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Galois theory of second order covering maps of simplicial sets [1]

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Abstract

A classical theory gives an equivalence between the category of covering maps of a space and the category of actions on sets of the fundamental groupoid of the space. We give a corresponding theory in dimension 2 for simplicial sets as a consequence of a Generalised Galois Theory. This yields an equivalence between a category of 2-covering maps of a simplicial set B and a category of actions on groupoids of a certain double groupoid constructed from B. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

There is a classical description of covering maps of a "good" space B in terms of actions of the fundamental group $\pi_1(B,b)$ of B. This description can instead be used as a definition of the fundamental group.

There are second order analogues of the fundamental group. These include not just the second homotopy group but also the crossed module formed by the second relative homotopy group and the fundamental group, as considered by Whitehead [12] and by Mac Lane and Whitehead [11]. Several closely related structures were proposed by Quillen (the crossed module of a fibration), Brown and Higgins (the double groupoid of a pair [3], crossed modules over groupoids [4]), Loday (the fundamental cat^1-group of a map [10, 5]), and others.

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However there has been no corresponding theory of second order covering maps. The purpose of this paper is to develop such a theory for simplicial sets, as a special case of Galois theory in categories [7]. The second order notion of fundamental groupoid arising here as the Galois groupoid of a fibration is slightly different from the above notions but it yields the same notion of the second relative homotopy group, considered as a crossed module.

The paper contains four sections. The first section recalls an appropriate simplified version of Galois theory in categories with respect to an adjunction [7]. The second section describes the connection with the classical theory of coverings. The main results are in the third section: this gives the Galois theory with respect to the adjunction between simplicial sets and groupoids. The fourth section gives the corresponding notion of second order covering and second order fundamental double groupoid; these second order coverings are described as the "fibrational" internal actions of this double groupoid in the category of groupoids.

1. Galois theory in categories

Let $\mathbb{C}$ be a category with pullbacks and $\mathscr{F}$ a class of morphisms in $\mathbb{C}$ containing all isomorphisms, closed under compositions, and pullback stable: $\mathscr{F}$ can be considered as a pseudofunctor

$$\mathscr{F} : \mathbb{C}^{\mathrm{op}} \to Cat$$

defined as follows: given an object B in $\mathbb{C}$, the objects of $\mathscr{F}(B)$ are all pairs (A, α), where $\alpha : A \to B$ is a morphism in $\mathscr{F}$ and morphisms $(A, \alpha) \to (A', \alpha')$ are all commutative triangles in $\mathbb{C}$

$$\begin{array}{ccc} A & \longrightarrow & A' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & B & \end{array}$$

We will write $\mathscr{F}(B) = (\mathbb{C} \downarrow B)$.

Note that for any morphism $p : E \to B$ in $\mathscr{F}$ the pullback functor

$$\mathscr{F}(p) = p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$$
$$(A, \alpha) \mapsto (E \times_B A, pr_1)$$

has a left adjoint $p_! : (\mathbb{C} \downarrow E) \to (\mathbb{C} \downarrow B)$ which is the composition with p, i.e. for a given object (D, δ) in $(\mathbb{C} \downarrow E)$, we have $p_!(D, \delta) = (D, p\delta)$. If in addition p^* is monadic, then we say that $p : E \to B$ is an *effective $\mathscr{F}$-descent morphism*.

Let

$$\mathbb{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbb{X}, \qquad \eta : 1_{\mathbb{C}} \to HI, \quad \varepsilon : IH \to 1_{\mathbb{X}}$$

be an adjunction between categories $\mathbb{C}$ and $\mathbb{X}$ with pullbacks and let $\mathscr{F}$ and $\mathscr{F}'$ be classes of morphisms in $\mathbb{C}$ and $\mathbb{X}$ respectively satisfying the conditions above. If $I(\mathscr{F}) \subseteq \mathscr{F}'$ and $H(\mathscr{F}') \subseteq \mathscr{F}$ then for any object $B \in \mathbb{C}$ we obtain an induced adjunction

$$(\mathbb{C} \downarrow B) \underset{H^B}{\overset{I^B}{\rightleftarrows}} (\mathbb{X} \downarrow I(B)), \qquad \eta^B : 1_{(\mathbb{C} \downarrow B)} \to H^B I^B, \;\; \varepsilon^B : I^B H^B \to 1_{(\mathbb{X} \downarrow I(B))}$$

