently observed by Norrie [14]. Crossed squares arose from a study of excision in algebraic K-theory [6]. They also form algebraic models of 3-types [9] and the fundamental crossed square functor satisfies a generalized Van Kampen theorem [3]. Part of the interest of our study is that 2-crossed modules and crossed squares are seen to arise from algebraic considerations.

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1. *Crossed modules and crossed complexes*

We begin with a review of the basic facts that we need on monoidal closed categories. Let $\mathbb{C}$ be a monoidal closed category with tensor product $-\otimes-$, identity object I, and internal hom functor HOM (see [12]). Then for all objects A, B, C of $\mathbb{C}$ there exists a natural isomorphism

$$\theta\colon \mathbb{C}(A\otimes B, C)\to \mathbb{C}(A, \mathrm{HOM}(B, C)), \tag{1.1}$$

which, together with the associativity of the tensor product, implies the existence in $\mathbb{C}$ of a natural isomorphism

$$\Theta\colon \mathrm{HOM}(A\otimes B, C)\to \mathrm{HOM}(A, \mathrm{HOM}(B, C)). \tag{1.2}$$

Further, the isomorphism

$$\theta\colon \mathbb{C}(\mathrm{HOM}(A, B)\otimes A, B)\to \mathbb{C}(\mathrm{HOM}(A, B), \mathrm{HOM}(A, B))$$

shows that there is a unique morphism $\varepsilon_A\colon \mathrm{HOM}(A, B)\otimes A\to B$ such that $\theta(\varepsilon_A)$ is the identity on $\mathrm{HOM}(A, B)$: ε_A is called the *evaluation morphism.* Then for all objects A, B, C of $\mathbb{C}$, there is a morphism

$$(\mathrm{HOM}(B, C)\otimes \mathrm{HOM}(A, B))\otimes A \xrightarrow{\alpha} \mathrm{HOM}(B, C)\otimes(\mathrm{HOM}(A, B)\otimes A)$$

$$\xrightarrow{1\otimes\varepsilon_A} \mathrm{HOM}(B, C)\otimes B \xrightarrow{\varepsilon_B} C.$$

This corresponds under θ to a morphism

$$\gamma_{ABC}\colon \mathrm{HOM}(B, C)\otimes \mathrm{HOM}(A, B)\to \mathrm{HOM}(A, C)$$

which is called *composition.*

We write END(C) for HOM(C, C). There is a morphism η_C: $I \to \text{END}(C)$ corresponding to the morphism λ: $I \otimes C \to C$. The main result we need is the following [**8**].

1.1. PROPOSITION. *The morphism* η_C *and the composition*

$$\mu_C = \gamma_{CCC}\colon \text{END}(C) \otimes \text{END}(C) \to \text{END}(C)$$

make END(C) *a monoid in* $\mathbb{C}$.

Recall that a *groupoid* is a small category in which every arrow is an isomorphism. We write a groupoid as (C_1, C_0), where C_0 is the set of vertices and C_1 is the set of arrows. The set of arrows $p \to q$ from p to q is written $C_1(p, q)$, and p, q are the *source* and *target* of such an arrow. The *source* and *target* maps are written s, t: $C_1 \to C_0$. If $a \in C_1(p, q)$ and $b \in C_1(q, r)$, their composite is written $a + b \in C_1(p, r)$. We write $C_1(p, p)$ as $C_1(p)$. For a survey of applications of groupoids and an introduction to their literature, see [**1**].

Recall from [**2**] that a *crossed complex*

$$C\colon \dots \to C_r \xrightarrow{\delta} C_{r-1} \to \dots \to C_3 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \rightrightarrows C_0$$

consists of a groupoid C_1 with vertex set C_0 and families of groupoids $C_n = \{C_n(p) \mid p \in C_0\}$ over C_0 with each $C_n(p)$ abelian for $n \geq 3$. We shall write the operations in C_n $(n \geq 1)$ additively. The groupoid C_1 acts on each C_n so that for $x \in C_n(p)$ and $a \in C_1(p, q)$ we have $x^a \in C_n(q)$. For $n \geq 2$, δ: $C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1, where C_1 acts on each $C_1(p)$ by conjugation, and for $n \geq 3$, $\delta\delta$: $C_n \to C_{n-1}$ is the zero map. Further, $\delta(C_2)$ acts trivially on C_n for $n \geq 3$, whilst if $x, y \in C_2(p)$ then $y^{\delta(x)} = -x + y + x$. A *morphism* of crossed complexes f: $C \to D$ is a family of morphisms of groupoids f_n: $C_n \to D_n$ $(n \geq 1)$ inducing the same map f_0: $C_0 \to D_0$ and compatible with the maps δ: $C_n \to C_{n-1}$, $D_n \to D_{n-1}$ and the actions of C_1 and D_1.

The above includes the definition of

$$C_2 \xrightarrow{\delta} C_1 \rightrightarrows C_0$$

as a *crossed module over the groupoid* (C_1, C_0), or a *crossed* C_1*-module.* Note in particular that for each $p \in C_0$, $C_2(p) \to C_1(p)$ is a crossed module of groups.

Let U be a monoid. A *biaction* of U on the crossed module

$$C_2 \xrightarrow{\delta} C_1 \rightrightarrows C_0$$

consists of a pair of commuting left and right actions of U on the set C_0 and on the groupoids C_1 and C_2 compatible with all the structure. Specifically we have functions $U \times C_i \to C_i$ and $C_i \times U \to C_i$ for $i = 0, 1, 2$, denoted by $(u, c) \mapsto u \cdot c$ and $(c, u) \mapsto c \cdot u$, such that

BA1: each function $U \times C_i \to C_i$ determines a left action of U and each function $C_i \times U \to C_i$ determines a right action of U and these actions commute;

BA2: each action of U preserves the groupoid structure of C_1 over C_0 and in particular the source and target maps s, t: $C_1 \to C_0$ are U-equivariant relative to each action;

BA3: each action of U preserves the group operations in C_2 and if $x \in C_2(p)$ and $u \in U$ then $u \cdot x \in C_2(u \cdot p)$ and $x \cdot u \in C_2(p \cdot u)$;

BA4: each action of U is compatible with the action of C_1 on C_2 so that if $x \in C_2(p)$, $a \in C_1(p, q)$, and $u \in U$ then

$$u \cdot (x^a) = (u \cdot x)^{u \cdot a} \in C_2(u \cdot q),$$
$$(x^a) \cdot u = (x \cdot u)^{a \cdot u} \in C_2(q \cdot u);$$

BA5: the boundary homomorphism $\delta\colon C_2 \to C_1$ is U-equivariant relative to each action.

The crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \rightrightarrows C_0$$

is *semiregular* if the vertex set C_0 is a monoid and there is a biaction of C_0 on C in which C_0 acts on itself in its left and right regular representations. A semiregular crossed module in which C_0 is a group is said to be *regular.* Note that every crossed module of groups is regular.

Let

$$C\colon C_2 \xrightarrow{\delta} C_1 \rightrightarrows C_0$$

be a semiregular crossed module. We write the monoid C_0 multiplicatively with identity element e. A *braiding* on C is a function $C_1 \times C_1 \to C_2$, written $(a, b) \mapsto \{a, b\}$, which satisfies the following axioms (here $a, a', b, b' \in C_1$, $x, y \in C_2$, and $p, q \in C_0$):

B1: $\{a, b\} \in C_2((ta)(tb))$, $\{0_e, b\} = 0_{tb}$, $\{a, 0_e\} = 0_{ta}$;

B2: $\{a, b + b'\} = \{a, b\}^{ta \cdot b'} + \{a, b'\}$;

B3: $\{a + a', b\} = \{a', b\} + \{a, b\}^{a' \cdot tb}$;

B4: $\delta\{a, b\} = -(ta \cdot b) - a \cdot sb + sa \cdot b + a \cdot tb$;

B5: $\{a, \delta y\} = -(ta \cdot y) + (sa \cdot y)^{a \cdot q}$ if $y \in C_2(q)$;

B6: $\{\delta x, b\} = -(x \cdot sb)^{p \cdot b} + x \cdot tb$ if $x \in C_2(p)$;

B7: $p \cdot \{a, b\} = \{p \cdot a, b\}$,
$\{a, b\} \cdot p = \{a, b \cdot p\}$,
$\{a \cdot p, b\} = \{a, p \cdot b\}$.

EXAMPLE. A braiding on a crossed module of groups

$$C_2 \xrightarrow{\delta} C_1$$

is a function $\{\ ,\ \}\colon C_1 \times C_1 \to C_2$ satisfying the following axioms:

(i) $\{a, b + b'\} = \{a, b\}^{b'} + \{a, b'\}$,
(ii) $\{a + a', b\} = \{a', b\} + \{a, b\}^{a'}$,
(iii) $\delta\{a, b\} = [b, a]$,
(iv) $\{a, \delta y\} = -y + y^a$,
(v) $\{\delta x, b\} = -x^b + x$,

where $a, a', b, b' \in C_1$ and $x, y \in C_2$.

In [7] A. Joyal and R. Street have defined a notion of braiding for an arbitrary monoidal category, and in particular have considered *braided categorical groups*. These amount to braided crossed modules, with the bracket operation in [7] given by $(a, b) \mapsto \{a^{-1}, b\}^a$. This difference is merely one of notational conventions.

The axioms B1, ..., B7 are evidently closely related to the axioms given by D. Conduché [**4**, Axioms 2.9] for the *Peiffer lifting* $M \times M \to L$ in a 2-crossed module $L \to M \to N$. We shall pursue this relationship in § 2.

In [**2**] the category $\mathscr{C}rs$ of crossed complexes is endowed with a tensor product $-\otimes-$ and an internal hom-functor $\mathrm{CRS}(-, -)$ giving $\mathscr{C}rs$ a symmetric, closed, monoidal structure. The tensor product $C \otimes D$ of crossed complexes C and D is generated as a crossed complex by elements $c \otimes d$ in dimension $m+n$ for all $c \in C_m$ and $d \in D_n$. A presentation of $C \otimes D$ is given in [**2**, Proposition 3.10]. An important notion of [**2**] for the present work is that of a *bimorphism* θ: $(A, B) \to C$ of crossed complexes. Here A, B, and C are crossed complexes and θ is a family of maps $A_m \times B_n \to C_{m+n}$. The conditions satisfied by θ are given as (3.4) of [**2**]. The tensor product transforms bimorphisms into morphisms of crossed complexes, so that there is a natural bijection between the set of morphisms of crossed complexes $A \otimes B \to C$ and the set of bimorphisms $(A, B) \to C$.