in which

$$I^B(A, \alpha) = (I(A), I(\alpha));$$
$$H^B(X, \phi) = (B \times_{HI(B)} H(X), pr_1)$$

via the pullback

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{pr_2} & H(X) \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

for any (X, ϕ) in $(\mathbb{X} \downarrow I(B))$;

$$\eta^B_{(A,\alpha)} = \langle \alpha, \eta_A \rangle : A \to B \times_{HI(B)} HI(A);$$
$$\varepsilon^B_{(X,\phi)} = \varepsilon_X I(pr_2),$$

i.e. it is the composition

$$I(B \times_{HI(B)} H(X)) \xrightarrow{I(pr_2)} IH(X) \xrightarrow{\varepsilon_X} X.$$

Let $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathscr{F}, \mathscr{F}')$ be the data above; as in [7] we will say that Γ is a *Galois structure*.

Let $p : E \to B$ be an effective $\mathscr{F}$-descent morphism, i.e. (E, p) is a *monadic extension* in the sense of [7, Definition 6.7] and let $Gal_I(E, p) =$

$$I(E \times_B E \times_B E) \rightrightarrows I(E \times_B E) \leftrightarrows I(E)$$

be its Galois pregroupoid in the sense of [7]. The fundamental theorem of Galois theory [7, Theorem 6.8] establishes a certain category equivalence

$$Spl_\Gamma(E, p) \sim Cospl_\Gamma(Gal_I(E, p), \mathbb{X}) \tag{1}$$

between a full subcategory $Spl_\Gamma(E, p)$ of $(\mathbb{C} \downarrow B)$ whose objects can be described as "coverings split over (E, p)" and a certain category $Cospl_\Gamma(Gal_I(E, p), \mathbb{X})$ of "co-split $Gal_I(E, p)$-actions" in $\mathbb{X}$. In this paper we will consider only a special case where

$$Spl_\Gamma(E, p) = \{(A, \alpha) \in (\mathbb{C} \downarrow B) \mid \eta^E_{(E \times_B A, pr_1)} \text{is an isomorphism}\} \tag{2}$$

and

$$Cospl_{\Gamma}(Gal_{\Gamma}(E,p),\mathbb{X}) = \mathbb{X}^{Gal_{\Gamma}(E,p)} \cap (\mathbb{X} \downarrow I(E)) \tag{3}$$

– see [7] for details.

According to the results of [7] a sufficient condition for that is:

Condition 1.1. *The morphisms* ε^{E}, $\varepsilon^{E\times_B E}$ *and* $\varepsilon^{E\times_B E\times_B E}$ *are isomorphisms.*

2. The "easiest case"; coverings of abstract families

Let $\mathbb{A}$ be a category and let $\mathbb{C} = Fam(\mathbb{A})$ be the category of families of objects in $\mathbb{A}$; a morphism $(f,\alpha):(A_\lambda)_{\lambda\in\Lambda} \to (A'_{\lambda'})_{\lambda'\in\Lambda'}$ consists of a map $f:\Lambda\to\Lambda'$ and morphisms $\alpha_\lambda : A_\lambda \to A'_{f(\lambda)}$ for all $\lambda\in\Lambda$. For example, $Sets = Fam(\mathbb{1})$, where $Sets$ is the usual category of sets and $\mathbb{1}$ is the category with exactly one morphism.

If $\mathbb{A}$ has pullbacks, then so also does $\mathbb{C}$, but the converse is not true; we will assume only that $\mathbb{C}$ has pullbacks.

Consider the following Galois structure $\Gamma = (\mathbb{C},\mathbb{X},I,H,\eta,\varepsilon,\mathscr{F},\mathscr{F}')$:

$\mathbb{C} = Fam(\mathbb{A})$ as above, assuming that $\mathbb{A}$ has a terminal object t, and $\mathbb{C}$ has pullbacks;

$\mathbb{X} = Sets$;

$I:(A_\lambda)_{\lambda\in\Lambda} \mapsto \Lambda$;

$H: X \mapsto \sum_{x\in X} t = (A_x)_{x\in X}$ where $A_x = t$ for all $x\in X$, with obvious η and ε;

$\mathscr{F}$ and $\mathscr{F}'$ are the classes of all morphisms in $\mathbb{C}$ and $\mathbb{X}$, respectively.