The crossed complex $\mathrm{CRS}(C, D)$ has as its vertex set $\mathrm{CRS}(C, D)_0$ the set $\mathscr{C}rs(C, D)$ of all morphisms of crossed complexes $C \to D$. For $m \geqslant 1$, $\mathrm{CRS}(C, D)_m$ consists of m-fold left homotopies h: $C \to D$ over morphisms f: $C \to D$: that is, h is a map of degree m such that h_1: $C_1 \to D_{m+1}$ is a derivation and h_r: $C_r \to D_{m+r}$ $(r \geqslant 2)$ is a morphism of groupoids compatible with the actions of C_1 and D_1. Full details of the notion of homotopy and of the crossed complex structure of $\mathrm{CRS}(C, D)$ are given in [**2**].

The basic properties of the tensor product and internal hom-functor are summarized in the following result:

1.2. THEOREM [**2**, Theorem 3.15]. (i) *The functor* $-\otimes B$ *is left adjoint to the functor* $\mathrm{CRS}(B, -)$ *from* $\mathscr{C}rs$ *to* $\mathscr{C}rs$.

(ii) *For crossed complexes* A, B, C *there are natural isomorphisms of crossed complexes*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$
$$\mathrm{CRS}(A \otimes B, C) \cong \mathrm{CRS}(A, \mathrm{CRS}(B, C)).$$

So if C is a crossed complex, we set $\mathrm{END}(C) = \mathrm{CRS}(C, C)$ and, by Proposition 1.1, this is a monoid in $\mathscr{C}rs$ with composition map

$$\gamma\colon\ \mathrm{END}(C) \otimes \mathrm{END}(C) \to \mathrm{END}(C).$$

Note that for any crossed complexes A and B we have $(A \otimes B)_0 = A_0 \times B_0$ and it is easy to see that the function

$$\gamma_0\colon\ \mathscr{C}rs(C, C) \otimes \mathscr{C}rs(C, C) \to \mathscr{C}rs(C, C)$$

is just composition of morphisms in $\mathscr{C}rs$. We can now define the automorphism structure $\mathrm{AUT}(C)$ of C to be the full subcrossed complex of $\mathrm{END}(C)$ on the vertex set $\mathrm{Aut}(C)$ of automorphisms of C in $\mathscr{C}rs$. We obtain by restriction the composition map γ: $\mathrm{AUT}(C) \otimes \mathrm{AUT}(C) \to \mathrm{AUT}(C)$.

If C is an n-truncated crossed complex, so that for $r > n$, C_r is the trivial groupoid on C_0, then maps of degree greater than n in $\mathrm{END}(C)$ are necessarily trivial and hence END(C) and also AUT(C) are n-truncated. In particular, if C is a crossed module (that is, a 2-truncated crossed complex) then AUT(C) is again a crossed module. However, even if C is a crossed module of groups, so that C_0 is a single point, then AUT(C) is a crossed module over a groupoid with vertex set Aut(C), the set of automorphisms of C in the category of crossed modules. So the theory over groupoids is indispensable.

1.3. THEOREM. *A crossed module* C: $C_2 \to C_1 \rightrightarrows C_0$ *over the groupoid* (C_1, C_0) *is a monoid in the category of crossed complexes if and only if it is braided and semiregular.*

Proof. Suppose that C is a monoid in $\mathscr{C}\!\mathscr{rs}$. Denote by $\mathbf{0}$ the crossed complex with one vertex $*$ and the trivial group $\{0\}$ in each dimension greater than or equal to 1. Then we have morphisms of crossed complexes η: $\mathbf{0} \to C$ and μ: $C \otimes C \to C$, where μ corresponds to a bimorphism consisting of a family of maps μ_{ij}: $C_i \times C_j \to C_{i+j}$.

If we rewrite the defining diagrams for the monoid structure on C in terms of the μ_{ij}, we obtain equivalent commutative diagrams

$$(1)\qquad \begin{array}{ccc} C_i \times C_j \times C_k & \xrightarrow{\mu_{ij} \times 1} & C_{i+j} \times C_k \\ {\scriptstyle 1 \times \mu_{jk}} \downarrow & & \downarrow {\scriptstyle \mu_{i+j,k}} \\ C_i \times C_{j+k} & \xrightarrow[\mu_{i,j+k}]{} & C_{i+j+k} \end{array}$$

$$(2)\qquad \begin{array}{ccccc} \mathbf{0}_i \times C_j & \xrightarrow{\eta_i \times 1} & C_i \times C_j & \xleftarrow{1 \times \eta_j} & C_i \times \mathbf{0}_j \\ & {\scriptstyle \lambda_{ij}} \searrow & \downarrow {\scriptstyle \mu_{ij}} & \swarrow {\scriptstyle \rho_{ij}} & \\ & & C_{i+j} & & \end{array}$$

Note that if $l > 2$ then C_l is the trivial groupoid on the vertex set C_0, so all maps to C_l (whether maps of crossed complexes or bimorphisms) are trivial but preserve basepoints in C_0.

Plainly the above diagrams with $i = j = k = 0$ exhibit a monoid structure on the set C_0, with associative multiplication μ_{00}: $C_0 \times C_0 \to C_0$, which we write as juxtaposition, and identity $\eta_0(*)$, which we write as e.

For $p \in C_0$, $a \in C_1$, and $x \in C_2$ we set

$$p \cdot a = \mu_{01}(p, a), \quad a \cdot p = \mu_{10}(a, p), \quad p \cdot x = \mu_{02}(p, x), \quad x \cdot p = \mu_{20}(x, p).$$

It follows from appropriate choices of i, j, and k in (1) and (2) that we have left and right monoid actions of C_0 on the sets C_1 and C_2, and that these actions commute. For example, setting $(i, j, k) = (2, 0, 0)$ in (1) shows that $x \cdot (pq) = (x \cdot p) \cdot q$ for all $x \in C_2$ and $p, q \in C_0$, whilst setting $(i, j) = (0, 1)$ in (1) gives $e \cdot a = a$ for all $a \in C_1$ in the left-hand part. Further, from (2) we find that the identities in C_1 and C_2 are transformed amongst themselves by the actions of C_0.

It follows from the defining formulae for a bimorphism given in **[2]** that μ does induce a biaction of C_0 on the crossed module C and that C is semiregular. Further, μ_{11}: $C_1 \times C_1 \to C_2$ does define a braiding on C. Axiom B7 is obtained from diagram (1) by choosing precisely one of i, j, k to be 0 and the remaining two to be 1. The second part of Axiom B1 is obtained from diagram (2) by choosing $i=1=j$. The remaining axioms for a braiding follow from the bimorphism formulae for μ_{11}.

Conversely, it is straightforward to check that given a semiregular crossed module with a braiding we can reverse the procedure outlined above to obtain a bimorphism $\{\mu_{ij}: C_i \times C_j \to C_{i+j}\}$ and a map of crossed complexes $\eta: \mathbf{0} \to C$ making (1) and (2) commute.

2. *Braided, regular crossed modules and simplicial groups*

Let $G.$ be a simplical group with face maps d_i and degeneracy maps s_j. Recall that the *Moore complex* $N(G.)$ of $G.$ is defined by

$$N(G.)_0 = G_0 \quad \text{and} \quad N(G.)_m = \bigcap_{i=0}^{m-1} \ker(d_i\colon G_m \to G_{m-1})$$

with boundary $N(G.)_m \to N(G.)_{m-1}$ given by restricting d_m. Simplicial groups form algebraic models of homotopy types via the functors $\bar{W}$ and geometric realization, and the homotopy groups of the CW-complex obtained from $G.$ are the homology groups of the Moore complex $N(G.)$. We refer to [**5**] for further details. If the Moore complex of $G.$ is trivial in dimensions greater than n, then $G.$ will model an $(n+1)$-type.

D. Conduché [**4**] gave necessary and sufficient conditions for a simplicial group to be determined by a truncated simplicial group.

2.1. THEOREM [**4**, Theorem 1.5]. *Let $G_.'$ be an n-truncated simplicial group. There exists a simplicial group $G.$ with $G_j = G_j'$ for $0 \leq j \leq n$ and with $N(G.)_m = 0$ for $m > n$ if and only if G_n' satisfies the following condition:*

(*) *for every partition of $\{0, \ldots, n\}$ into non-empty subsets I and J, the subgroups $\bigcap_{i\in I} \ker d_i$ and $\bigcap_{j\in J} \ker d_j$ commute elementwise. Further, such a $G.$ is unique up to isomorphism.*

In what follows we shall construct functors which are naturally defined on, or take values in, the category of 2-truncated simplicial groups. Conduché's theorem tells us that these functors extend to, or restrict from, the category of simplicial groups, provided Condition (*) holds where appropriate.

We now state the main theorem of this section.

2.2. THEOREM. *The category $\mathcal{BRCM}$ of braided, regular crossed modules is equivalent to the category $\mathcal{SG}^{(2)}$ of simplicial groups with Moore complex of length* 2.

The proof of this theorem will occupy us for some time, and we shall approach it through a series of subsidiary results. We begin with a simplicial group $G.$ and define the structural components of a braided, regular crossed module; these components are seen to satisfy the axioms B1, ..., B7 if the 2-truncation of $G.$ satisfies Conduché's condition (*).

Let $G.$ be a simplicial group as above. Then G_1 is the semidirect product $\ker d_0 \rtimes s_0G_0$. An element $g \in G_1$ can be written $g = g(s_0d_0g^{-1})(s_0d_0g)$ where $g(s_0d_0g^{-1}) \in \ker d_0$ and $s_0d_0g \in s_0G_0$. Let us write $\tilde{g}$ (or $(g)^{\sim}$ if g is a lengthy expression) for the element $g(s_0d_0g^{-1})$: then we observe the following two results.

2.3. LEMMA. *The set G_1 admits a groupoid structure with vertex set G_0, source and target maps d_1 and d_0 respectively, and composition $g+h=\bar{g}h$ defined if $d_0g=d_1h$.*

2.4. LEMMA. *The group G_0 acts on the set G_1 via the degeneracy s_0: for $p \in G_0$ and $g \in G_1$ we set*

$$p \cdot g=(s_0p)g, \quad g \cdot p=g(s_0p).$$

Then $(p \cdot g)\bar{}=(s_0p)\bar{g}(s_0p)^{-1}$, $(g \cdot p)\bar{}=\bar{g}$, and these actions, together with the left and right regular actions of G_0 on itself, give an action of G_0 on the groupoid (G_1, G_0).

If $x \in G_2$, we write $\bar{x}$ for $x(s_0d_0x)^{-1}$. We set

$$C_2=(\ker d_0 \cap \ker d_1) \rtimes s_0s_0G_0 \subseteq G_2:$$

then C_2 is partitioned into parts $C_2(p)$ where $p \in G_0$ and

$$C_2(p)=\{x \in G_2 |\ d_0x=s_0p=d_1x\}.$$

2.5. LEMMA. *The set $C_2(p)$ becomes a group if we define $x+y=\bar{x}y$ with identity element s_0s_0p and with $-x=\bar{x}^{-1}(s_0d_0x)$. Furthermore, G_0 acts on the left and on the right of the groupoid (C_2, G_0) where $q \cdot x=(s_0s_0q)x$ and $x \cdot q=x(s_0s_0q)$, so that if $x \in C_2(p)$ then $q \cdot x \in C_2(qp)$ and $x \cdot q=C_2(pq)$.*

2.6. PROPOSITION. *Let $g \in G_1(p, q)$ and let $x \in C_2(p)$. Set*

$$x^g=d_0g \cdot ((s_1g)^{-1}\bar{x}(s_1g)).$$

Then $(x, g) \mapsto x^g$ is an action of G_1 on C_2 and C_2 is a regular, precrossed G_1-module with boundary homomorphism $d_2|C_2$ and biaction of G_0 given by Lemma 2.5.

Proof. It is easy to check that d_2 maps $C_2(p)$ into $G_1(p)$ and is a homomorphism of groupoids and, further, that $d_0(x^g)=s_0q=d_1(x^g)$ so that $x^g \in C_2(q)$. We now verify that we do have an action of G_1 on C_2. So if $x, y \in C_2(p)$, $g \in G_1(p, q)$, and $h \in G_1(q, r)$, we have

$$\begin{aligned}
x^g+y^g &= d_0g \cdot ((s_1g)^{-1}\bar{x}(s_1g))+d_0g \cdot ((s_1g)^{-1}\bar{y}(s_1g)) \\
&= d_0g \cdot ((s_1g)^{-1}\bar{x}(s_1g)+(s_1g)^{-1}\bar{y}(s_1g)) \\
&= d_0g \cdot (((s_1g)^{-1}\bar{x}(s_1g))\bar{}(s_1g)^{-1}\bar{y}(s_1g)) \\
&= d_0g \cdot ((s_1g)^{-1}(\bar{x}(s_1g))\bar{}(s_1g)(s_1g)^{-1}(s_1g)\bar{}^{-1}\bar{y}(s_1g)) \\
&= d_0g \cdot ((s_1g)^{-1}\bar{x}(s_1g)\bar{}\bar{x}^{-1}\bar{\bar{x}}(s_1g)(s_1g)^{-1}(s_1g)\bar{}^{-1}(s_1g)(s_1g)^{-1}\bar{y}(s_1g)) \\
&= d_0g \cdot ((s_1g)^{-1}(x+y)\bar{}(s_1g)) \\
&= (x+y)^g.
\end{aligned}$$

Similarly, we check that $x^{g+h}=(x^g)^h$. It is then straightforward to verify the remaining assertions of the proposition.

2.7. Proposition. *The precrossed G_1-module d_2: $C_2 \to G_1$ is a crossed module if and only if*

$$[\ker d_0 \cap \ker d_1, (\ker d_0 \cap \ker d_1)s_1G_1 \cap \ker d_2] = 1$$

in G_2.

Proof. Let $x, y \in C_2(p)$. Calculation shows that $-x + y + x = p \cdot (x^{-1}\bar{y}x)$ whereas $y^{d_2x} = p \cdot (s_1d_2x^{-1}\bar{y}s_1d_2x)$. So we have a crossed module if and only if, for each $p \in G_0$ and for all $x, y \in C_2(p)$, $[xs_1d_2x^{-1}, \bar{y}] = 1$. Now $xs_1d_2x^{-1} = \bar{x}s_1d_2\bar{x}^{-1}$, so equivalently, we have a crossed module if and only if $[\bar{x}s_1d_2\bar{x}^{-1}, \bar{y}] = 1$. Observe that

$$\{\bar{y} \mid y \in C_2(p)\} = \ker d_0 \cap \ker d_1$$

and that the set $\{\bar{x}s_1d_2\bar{x}^{-1} \mid x \in C_2(p)\}$ consists of those elements of $\ker d_2$ occurring in expressions of elements of $\ker d_0 \cap \ker d_1$ relative to the semidirect product decomposition of G_2 as $\ker d_2 \rtimes s_1G_1$. It follows that

$$\{\bar{x}s_1d_2\bar{x}^{-1} \mid x \in C_2(p)\} = (\ker d_0 \cap \ker d_1)s_1G_1 \cap \ker d_2.$$

2.8. Proposition. *For $g, h \in G_1$, denote by $\{g, h\}$ the element*

$$d_0g \cdot [s_1\bar{h}, s_0g^{-1}s_1g] \cdot d_0h$$

of G_2. Then the function $(g, h) \mapsto \{g, h\}$ satisfies Axioms B1, B2, B4, *and* B7.

Proof. For B1, observe that $d_0\{g, h\} = s_0(d_0gd_0h) = d_1\{g, h\}$ since d_0 kills the first term of the commutator in $\{g, h\}$ and d_1 kills the second term. So $\{g, h\} \in C_2(d_0gd_0h)$ and plainly $\{1, h\} = s_0s_0d_0h$ and $\{g, 1\} = s_0s_0d_0g$.

We shall verify B2, leaving B4 and B7 to the reader. Let $g, h, k \in G_1$ with $d_0h = d_1k$. Then

$$\begin{aligned}
\{g, h + k\} &= \{g, \bar{h}k\} \\
&= d_0g \cdot [s_1\bar{h}s_1\bar{k}, s_0g^{-1}s_1g] \cdot d_0(h + k) \\
&= d_0g \cdot (s_1\bar{k}^{-1}[s_1\bar{h}, s_0g^{-1}s_1g]s_1\bar{k}[s_1\bar{k}, s_0g^{-1}s_1g]) \cdot d_0k \\
&= d_0g \cdot (s_1\bar{k}^{-1}[s_1\bar{h}, s_0g^{-1}s_1g]s_1\bar{k} + [s_1\bar{k}, s_0g^{-1}s_1g]) \cdot d_0k \\
&= d_0g \cdot (s_1\bar{k}^{-1}[s_1\bar{h}, s_0g^{-1}s_1g]s_1\bar{k}) \cdot d_0k + \{g, k\} \\
&= d_0g \cdot (s_1\bar{k}^{-1}(d_0g^{-1} \cdot \{g, h\} \cdot d_0h^{-1})s_1\bar{k}) \cdot d_0k + \{g, k\} \\
&= d_0g \cdot (s_1\bar{k}^{-1}s_0s_0d_0g^{-1}(\{g, h\} \cdot d_0h^{-1} \cdot d_0g^{-1})s_0s_0d_0gs_1\bar{k}) \cdot d_0k + \{g, k\} \\
&= d_0g \cdot (s_1(d_0g \cdot \bar{k})^{-1}\{g, h\}^{-}s_1(d_0g \cdot \bar{k})) \cdot d_0k + \{g, k\} \\
&= d_0g \cdot (s_1(d_0g \cdot \bar{k})^{-1}(\{g, h\} \cdot d_0k^{-1})^{-}s_1(d_0g \cdot \bar{k})) \cdot d_0k + \{g, k\} \\
&= ((\{g, h\} \cdot d_0k^{-1})^{d_0g \cdot \bar{k}}) \cdot d_0k + \{g, k\} \\
&= (\{g, h\} \cdot d_0k^{-1} \cdot d_0k)^{d_0g \cdot \bar{k} \cdot d_0k} + \{g, k\} \\
&= \{g, h\}^{d_0g \cdot k} + \{g, k\},
\end{aligned}$$

which completes the verification of B2.

We now determine the additional assumptions on G. that will ensure that the function $(g, h) \mapsto \{g, h\}$ is a braiding. Let $K_1 \subseteq G_2$ be the subgroup of $\ker d_1$

generated by all elements $s_0g^{-1}s_1g$ where $g \in G_1$, and let $K_2 \subseteq G_2$ be the subgroup of $\ker d_2$ generated by all elements $s_0g^{-1}s_1s_0d_1g$ where $g \in G_1$.

2.9. PROPOSITION. *If in* G_2 *we have*

$$[K_1, \ker d_0 \cap \ker d_2] = 1 \quad \textit{and} \quad [\ker d_0 \cap \ker d_1, K_2] = 1$$

then the function $(g, h) \mapsto \{g, h\}$ *satisfies Axiom* B5. *If*

$$[s_1(\ker d_0), \ker d_1 \cap \ker d_2] = 1$$

in G_2 *then the function* $(g, h) \mapsto \{g, h)$ *satisfies Axiom* B6.