If $p:E\to B$ is an effective descent morphism in $\mathbb{C}$ with *connected* E, i.e. with E in $\mathbb{A}$, then $Spl_{\Gamma}(E,p)$ consists of those $(A,\alpha)\in(\mathbb{C}\downarrow B)$ for which there exists an isomorphism in $(\mathbb{C}\downarrow E)$ of the form

$$\begin{array}{ccc} E\times_B A & \approx & \sum E \\ & \searrow \quad \swarrow & \\ & E & \end{array} \tag{4}$$

where $\sum E$ is a (possibly infinite) coproduct of copies of E with the canonical morphism to E – in fact this coproduct is just a family each member of which is E.

Clearly (4) agrees with the ordinary notion of covering space. Moreover, under an appropriate choice of $\mathbb{C} = Fam(\mathbb{A})$ and $p:E\to B$, the category equivalence (1) gives the classical equivalence

$$Cov(B) \sim Sets^{\pi_1(B)} \tag{5}$$

between the category $Cov(B)$ of covering spaces over a "good" topological space B, and the category $Sets^{\pi_1(B)}$ of its fundamental group actions. In fact (5) is a special case of the covering theory in a molecular topos – see [1], which itself is a special case of the situation considered here as explained in detail in [8].

Note that the category $Sets^{\Delta^{op}}$ of simplicial sets can also be used as $\mathbb{C}$ (any category of the form $Sets^D$, where D is a small category, is a molecular topos and in particular is the category of families of its connected objects), which again will give (5) as a special case of (1).

3. The Galois structure for second order coverings

Consider the following Galois structure $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathscr{F}, \mathscr{F}')$:

$\mathbb{C} = Sets^{\Delta^{op}}$ is the category of simplicial set – here and below we use as far as possible the terminology and notation of Gabriel and Zisman [6] for simplicial sets;

$\mathbb{X}$ is the category of groupoids;

$H : \mathbb{X} \to \mathbb{C}$ is the canonical inclusion, often called the nerve functor, and written as D^1 in [6];

$I = \pi_1 : \mathbb{C} \to \mathbb{X}$ (written as $\Pi : \Delta^o\mathscr{E} \to \mathscr{G}r$ in [6]) is the left adjoint of the canonical inclusion $H : \mathbb{X} \to \mathbb{C}$, with obvious η and ε;

$\mathscr{F}$ is the class of *fibrations* in the sense of Kan [6, p. 65] and so $\mathscr{F}' = \mathscr{F} \cap \mathbb{X}$ is the class of fibrations of groupoids in the sense of [2] – so $H(\mathscr{F}') \subset \mathscr{F}$ by the definition, and clearly also $I(\mathscr{F}) = \mathscr{F}'$.

Recall [6, p. 65] that a simplicial set B is a Kan complex if and only if the unique map $B \to \mathbb{1}$ is a fibration.

Proposition 3.1. *If B is a Kan complex, then $\varepsilon^B : I^B H^B \to 1_{(\mathbb{X} \downarrow I(B))}$ is an isomorphism.*

Proof. Since $\varepsilon : IH \to 1_{\mathbb{X}}$ is an isomorphism, we have to show that for any fibration of the form $\phi : X \to I(B)$ in $\mathbb{X}$ the morphism $I(pr_2) : I(B \times_{HI(B)} H(X)) \to IH(X)$ is an isomorphism of groupoids. Furthermore, since the functors I and H do not change vertices of simplicial sets, it suffices to show that for any vertex (b, x) of $B \times_{HI(B)} H(X)$ the homomorphism

$$\pi_1(B \times_{HI(B)} H(X), (b, x)) \to \pi_1(H(X), x) \tag{6}$$

is an isomorphism.