Proof. Note that $\ker d_0 \cap \ker d_2$ consists of elements $zs_1d_2z^{-1}$ where $z \in \ker d_0 \cap \ker d_1$: then if $g \in G_1$ and $y \in C_2(q)$,

$$\begin{aligned}
\{g, d_2y\} &= d_0g \cdot [s_1d_2y, s_0g^{-1}s_1g] \cdot d_0d_2y \\
&= d_0g \cdot [s_1d_2\tilde{y}, s_0g^{-1}s_1g] \cdot q \\
&= d_0g \cdot [\tilde{y}, s_0g^{-1}s_1g] \cdot q \\
&= s_0s_0d_0g\tilde{y}^{-1}s_1g^{-1}s_0g\tilde{y}s_0g^{-1}s_1gs_0s_0q \\
&= s_0s_0d_0g\tilde{y}^{-1}s_1g^{-1}s_1s_0d_1g\tilde{y}s_1s_0d_1g^{-1}s_1gs_0s_0q \\
&= s_0s_0d_0g(-y)\tilde{}s_0s_0d_0g^{-1}s_0s_0d_0gs_0s_0qs_1s_0q^{-1}s_1g^{-1}(d_1g \cdot y)\tilde{}s_1gs_0s_0q \\
&= (d_0g \cdot (-y))\tilde{}(d_0g \cdot q \cdot (s_1(g \cdot q)^{-1}(d_1g \cdot y)\tilde{}s_1(g \cdot q))) \\
&= -(d_0g \cdot y) + (d_1g \cdot y)^{g \cdot q},
\end{aligned}$$

and this is Axiom B5.

The verification of Axiom B6 under the condition

$$[s_1(\ker d_0), \ker d_1 \cap \ker d_2] = 1$$

proceeds similarly, and so we omit the details.

2.10. PROPOSITION. *If*

$$[s_0(\ker d_1), [s_1(\ker d_0), K_1]] = 1 = [s_1(\ker d_0), [s_0(\ker d_1), K_1]]$$

in G_2, *then the function* $(g, h) \mapsto \{g, h\}$ *satisfies Axiom* B3.

Proof. Let $g, h, k \in G_1$ with $d_0g = d_1h$. Then

$$\begin{aligned}
\{g + h, k\} &= \{\bar{g}h, k\} \\
&= d_0(\bar{g}h) \cdot [s_1\bar{k}, s_0(\bar{g}h)^{-1}s_1(\bar{g}h)] \cdot d_0k \\
&= d_0h \cdot [s_1\bar{k}, s_0h^{-1}s_0\bar{g}^{-1}s_1\bar{g}s_0hs_0h^{-1}s_1h] \cdot d_0k \\
&= d_0h \cdot ([s_1\bar{k}, s_0h^{-1}s_1h]s_1h^{-1}s_0h[s_1\bar{k}, s_0h^{-1}s_0\bar{g}^{-1}s_1\bar{g}s_0h]s_0h^{-1}s_1h) \cdot d_0k \\
&= d_0h \cdot ([s_1\bar{k}, s_0h^{-1}s_1h] + s_1h^{-1}s_0h[s_1\bar{k}, s_0h^{-1}s_0\bar{g}^{-1}s_1\bar{g}s_0h]s_0h^{-1}s_1h) \cdot d_0k \\
&= \{h, k\} + d_0h \cdot (s_1h^{-1}s_0h[s_1\bar{k}, s_0h^{-1}s_0\bar{g}^{-1}s_1\bar{g}s_0h]s_0h^{-1}s_1h) \cdot d_0k.
\end{aligned}$$

Now consider the commutator in the second term:

$$\begin{aligned}
&[s_1\bar{k}, s_0h^{-1}s_0\bar{g}^{-1}s_1\bar{g}s_0h] \\
&\quad = [s_1\bar{k}, s_0h^{-1}s_0s_0d_0gs_0g^{-1}s_1gs_1s_0d_0g^{-1}s_0h] \\
&\quad = [s_1\bar{k}, s_0h^{-1}s_0s_0d_1hs_0g^{-1}s_1gs_0s_0d_1h^{-1}s_0h] \\
&\quad = [s_1\bar{k}, [s_0s_0d_1h^{-1}s_0h, s_1g^{-1}s_0g]s_0g^{-1}s_1g] \\
&\quad = [s_1\bar{k}, s_0g^{-1}s_1g]s_1g^{-1}s_0g[s_1\bar{k}, [s_0s_0d_1h^{-1}s_0h, s_1g^{-1}s_0g]]s_0g^{-1}s_1g \\
&\quad = [s_1\bar{k}, s_0g^{-1}s_1g],
\end{aligned}$$

since

$$[s_1\tilde{k}, [s_0s_0d_1h^{-1}s_0h, s_1g^{-1}s_0g]] \in [s_1(\ker d_0), [s_0(\ker d_1), K_1]].$$

Thus

$$\{g+h, k\} = \{h, k\} + d_0h \cdot (s_1h^{-1}s_0h[s_1\tilde{k}, s_0g^{-1}s_1g]s_0h^{-1}s_1h) \cdot d_0k$$

$$= \{h, k\} + d_0h \cdot (s_1h^{-1}s_0s_0d_1h[s_1\tilde{k}, s_0g^{-1}s_1g]s_0s_0d_1h^{-1}s_1h) \cdot d_0k,$$

since $[s_0(\ker d_1), [s_1(\ker d_0), K_1]] = 1$. But since $d_0g = d_1h$, we have

$$\{g+h, k\} = \{h, k\} + d_0h \cdot (s_1h^{-1}s_0s_0d_0g[s_1\tilde{k}, s_0g^{-1}s_1g]s_0s_0d_0g^{-1}s_1h) \cdot d_0k$$

$$= \{h, k\} + d_0h \cdot (s_0s_0d_0ks_0s_0d_0k^{-1}s_1h^{-1}(d_0g \cdot [s_1\tilde{k}, s_0g^{-1}s_1g]$$

$$\cdot d_0k \cdot d_0k^{-1} \cdot d_0g^{-1})s_1hs_0s_0d_0k)$$

$$= \{h, k\} + (d_0hd_0k) \cdot (s_1(h \cdot d_0k)^{-1}(\{g, k\} \cdot d_0k^{-1} \cdot d_0g^{-1})s_1(h \cdot d_0k))$$

$$= \{h, k\} + (d_0hd_0k) \cdot (s_1(h \cdot d_0k)^{-1}\{g, k\}s_1(h \cdot d_0k))$$

$$= \{h, k\} + \{g, k\}^{h \cdot d_0k},$$

which is Axiom B3.

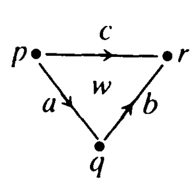
We have now described a functor Θ: $\mathcal{SG}^{(2)} \to \mathcal{BRCM}$ (by virtue of Theorem 2.1). We shall now go on to explain the construction of a functor Δ: $\mathcal{BRCM} \to \mathcal{SG}^{(2)}$ and then prove that Θ and Δ give an equivalence of categories.

Let C: $(C_2 \to C_1 \rightrightarrows C_0)$ be a braided, regular crossed module. Then $C_1 \rightrightarrows C_0$ is a 1-truncated simplicial group with degeneracy s_0: $C_0 \to C_1$ taking $p \mapsto 0_p$ and group structure on C_1 given by $ab = a \cdot sb + ta \cdot b$.

We set

$$G_2 = \{(w\,; a, b, c) \mid w \in C_2, a, b, c \in C_1, \delta w = a + b - c, sa = sc\}.$$

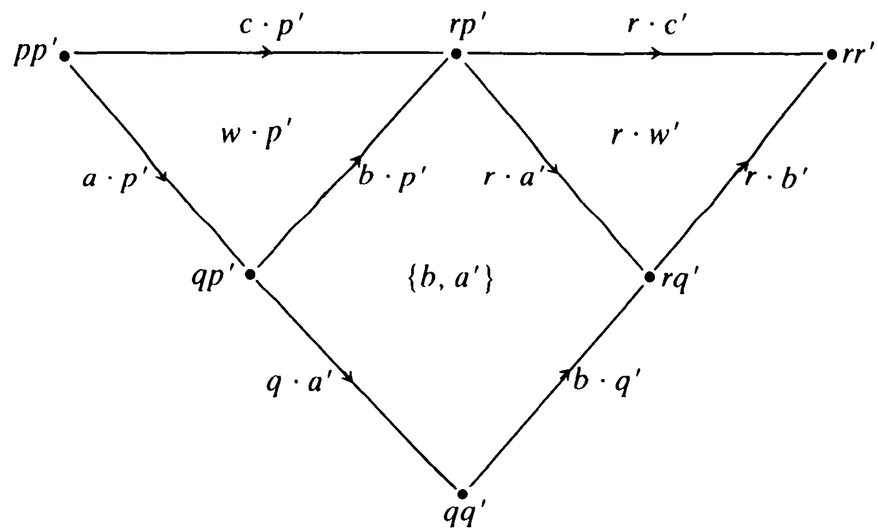
Face maps $G_2 \to C_1$ are defined by $d_0(w\,; a, b, c) = b$, $d_1(w\,; a, b, c) = c$, and $d_2(w\,; a, b, c) = a$, with degeneracies $s_0g = (0_{sg}\,; 0_{sg}, g, g)$ and $s_1g = (0_{tg}\,; g, 0_g, g)$. We picture an element of G_2 as



where $sa = p = sc$, $ta = q = sb$, and $tb = r = tc$. A multiplication on G_2 is given by

$$(w\,; a, b, c)(w'\,; a', b', c') = (w''\,; aa', bb', cc'),$$

where w'' is essentially defined by the diagram



More precisely, we require that $\delta w''$ is the boundary of the diagram, and this condition is satisfied when we take

$$w'' = w \cdot p' + \{b, a'\}^{-(c \cdot p' + r \cdot a')} + (r \cdot w')^{-c \cdot p'}.$$

It is clear that $(0_e\,; 0_e, 0_e, 0_e)$ is an identity, and that $(w\,; a, b, c)^{-1} = (\bar{w}\,; a^{-1}, b^{-1}, c^{-1})$ where $\bar{w}$ is determined by

$$0_e = w \cdot p^{-1} + \{b, a^{-1}\}^{-(c \cdot p^{-1} + r \cdot a^{-1})} + (r \cdot \bar{w})^{-c \cdot p^{-1}}.$$

It is not immediately apparent that the multiplication just defined is associative. Let $x_i = (w_i\,; a_i, b_i, c_i) \in G_2$ where $i = 0, 1, 2$. Then

$$x_0(x_1x_2) = (u\,; a_0a_1a_2, b_0b_1b_2, c_0c_1c_2),$$
$$(x_0x_1)x_2 = (v\,; a_0a_1a_2, b_0b_1b_2, c_0c_1c_2),$$

where

$$\begin{aligned} u = {} & w_0 \cdot p_1p_2 + \{b_0, a_1 \cdot p_2 + q_1 \cdot a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot a_1 \cdot p_2 + r_0q_1 \cdot a_2)} \\ & + (r_0 \cdot w_1 \cdot p_2)^{-c_0 \cdot p_1p_2} + \{r_0 \cdot b_1, a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot a_1 \cdot p_2 + r_0q_1 \cdot a_2)} \\ & + (r_0r_1 \cdot w_2)^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2)} \end{aligned}$$

and

$$\begin{aligned} v = {} & w_0 \cdot p_1p_2 + \{b_0, a_1 \cdot p_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot a_1 \cdot p_2)} + (r_0 \cdot w_1 \cdot p_2)^{-c_0 \cdot p_1p_2} \\ & + \{b_0 \cdot q_1 + r_0 \cdot b_1, a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)} \\ & + (r_0r_1 \cdot w_2)^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2)}. \end{aligned}$$

Expanding the $\{\ ,\ \}$-terms using B2 and B3, we find that $u = v$ if and only if

$$\begin{aligned} & (r_0 \cdot w_1 \cdot p_2)^{-c_0 \cdot p_1p_2} + \{r_0 \cdot b_1, a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)} \\ & \qquad + \{b_0 \cdot q_1, a_2\}^{r_0 \cdot b_1 \cdot q_2 - (c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)} \\ & \quad = \{b_0 \cdot q_1, a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)} \\ & \qquad + (r_0 \cdot w_1 \cdot p_2)^{-c_0 \cdot p_1p_2} + \{r_0 \cdot b_1, a_2\}^{-(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)}. \end{aligned}$$

Write y for the sum of the first two terms on the left-hand side and $g = -(c_0 \cdot p_1p_2 + r_0 \cdot c_1 \cdot p_2 + r_0r_1 \cdot a_2)$. Then $u = v$ if and only if

$$y + \{b_0 \cdot q_1, a_2\}^{r_0 \cdot b_1 \cdot q_2 + g} = \{b_0 \cdot q_1, a_2\}^{g} + y,$$

that is, if and only if

$$\{b_0 \cdot q_1, a_2\}^{r_0 \cdot b_1 \cdot q_2 + g} = -y + \{b_0 \cdot q_1, a_2\}^g + y = \{b_0 \cdot q_1, a_2\}^{g + \delta y}.$$

It is now a trivial matter to check, using CM1 and B4, that $r_0 \cdot b_1 \cdot q_2 + g = g + \delta y$.

This completes the definition of a 2-truncated simplicial group from a braided, regular crossed module. It remains to check the commutator conditions of Theorem 2.1. Let $x, y \in G_2$: there are three cases to consider:

Case 1. $d_0 x = 0_e$, $d_1 y = 0_e = d_2 y$.

Case 2. $d_1 x = 0_e$, $d_0 y = 0_e = d_2 y$.

Case 3. $d_2 x = 0_e$, $d_0 y = 0_e = d_1 y$.

We give the details for Case 1. So $x = (w\,; a, 0_e, c)$, $y = (w'\,; 0_e, b, 0_e)$, and $xy = yx$ if and only if

$$w + w'^{-c} = w' \cdot p + \{b, a\}^{-a} + w.$$

Now $\delta w = a - c$ and $\delta w' = b$ and using B6 and CM2 we see that

$$\begin{aligned} w'^{-c} &= -w + w'^{-a} + w \\ &= -w + w' \cdot p + (-(w' \cdot p)^a + w')^{-a} + w \\ &= -w + w' \cdot p + \{b, a\}^{-a} + w, \end{aligned}$$

and the desired conclusion follows.

Proof of Theorem 2.2. We shall show that the functors Θ: $\mathcal{SG}^{(2)} \to \mathcal{BRCM}$ and Δ: $\mathcal{BRCM} \to \mathcal{SG}^{(2)}$ give an equivalence of categories. Let

$$C\colon\ (C_2 \to C_1 \rightrightarrows C_0)$$

be a braided, regular crossed module. Then $\Delta(C)$ is a simplicial group with Moore complex of length 2: we write $\Delta(C) = G.$ and $\Theta(G.)$ as B: $(B_2 \to B_1 \rightrightarrows B_0)$. From the definitions of Θ and Δ we see at once that $C_0 = B_0$, $C_1 = B_1$ and that the source and target maps in B are the same as those in C. Now $G_1 = C_1$ with group operation $ab = a \cdot sb + ta \cdot b$ and the groupoid operation in B is

$$\begin{aligned} a + b &= a s_0 d_0(a)^{-1} b \\ &= a 0_{ta^{-1}} b \\ &= (a \cdot ta^{-1}) b \\ &= a \cdot ta^{-1} \cdot sb + t(a \cdot ta^{-1}) \cdot b \\ &= a + b, \end{aligned}$$

where + on the right-hand side denotes the groupoid operation in C. Hence the groupoids (C_1, C_0) and (B_1, B_0) are identical. Now

$$G_2 = \{(w\,; a, b, c) \mid w \in C_2,\ a, b, c \in C_1,\ \delta w = a + b - c,\ sa = sc\}$$

and so

$$\begin{aligned} B_2(p) &= \{(w\,; a, 0_p, 0_p) \mid w \in C_2,\ a \in C_1,\ \delta w = a,\ sa = p = ta\} \\ &\cong \{w \in C_2 \mid s\delta(w) = p = t\delta(w)\} \\ &= \{w \in C_2 \mid \delta s(w) = p = \delta t(w)\} \\ &= \{w \in C_2 \mid s(w) = p = t(w)\} \\ &= C_2(p). \end{aligned}$$

The isomorphism here is *a priori* a bijection of sets, but is readily seen to be an isomorphism of groups: we shall show that it is an isomorphism of crossed modules over (C_1, C_0). We consider θ: $C_2 \to B_2$ where $\theta|C_2(p)$ maps w to $(w\,; \delta w, 0_p, 0_p)$. The boundary of B is β: $(w\,; a, b, c) \mapsto a$ so that $\beta\theta = \delta$. Recall that in G_2,

$$(w\,; a, b, c)(w'\,; a', b', c') = (w''\,; aa', bb', cc'),$$

where $w'' = w \cdot p' + \{b, a'\}^{-(c\cdot p' + r\cdot a')} + (r \cdot w')^{-c\cdot p'}$. Let $x \in C_2(p)$ and $a \in C_1(p, q)$. Then

$$\theta(x)^a = ta \cdot (s_1(a)^{-1}(\theta(x) \cdot t\theta(x)^{-1})s_1(a)).$$

Now $t\theta(x) = p$ and $s_0s_0p = (0_p\,; 0_p, 0_p, 0_p)$ so that

$$\theta(x) \cdot t\theta(x)^{-1} = (x \cdot p^{-1}\,; \delta x \cdot p^{-1}, 0_e, 0_e):$$

further, $s_1(a) = (0_p\,; a, 0_q, a)$ and $s_1(a)^{-1} = (0_{p^{-1}}\,; a^{-1}, 0_{q^{-1}}, a^{-1})$. It follows that

$$s_1(a)^{-1}(\theta(x) \cdot t\theta(x)^{-1})s_1(a) = (q^{-1} \cdot x^a\,; q^{-1} \cdot (-a + \delta x + a), 0_e, 0_e)$$

and thus that $\theta(x)^a = (x^a\,; -a + \delta x + a, 0_q, 0_q) = \theta(x^a)$, whence θ is an isomorphism of crossed modules. It is an easy matter to verify that θ preserves the actions of $C_0 = B_0$ and so is an isomorphism of regular crossed modules. Finally, we check that θ preserves the braiding, that is, $\theta\{a, b\}_C = \{\theta(a), \theta(b)\}_B$ (where $\{\ ,\ \}_C$ and $\{\ ,\ \}_B$ are the braidings on C and B). Now $\{a, b\}_B = ta \cdot [s_1(b \cdot tb^{-1}), s_0a^{-1}s_1a] \cdot tb$, where the commutator is evaluated in the group G_2. We find that

$$\begin{aligned} s_1(b \cdot tb^{-1}) &= (0_{sb}\,; b \cdot tb^{-1}, 0_e, b \cdot tb^{-1}),\\ s_1(b \cdot tb^{-1})^{-1} &= (0_{sb^{-1}}\,; -b \cdot sb^{-1}, 0_e, -b \cdot sb^{-1}),\\ s_0as_1a^{-1} &= (\{a^{-1}, a\}\,; sa^{-1} \cdot a, a^{-1} \cdot ta, 0_e),\\ s_1as_0a^{-1} &= (0_e\,; a^{-1} \cdot sa, ta^{-1} \cdot a, 0_e), \end{aligned}$$

and that

$$\begin{aligned}{} [s_1(b \cdot tb^{-1}), s_0a^{-1}s_1a] = (\{ta^{-1} \cdot a, b \cdot tb^{-1}\}\,; &-b \cdot tb^{-1} - ta^{-1} \cdot a \cdot sb \cdot tb^{-1}\\ &+ ta^{-1} \cdot sa \cdot b \cdot tb^{-1} + ta^{-1} \cdot a, 0_e, 0_e), \end{aligned}$$

whence

$$\{a, b\}_B = (\{a, b\}_C\,; -ta \cdot b - a \cdot sb + sa \cdot b + a \cdot tb, 0_{tatb}, 0_{tatb}) = \theta(\{a, b\}_C).$$

We now have an isomorphism θ: $C \to \Theta\Delta(C)$ in $\mathcal{BRCM}$, differing from the identity only on C_2 and there defined using only the boundary of C, so naturality is immediate.