The pullback diagram

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{pr_1} & B \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ H(X) & \xrightarrow[H(\phi)]{} & HI(B) \end{array}$$

gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\pi_2(B,b) & \longrightarrow & \pi_1(\{b\}F,(b,x)) & \longrightarrow & \pi_1(B\times_{HI(B)}H(X),(b,x)) & \longrightarrow & \pi_1(B,b) & \longrightarrow & \pi_0(\{b\}\times F,(b,x)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_2(HI(B),\eta_B(b)) & \longrightarrow & \pi_1(F,x) & \longrightarrow & \pi_1(H(x),x) & \longrightarrow & \pi_1(HI(B),\eta_B(b)) & \longrightarrow & \pi_0(F,x)
\end{array}$$

where $F = H(\phi)^{-1}(\eta_B(b))$, and the homomorphism (6) is an isomorphism by the standard (non-abelian) five-lemma (see, e.g., [2]) since we know that

(i) all arrows not involving π_0 are group homorphisms;

(ii) $\pi_2(HI(B),\eta_B(b))=0$ since $HI(B)$ is a groupoid considered as a simplicial set;

(iii) the projection $(\{b\}\times F,(b,x))\to(F,x)$ is an isomorphism and hence so also are the induced maps on π_1 and π_0;

(iv) $\pi_1(B,b)\to\pi_1(HI(B),\eta_B(b))$ is an isomorphism by the definition of π_1.

The exact sequence for a fibration used above is described in [6, p. 117] – we may use it here since B is a Kan complex and $H(X)\to HI(B)$ and $pr_1 : B\times_{HI(B)}H(X)\to B$ are fibrations. □

Proposition 3.2. *Let $p:E\to B$ be a surjective fibration of Kan complexes. Then p is an effective descent morphism in $\mathbb{C}$ satisfying the Condition* 1.1.

Proof. Since p is surjective and $\mathbb{C}$ is a topos, p is an effective global-descent morphism and in order to show that p is an effective $\mathscr{F}$-descent morphism we need only show that if

$$\begin{array}{ccc}
E\times_B A & \xrightarrow{pr_2} & A \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\
E & \xrightarrow[p]{} & B
\end{array}$$

is a pullback diagram in which $pr_1 : E\times_B A\to E$ is a fibration, then so also is $\alpha : A\to B$ (see [9] for details).

Consider a commutative diagram

$$\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{g} & A \\
{\scriptstyle i}\downarrow & \overset{f'}{\nearrow} & \downarrow{\scriptstyle \alpha} \\
\Delta^n & \xrightarrow[f]{} & B
\end{array} \qquad (7)$$

in which i is the inclusion of the k-horn of Δ^n. We have to prove that there exists a completion $f' : \Delta^n\to A$ such that $\alpha f' = f$, $f'i = g$.

Choose $f_1 : \Delta^n \to E$ such that $pf_1 = f$. This is possible since p is surjective. Let $h = \langle f_1 i, g \rangle : \Lambda^n_k \to E \times_B A$:

$$
\begin{array}{ccccc}
\Lambda^n_k & \xrightarrow{\;h\;} & E \times_B B & \xrightarrow{\;pr_2\;} & A \\
{\scriptstyle i}\downarrow & \nearrow_{f'_1} & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \alpha} \\
\Delta^n & \xrightarrow[f_1]{} & E & \xrightarrow[p]{} & B
\end{array}
$$

Since pr_1 is a fibration, the lift f'_1 exists, and then $pr_2 f'_1$ is the required completion.

We can apply Proposition 3.1 to complete the proof once we know that p satisfies the Condition 1.1. For this it suffices to show that E, $E \times_B E$, $E \times_B E \times_B E$ are Kan complexes. This follows from the assumptions that E is Kan and $p : E \to B$ is a fibration, since then $E \times_B E \times_B E \to E \times_B E \to E$ are also fibrations. □

From this and the results of [7] described in the first section we obtain

Corollary 3.3. *If $p : E \to B$ is a surjective fibration of Kan complexes, then there is a category equivalence*

$$Spl_\Gamma(E, p) \sim \mathbb{X}^{Gal_I(E,p)} \cap (\mathbb{X} \downarrow I(E))$$

in which $Spl_\Gamma(E, p)$ is the full subcategory of $(\mathbb{C} \downarrow B)$ with objects those pairs (A, α) for which the diagram

$$
\begin{array}{ccc}
E \times_B A & \xrightarrow{\;\eta_{E \times_B A}\;} & HI(E \times_B A) \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle HI(pr_1)} \\
E & \xrightarrow[\eta_E]{} & HI(E)
\end{array}
\qquad (8)
$$

is a pullback.