Now let $G.$ be a simplicial group with Moore complex of length 2. We write $\Theta(G.) = (C_2 \to C_1 \rightrightarrows C_0)$ and $\Delta\Theta(G.) = H.$. It is easy to check that $G_0 = H_0$, $G_1 = H_1$, and that the simplicial group structures of the 1-truncations of $G.$ and $H.$ are identical. We now define a function $G_2 \to H_2$. This will 'fold' an element $x \in G_2$ into the group $C_2(d_1d_1x)$: the folding of x, together with the faces of x, then gives an element of H_2.

Let $x \in G_2$. We define

$$\phi(x) = xs_0d_0x^{-1}s_1d_0xs_1d_1x^{-1}s_0s_0d_1d_1x.$$

Then $d_0\phi(x) = s_0d_1d_1x = d_1\phi(x)$ and

$$d_2\phi(x) = d_2xs_0d_0d_2x^{-1}d_0xd_1x^{-1}s_0d_1d_1x = d_2x + d_0x - d_1x \in G_1.$$

Now we define Φ: $G_2 \to H_2$ by $\Phi(x) = (\phi(x)\,; d_2x, d_0x, d_1x)$ and we claim that Φ is a homomorphism of groups. Take $x \in G_2$ as above: we write $g = d_2x$, $h = d_0x$, $k = d_1x$, $p = d_1g = d_1k$, $q = d_1h = d_0g$, and $r = d_0k = d_0h$. Similarly, for $x' \in G_2$ we have g', h', k', p', q', and r'. Then

$$\begin{aligned}\Phi(x)\Phi(x') &= (\phi(x)\,; d_2x, d_0x, d_1x)(\phi(x')\,; d_2x', d_0x', d_1x')\\ &= (x''\,; d_2(xx'), d_0(xx'), d_1(xx')),\end{aligned}$$

where

$$\begin{aligned}x'' &= \phi(x)\cdot p' + \{h, g'\}^{-(k\cdot p' + r\cdot g')} + (r\cdot\phi(x'))^{-k\cdot p'}\\
&= \phi(x)\cdot p' + \{h, g'\}^{rq'\cdot(kg')^{-1}\cdot pp'} + (r\cdot\phi(x'))^{r\cdot k^{-1}\cdot pp'}\\
&= \phi(x)\cdot p' + pp'\cdot(s_1(s_0(pp')^{-1}kg's_0(rq')^{-1})(\{h, g'\}\cdot(rq')^{-1})\\
&\quad\cdot s_1(s_0(rq')g'^{-1}k^{-1}s_0(pp')) + s_0s_0rs_0s_0r^{-1}s_0s_0(pp')\\
&\quad\cdot s_1(s_0(pp')^{-1}k)x's_0h'^{-1}s_1h's_1k'^{-1}s_1(k^{-1}s_0(pp'))\\
&= \phi(x)\cdot p' + s_1ks_1g's_1s_0q'^{-1}s_1s_0r^{-1}(s_0s_0rs_1(g's_0q'^{-1})^{-1}\\
&\quad\cdot s_1h^{-1}s_0hs_1(g's_0q'^{-1})s_0h^{-1}s_1h)s_0s_0q's_1g'^{-1}s_1k^{-1}s_1s_0(pp')\\
&\quad + s_1kx's_0h'^{-1}s_1h's_1k'^{-1}s_1k^{-1}s_1s_0(pp')\\
&= \phi(x)\cdot p' + s_1ks_1h^{-1}s_0hs_1(g's_0q'^{-1})s_0h^{-1}s_1hs_1(g'^{-1}s_0q'^{-1})s_1k^{-1})\cdot pp'\\
&\quad + (s_1kx's_0h'^{-1}s_1h's_1k'^{-1}s_1k^{-1})\cdot pp'\\
&= ((xs_0h^{-1}s_1hs_1k^{-1})(s_1ks_1h^{-1}s_0hs_1(g's_0q'^{-1})s_0h^{-1}s_1h\\
&\quad\cdot s_1(g's_0q'^{-1})^{-1}s_1k^{-1})(s_1kx's_0h'^{-1}s_1h's_1k'^{-1}s_1k^{-1}))\cdot pp'\\
&= (xs_1(g's_0q'^{-1})s_0h^{-1}s_1hs_1(g's_0q'^{-1})^{-1}x's_0h'^{-1}s_1h's_1k'^{-1}s_1k^{-1})\cdot pp'.\end{aligned}$$

Now $s_1(g's_0q'^{-1})x's_0h'^{-1} \in \ker d_0 \cap \ker d_2$, whilst $s_0h^{-1}s_1h \in \ker d_1$, so that the right-hand side becomes

$$(xx's_0h'^{-1}s_0h^{-1}s_1hs_1h's_1k'^{-1}s_1k)\cdot pp' = \phi(xx').$$

So Φ is a homomorphism, and since x is determined by $\phi(x)$, d_0x, and d_1x, then Φ is injective. Now explicitly H_2 is the set

$$\begin{aligned}H_2 = \{(w\,; a, b, c) \mid\ & w \in G_2,\ a, b, c \in G_1,\ d_0w = s_0d_1a = d_1w,\ d_1a = d_1c,\\ & d_0a = d_1b,\ d_0b = d_0c,\ d_2w = as_0d_0a^{-1}bc^{-1}s_0d_1c\}\end{aligned}$$

and it is easy to check that $(w\,; a, b, c) = \Phi(ws_0s_0d_1c^{-1}s_1cs_1b^{-1}s_0b)$ so that Φ is surjective and therefore an isomorphism.

It is immediate that Φ is compatible with the faces and degeneracies of the 2-truncations of $G.$ and $H.$ and thus we have determined an isomorphism $G. \to H.$ of simplicial groups. Furthermore, naturality is clear since Φ is defined using only the face and degeneracy maps of $G.$

We have now shown the existence of natural isomorphisms of functors $\Theta\Delta \cong \mathrm{id}$ and $\Delta\Theta \cong \mathrm{id}$, so that Δ and Θ do give an equivalence of categories.

Recall from [4] that a *2-crossed module* consists, in the first instance, of a complex of N-groups

$$L \xrightarrow{\partial} M \xrightarrow{\partial} N$$

and N-equivariant homomorphisms, where the group N acts on itself by conjugation, such that

$$L \xrightarrow{\partial} M$$

is a crossed module. Thus M acts on L and we require that for all $l \in L$, $m \in M$, and $n \in N$ that $(l^m)^n = (l^n)^{m^n}$. Further, there is a function $\langle\ ,\ \rangle$: $M \times M \to L$, called a *Peiffer lifting*, which satisfies the following axioms:

PL1: $\partial\langle m_0, m_1\rangle = m_0^{-1}m_1^{-1}m_0m_1^{\partial m_0}$,

PL2: $\langle \partial l, m\rangle = l^{-1}l^m$,

PL3: $\langle m, \partial l\rangle = l^{-m}l^{\partial m}$,

PL4: $\langle m_0, m_1m_2\rangle = \langle m_0, m_2\rangle\langle m_0, m_1\rangle^{m_2^{\partial m_0}}$,

PL5: $\langle m_0m_1, m_2\rangle = \langle m_0, m_2\rangle^{m_1}\langle m_1, m_2^{\partial m_0}\rangle$,

PL6: $\langle m_0, m_1\rangle^n = \langle m_0^n, m_1^n\rangle$.

Let 2-$\mathscr{CM}$ denote the category of 2-crossed modules. Then the equivalence of Theorem 2.2, together with Conduché's equivalence [4] between the categories 2-$\mathscr{CM}$ and $\mathscr{SG}^{(2)}$, yields a composite equivalence between 2-$\mathscr{CM}$ and $\mathscr{BRCM}$. We shall indicate how to pass back and forth between 2-$\mathscr{CM}$ and $\mathscr{BRCM}$, leaving the interested reader to supply the details.

Let C: $(C_2 \to C_1 \rightrightarrows C_0)$ be a regular crossed module. The 2-crossed module associated to C is the Moore complex of the simplicial group $\Delta(C)$. Denote by K the costar in C_1 at the vertex $e \in C_0$, that is, $K = \{a \in C_1 |\ ta = e\}$. Then K is the subgroup $\ker d_0$ of $\Delta(C)_1$ with group operation given for any $a, b \in K$ by

$$ab = (a \cdot sb) + b.$$

The source map s: $K \to C_0$ is a homomorphism of groups and is C_0-equivariant relative to the biaction of C_0 on C_1. Note that the new composition extends the group structure on the vertex group $C_1(e)$ so that $C_1(e)$ is a subgroup of K: it is plainly the kernel of s. Further, C_0 acts diagonally on K: for all $a \in K$ and $p \in C_0$ we set $a^p = p^{-1} \cdot a \cdot p$. (There should be no confusion with the given action of C_0 on C_2 which we denote in a similar way.) Then the homomorphism s: $K \to C_0$ is C_0-equivariant relative to the diagonal action on K and the conjugation action of the group C_0 on itself. Now C_0 also acts diagonally on the vertex group $C_2(e)$ and so we have a complex of groups

$$C_2(e) \xrightarrow{\delta} K \xrightarrow{s} C_0$$

in which δ and s are C_0-equivariant. We know that δ: $C_2(e) \to C_1(e)$ is a crossed module: we claim that K acts on $C_2(e)$, extending the action of $C_1(e) \subseteq K$, so that δ: $C_2(e) \to K$ is a crossed module.

We define an action $(x, a) \mapsto x \wr a$ by $x \wr a = (x \cdot sa)^a$ where $x \in C_2(e)$ and $a \in K$. This is indeed a group action and δ is K-equivariant. Moreover, the actions of $C_2(e)$ on itself via K and by conjugation coincide, for δ: $C_2(e) \to C_1(e)$ is a crossed module and so for all $x, y \in C_2(e)$,

$$x \wr \delta y = (x \cdot s(\delta y))^{\delta y} = (x \cdot e)^{\delta y} = x^{\delta y} = -y + x + y.$$

Therefore the map δ: $C_2(e) \to K$ is a crossed module. Further, the action of C_0 on $C_2(e)$ is compatible with that of K.