In addition we have

Proposition 3.4. *Let $p : E \to B$ and $p' : E' \to B$ be surjective fibrations of Kan complexes such that each connected component of E is contractible. Then*

$$Spl_\Gamma(E', p') \subset Spl_\Gamma(E, p).$$

Proof. We need to show only that there exists a morphism $f : E \to E'$ with $p'f = p$. This is a standard lifting argument on each component of E. □

Proposition 3.5. *If $p: E \to B$ is a surjective fibration of Kan complexes then the Galois pregroupoid $Gal_I(E, p)$ is an internal groupoid in groupoids, i.e. is a double groupoid.*

Proof. As mentioned in [7, 5.5c] it suffices to show that the canonical morphisms

$$I((E \times_B E) \times_E (E \times_B E)) \to I(E \times_B E) \times_{I(E)} I(E \times_B E)$$
$$I((E \times_B E) \times_E (E \times_B E) \times_E (E \times_B E)) \to I(E \times_B E) \times_{I(E)} I(E \times_B E) \times_{I(E)} I(E \times_B E)$$

are isomorphisms. However, this follows from the more general known statement (which can be easily proved using again standard arguments involving the exact sequence of a fibration): The functor $I = \pi_1$ preserves all pullbacks

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array}$$

in which L, N are Kan complexes and f is both a fibration and a split surjection. □

Clearly $Gal_I(E, p)$ contains as an "object group" Loday's cat^1-group

$$\pi_1(E \times_B E, *) \rightrightarrows \pi_1(E, *)$$

of the fibration $p: E \to B$ [10]. This cat^1-group is known to be equivalent to other similar structures, for example the crossed module $\pi_1(F, *) \to \pi_1(E, *)$, due to Quillen.

4. Second order coverings and the second order fundamental groupoid

The general results of [7] applied to the Galois structure described in the previous section suggest the following:

Definition 4.1. A fibration $\alpha: A \to B$ of Kan complexes is said to be a *second order covering* if there exists a surjective fibration $p: E \to B$ such that the diagram (8) is a pullback.

Note that instead of saying that the diagram (8) is a pullback we could say that $E \times_B A \to E$ can be obtained from a fibration of groupoids $X \to HI(E)$, (where $HI(E)$ is the ordinary fundamental groupoid $I(E) = \pi_1(E)$ of E, considered as simplicial set) by pulling back along $E \to HI(E)$. So our "second order coverings" are in the same relationship with the usual coverings, as groupoids are with sets.

Let 2-$Cov(B)$ be the category of second order coverings of B. From Proposition 3.4 and Corollary 3.3 we obtain

Theorem 4.2. *Let $p: E \to B$ be a surjective fibration of Kan complexes such that each connected component of E is contractible. Then*

(a) 2-$Cov(B) = Spl_\Gamma(E, p)$, *where Γ is the Galois structure described in Section* 3;

(b) 2-$Cov(B)$ *is equivalent to the category of those internal actions $F = (F_0, \pi, \xi)$ of the Galois groupoid $Gal_I(E, p) = Gal_{\pi_1}(E, p) =$*

$$\pi_1(E \times_B E \times_B E) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \pi_1(E \times_B E) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \pi_1(E)$$

in the category of groupoids for which the projection functor $\pi: F_0 \to \pi_1(E)$ is a fibration.

Remark 4.3. As mentioned in Proposition 3.5, $Gal_I(E, p)$ is just a double groupoid – but it is better to consider it as an internal groupoid in the category of groupoids, since there are two ways to consider a double groupoid as such an internal groupoid.

Now we can define the second order fundamental groupoid of a Kan complex B as $Gal_I(E, p)$, where $p: E \to B$ is as in the theorem above. It is determined uniquely up to certain "Morita equivalence", and naturally contains the fundamental groupoid $\pi_1(B)$ together with the action of this groupoid on the family $\{\pi_2(B, b)\}_{b \in B}$ of second homotopy groups of B.

Note that all fibres of second order coverings are groupoids, so that this theory is related to that of the classification of fibre bundles with fibre a $K(G, 1)$, where G is a groupoid.

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