The final structural component of a 2-crossed module that we need is the Peiffer lifting, which is provided by the braiding. For suppose that C has a braiding $\{\ ,\ \}\colon C_1 \times C_1 \to C_2$. Then the map $K \times K \to C_2(e)$ given by $(a, b) \mapsto \{a^{-1}, b\} \wr a = \langle a, b \rangle$ is a Peiffer lifting. Therefore we have the 2-crossed module

$$C_2(e) \to K \to C_0,$$

which is indeed the Moore complex of $\Delta(C)$.

We now show how the construction of the 2-crossed module just described can be reversed, up to natural isomorphism. So we begin with a 2-crossed module

$$L \xrightarrow{\partial} G \xrightarrow{\partial} P$$

and construct from it, in a functorial way, a regular, braided crossed module $P_2 \to P_1 \rightrightarrows P_0$.

The group of vertices of P_0 is just the group P. The underlying set of elements of P_1 is $G \times P$ with source and target maps $s(g, p) = \partial(g)p$ and $t(g, p) = p$. The groupoid composition in P_1 is given by $(g_1, p_1) + (g_2, p_2) = (g_1 g_2, p_2)$ if $p_1 = \partial(g_2)p_2$. The underlying set of elements of P_2 is $L \times P$ with composition $(l_1, p) + (l_2, p) = (l_1 l_2, p)$. The boundary map $\delta\colon P_2 \to P_1$ is given by $\delta(l, p) = (\partial l, p)$ and the action of P_1 on P_2 is given by $(l, p)^{(g,q)} = (l^g, q)$ if $p = \partial(g)q$. This does define a crossed module over (P_1, P_0) and a biaction of P_0 on $P_2 \to P_1 \rightrightarrows P_0$ is obtained if we define

$$p \cdot (g, q) = (g^{p^{-1}}, pq), \quad (g, q) \cdot p = (g, qp),$$
$$p \cdot (l, q) = (l^{p^{-1}}, pq), \quad (l, q) \cdot p = (l, qp),$$

where $(g, q) \in P_1$, $(l, q) \in P_2$ and $p \in P_0 = P$ and therefore $P_2 \to P_1 \rightrightarrows P_0$ is regular. The braiding on P is given by $\{(g_1, p_1), (g_2, p_2)\} = (\langle g_1^{-1}, g_2^{g_1} \rangle^{g_1}, p_1 p_2)$ where $\langle\ ,\ \rangle\colon G \times G \to L$ is the Peiffer lifting.

This concludes the description of the functor $2\text{-}\mathscr{CM} \to \mathscr{BRCM}$ and it is straightforward to complete the verification of the equivalence between the categories $2\text{-}\mathscr{CM}$ and $\mathscr{BRCM}$ implied by Theorem 2.2.

We thus have a commutative diagram of equivalences of categories,

$$\begin{array}{ccc} \mathscr{BRCM} & \longrightarrow & \mathscr{SG}^{(2)} \\ & \searrow \quad \swarrow & \\ & 2\text{-}\mathscr{CM} & \end{array}$$

The equivalence $\mathscr{SG}^{(2)} \to 2\text{-}\mathscr{CM}$ was established by Conduché in [4], and so the equivalence $\mathscr{BRCM} \to \mathscr{SG}^{(2)}$ could have been established by using Conduché's result and proving the equivalence $\mathscr{BRCM} \to 2\text{-}\mathscr{CM}$. We have preferred to emphasise the equivalence $\mathscr{BRCM} \to \mathscr{SG}^{(2)}$ for two reasons. Firstly, we wished directly to relate braided, regular crossed modules to a category whose use is well established; this rôle is fulfilled by simplicial groups, whereas 2-crossed modules are less familiar. Secondly, the functor $\mathscr{BRCM} \to \mathscr{SG}^{(2)}$ has a clear geometric meaning, whilst Conduché's functor $2\text{-}\mathscr{CM} \to \mathscr{SG}^{(2)}$ is geometrically more obscure.

3. *Automorphism structures for crossed modules*

We now come to the motivating example for the ideas developed in § 2. Let $\partial\colon M \to P$ be a crossed module of groups, regarded as a 2-truncated crossed complex with one vertex $*$ (though we shall write M and P multiplicatively).

Form the crossed complex CRS$((M, P, \partial), (M, P, \partial))$: this is again 2-truncated and we denote it by E: $E_2 \to E_1 \rightrightarrows E_0$.

An explicit description of E may be extracted from [2]. The vertex set E_0 is just $\mathscr{C}_{\mathscr{M}}((M, P, \partial), (M, P, \partial)) = \mathrm{End}(M, P, \partial)$, the set of endomorphisms of the crossed module (M, P, ∂). We shall usually denote elements of E_0 by a single letter and use this same letter for either of its components, that is, for the endomorphism of M or of P.

Now E_1 consists of all *1-fold homotopies* $(M, P, \partial) \to (M, P, \partial)$. Since (M, P, ∂) is trivial in dimensions greater than 2, a 1-fold homotopy is completely specified by a triple (u, h, f) where $u \in P$, $f \in E_0$, and h: $P \to M$ is an *f-derivation*, so that for all $v, v' \in P$, $h(v'v) = h(v')^{f(v)}h(v)$. The source and target maps are given by $s(u, h, f) = f^0$ and $t(u, h, f) = f$ where f^0 is defined by

$$f^0(v) = uf(v)\partial h(v)u^{-1}, \quad f^0(m) = (f(m)h\partial(m))^{u^{-1}}$$

for all $v \in P$ and $m \in M$. It is straightforward to check that $f^0 \in E_0$ as required. The groupoid structure on E_1 is given by

$$(u_1, h_1, f^0) + (u_2, h_2, f) = (u_1u_2, h_1 + h_2, f),$$

where, for $v \in P$, $(h_1 + h_2)(v) = h_2(v)h_1(v)^{u_2}$.

An element of E_2 is a *2-fold homotopy* $(M, P, \partial) \to (M, P, \partial)$. Each consists of a pair (m, f) where $m \in M$ and $f \in E_0$. The groupoid structure on E_2 is $(m_1, f) + (m_2, f) = (m_1m_2, f)$. The boundary map δ: $E_2 \to E_1$ is $(m, f) \mapsto (\partial(m), h_m, f)$ where $h_m(v) = m^{-f(v)}m$. It is easy to check that h_m is an f-derivation. Finally, the action of E_1 on E_2 is

$$(m, f^0)^{(u,h,f)} = (m^u, f)$$

and this makes δ: $E_2 \to E_1$ a crossed E_1-module.

3.1. PROPOSITION. *The composition map γ: $E \otimes E \to E$ together with the map η: $\mathbf{0} \to E$ adjoint to λ: $\mathbf{0} \otimes (M, P, \partial) \to (M, P, \partial)$ make E a monoid in the category of crossed complexes.*

Proof. This is merely a special case of Proposition 1.1.

So by Theorem 3.2, E is semiregular and braided. To determine the biaction of E_0 and the braiding we have to understand the composition map γ explicitly. A direct calculation leads to the following non-trivial components for the bimorphism determining γ:

$$\begin{aligned}
&E_0 \times E_0 \to E_0\colon\ (f_1, f_2) \mapsto f_1f_2,\\
&E_0 \times E_1 \to E_1\colon\ (f_1, (u, h, f)) \mapsto (f_1(u), f_1h, f_1f),\\
&E_1 \times E_0 \to E_1\colon\ ((u, h, f), f_2) \mapsto (u, hf_2, ff_2),\\
&E_1 \times E_1 \to E_2\colon\ ((u, h, f), (u_1, h_1, f_1)) \mapsto (h(u_1), ff_1),\\
&E_0 \times E_2 \to E_2\colon\ (f_1, (m, f)) \mapsto (f_1(m), f_1f),\\
&E_2 \times E_0 \to E_2\colon\ ((m, f), f_2) \mapsto (m, ff_2).
\end{aligned}$$

These maps give a biaction of E_0 on E and a braiding $E_1 \times E_1 \to E_2$. The monoid structure on E_0 is the usual composition of maps.

Let $A = \mathrm{AUT}(M, P, \partial)$, the full subcrossed complex of E on the vertex set $A = \mathrm{Aut}(M, P, \partial)$ of automorphisms of the crossed module (M, P, ∂). Thus A_0 is the group of units of E_0 and A inherits from E the structure of a regular, braided crossed module.

Now an element of A_2 is a 2-fold homotopy over an automorphism of (M, P, ∂) and consists of a pair (m, f) where $m \in M$ and $f \in A_0$. An element of A_1 is a 1-fold homotopy over an automorphism of (M, P, ∂) and consists of a triple (u, h, f) where $u \in P$, $f \in A_0$, and h is an f-derivation $P \to M$ such that the endomorphism f^0 of (M, P, ∂) which gives the source vertex of (u, h, f) is actually an automorphism. Clearly f^0 is an automorphism of (M, P, ∂) if and only if

$$g(v) = f(v)\partial h(v), \quad g(m) = f(m)h\partial(m)$$

for all $v \in P$ and $m \in M$, defines an automorphism of (M, P, ∂). Here we make use of results due to K. J. Norrie [14] which extend results of J. H. C. Whitehead [15] (see also [11]). For $f \in E_0$, denote by $\mathrm{Der}_f(P, M)$ the set of f-derivations $P \to M$.

3.2. Proposition [14]. *If f is an automorphism of P then $\mathrm{Der}_f(P, M)$ is a monoid with composition*

$$(h_1 \circ h_2)(v) = h_1(v)h_2(vf^{-1}\partial h_1(v)) = h_2(v)h_1(v)h_2(f^{-1}\partial h_1(v))$$

and identity element $0\colon v \mapsto 1$ *for all $v \in P$.*

Proof. In [15] Whitehead defines a monoid structure on the set $\mathrm{Der}(P, M)$ of derivations $P \to M$. Now if f is an automorphism of P and h is an f-derivation, then hf^{-1} is a derivation: hence we can use f to transport Whitehead's composition on $\mathrm{Der}(P, M)$ to $\mathrm{Der}_f(P, M)$ and the result is as stated. Whitehead's composition is of course recovered by taking $f = \mathrm{id}_P$.

3.3. Proposition [14]. *Let f be an automorphism of the crossed module (M, P, ∂) and let $h\colon P \to M$ be an f-derivation. Then the following are equivalent:*

(i) *h is a unit in the monoid $\mathrm{Der}_f(P, M)$,*

(ii) *$g\colon v \mapsto f(v)\partial h(v)$ is an automorphism of P,*

(iii) *$g\colon m \mapsto f(m)h\partial(m)$ is an automorphism of M.*

Proof. For f equal to the identity automorphism of (M, P, ∂), this result is due to Whitehead [15] (see Lue's account in [11]). Now h is a unit in $\mathrm{Der}_f(P, M)$ if and only if hf^{-1} is a unit in $\mathrm{Der}(P, M)$ and by Whitehead's result, this is equivalent to gf^{-1} being an automorphism of P or of M: since f is an automorphism of (M, P, ∂), this is in turn equivalent to g being an automorphism of P or of M.

We write $\mathrm{Der}_f^*(P, M)$ for the group of units of $\mathrm{Der}_f(P, M)$ and h^* for the inverse of $h \in \mathrm{Der}_f^*(P, M)$. If f is the identity, we write Der^* for Der_f^*. An element of A_1 is now seen to consist of a triple (u, h, f) where $u \in P$, $f \in \mathrm{Aut}(M, P, \partial)$, and $h \in \mathrm{Der}_f^*(P, M)$.

3.4. THEOREM. *The regular crossed module* $A = \mathrm{AUT}(M, P, \partial)$ *corresponds via the equivalence of Theorem* 2.2 *to the* 2-*crossed module*

$$M \xrightarrow{\delta} P \ltimes \mathrm{Der}^*(P, M) \xrightarrow{s} \mathrm{Aut}(M, P, \partial)$$

in which $\delta(m) = (\partial(m), h_m)$ *where* $h_m(v) = m^{-v}m$ *and* $s(u, h) = f$ *where* $f(m) = (mh\partial(m))^{u^{-1}}$ *for* $m \in M$ *and* $f(v) = uv\partial h(v)u^{-1}$ *for* $v \in P$, *and where* P *acts diagonally on* $\mathrm{Der}^*(P, M)$ *in the semidirect product.*

Proof. The costar in the groupoid A_1 at the identity automorphism 1 of (M, P, ∂) may be identified as a set with $P \times \mathrm{Der}^*(P, M)$ and the source map s: $P \times \mathrm{Der}^*(P, M) \to \mathrm{Aut}(M, P, \partial)$ is then as claimed in the theorem. The group structure on the costar is given by $(u_1, h_1)(u_2, h_2) = (u_1u_2, h_3)$ where

$$\begin{aligned} h_3(v) &= (h_1 s(u_2, h_2) + h_2)(v) \\ &= h_2(v)h_1 s(u_2, h_2)(v)^{u_2} \\ &= h_2(v)h_1(u_2 v\partial h_2(v)u_2^{-1})^{u_2} \\ &= h_2(v)h_1(u_2 v\partial h_2(v))h_1(u_2)^{-1} \\ &= h_2(v)h_1(u_2 v)^{\partial h_2(v)}h_1\partial h_2(v)h_1(u_2)^{-1} \\ &= h_1(u_2 v)h_2(v)h_1\partial h_2(v)h_1(u_2)^{-1}. \end{aligned}$$

Now P acts on $\mathrm{Der}^*(P, M)$ by

$$h^u(v) = h(uv)h(u)^{-1} = h(uvu^{-1})^u,$$

that is diagonally, and we see that $h_3 = h_2 \circ h_1^{u_2}$ where $\circ$ denotes Whitehead's composition of derivations as in Proposition 3.2. Hence the group structure in the costar is

$$(u_1, h_1)(u_2, h_2) = (u_1u_2, h_2 \circ h_1^{u_2})$$

and we have the semidirect product $P \ltimes \mathrm{Der}^*(P, M)$.

The vertex group $A_2(1)$ is identified with the group M with $\delta(m) = (\partial(m), h_m)$ as required.

Note that $\mathrm{Aut}(M, P, \partial)$ acts on $P \ltimes \mathrm{Der}^*(P, M)$ by

$$(u, h)^f = (f^{-1}(u), f^{-1}hf)$$

and on M by $m^f = f^{-1}(m)$. The action of $P \ltimes \mathrm{Der}^*(P, M)$ on M is simply $m^{(u,h)} = m^u$ and the Peiffer lifting is given by

$$\begin{aligned} \langle (u_1, h_1), (u_2, h_2) \rangle &= \{(u_1, h_1)^{-1}, (u_2, h_2)\} \wr (u_1, h_1) \\ &= (\{(u^{-1}, h_1^{*u_1^{-1}}), (u_2, h_2)\} \cdot s(u_1, h_1))^{(u_1, h_1)} \\ &= h_1^{*u_1^{-1}}(u_2)^{u_1} \\ &= h_1^*(u_1^{-1}u_2u_1). \end{aligned}$$

Loday shows in [9] that the homotopy groups of the CW-complex modelled by a crossed square may be computed as the homology groups of a certain complex of non-abelian groups: Conduché (private communication, 1984) has observed that this complex is a 2-crossed module. The form of the 2-crossed module identified in Theorem 3.4 suggests that it may be obtained as the non-abelian complex associated to a crossed square. In this way our results concur with those of K. J. Norrie in **[14]**. She treats the crossed module Ξ: $\mathrm{Der}^*(P, M) \to \mathrm{Aut}(M, P, \partial)$ in which $\Xi(h) = g_h^{-1}$, where for all $h \in \mathrm{Der}^*(P, M)$, $m \in M$, and $v \in P$, $g_h(m) = mh\partial(m)$, and $g_h(v) = v\partial h(v)$, as defined by Lue **[11]**, as an analogue for the automorphism group of a group and shows that there is a crossed square

$$\begin{array}{ccc} M & \xrightarrow{\phi} & \mathrm{Der}^*(P, M) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\Xi} \\ P & \xrightarrow[\psi]{} & \mathrm{Aut}\,(M, P, \partial) \end{array}$$

in which $\phi(m) = h_m$, $\psi(u)(m) = m^{u^{-1}}$, $\psi(u)(v) = uvu^{-1}$ for all $m \in M$, $u, v \in P$ and with h-function ξ: $\mathrm{Der}^*(P, M) \times P \to M$ given by evaluation.

3.5. Theorem. *The crossed square above has, as associated* 2-*crossed module, that already obtained from* $\mathrm{AUT}(M, P, \partial)$ *in Theorem* 3.4.

Proof. Certainly the associated 2-crossed module consists of the complex shown: we need only verify that the evaluation map ξ gives rise to the Peiffer lifting given in the proof of Theorem 3.4. The Peiffer lifting determined by ξ is

$$\langle (u_1, h_1), (u_2, h_2) \rangle = \xi(h_1^*, u_1^{-1}u_2u_1) = h_1^*(u_1^{-1}u_2u_1),$$

as we required.

We conclude with some sample computations of the 2-crossed modules of Theorem 3.4. The homology groups of the 2-crossed module are of particular interest since they are also the homotopy groups π_1, π_2, and π_3 of the corresponding 3-type.

Example 1. Let M be a P-module, considered as a crossed P-module with trivial boundary map. In this case $\mathrm{Der}^*(P, M)$ is just the usual abelian group $\mathrm{Der}(P, M)$ of derivations $P \to M$. Then if $(u, h) \in P \ltimes \mathrm{Der}(P, M)$, we have that $s(u, h)(m) = m^{u^{-1}}$ and $s(u, h)(v) = uvu^{-1}$: the cokernel of s is written $\mathrm{Out}(M, P)$ and this is π_1. The kernel of s consists of elements (u, h) such that $u \in Z(P)$, the centre of P, and acts trivially on M, whilst h is any derivation. The homomorphism δ is given by $\delta(m) = (1, h_m)$ and thus the second homotopy group π_2 is

$$(Z(P) \cap \mathrm{stab}_P(M)) \times H^1(P, M).$$

Finally, π_3 is the fixed point subgroup M^P of M.

Example 2. Let M be a normal subgroup of P and let ∂: $M \hookrightarrow P$ be the inclusion. Now

$$\mathrm{Aut}(M, P) = \{\alpha \in \mathrm{Aut}\, P \mid \alpha(M) \subseteq M\}$$

and $s(u, h)(v) = uvh(v)u^{-1}$, with π_1 the cokernel of s. Now $s(u, h) = 1$ if and only if $h(v) = v^{-1}u^{-1}vu$ and $h(v) \in M$ for all $v \in P$. So $\ker s \cong \{u \in P \mid [x, u] \in M$ for all $x \in P\}$ and $\mathrm{Im}\, \delta \cong M$, whence $\pi_2 \cong Z(P/M)$. Further, π_3 is trivial.

The computation of the groups π_1 and π_2 depends upon a characterization of those automorphisms of a crossed module ∂: $M \to P$ which are induced by derivations $P \to M$, and of those derivations inducing the identity automorphism: convenient characterizations remain to be found for the general case. However, π_3 is easily described as $\ker \partial \cap M^P$.

EXAMPLE 3. In our final example, we point out a substructure of $\mathrm{AUT}(M, P, \partial)$. Consider the sub-2-crossed module determined by the subgroup $\mathrm{Aut}_P(M, P, \partial)$ of $\mathrm{Aut}(M, P, \partial)$ consisting of all automorphisms of ∂: $M \to P$ which are the identity on P. If $(u, h) \in P \ltimes \mathrm{Der}^*(P, M)$ and $s(u, h)(v) = v$ for all $v \in P$, then $\partial h(v) = [v, u]$. Thus the sub-2-crossed module is

$$M \to D \to \mathrm{Aut}_P(M, P, \partial),$$

where $D = \{(u, h) \in P \ltimes \mathrm{Der}^*(P, M) \mid \partial h(v) = [v, u] \text{ for all } v \in P\}$. This we recognise as the group considered by Lue in **[10]** and there denoted $\mathrm{DER}(P, M)$. The group operation on D takes the simple form $(u_1, h_1)(u_2, h_2) = (u_1u_2, h_3)$ where $h_3(v) = h_2(v)h_1(v)^{u_2}$.

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