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A van Kampen theorem for unions of non-connected spaces

By

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1. Introduction. The object of this note is to give a van Kampen theorem for the fundamental group of a union of spaces under as general conditions as we have found. In particular, we consider arbitrary unions of not necessarily connected spaces.

Our approach follows [2] in using groupoids. That is, we replace the fundamental group $\pi_1(X, x_0)$ of a space X with base point x_0 by the *fundamental groupoid* $\pi_1(X, X_0)$ of a space X with *set* X_0 of base points; this groupoid consists of the homotopy classes rel end points of maps $(I, \dot{I}) \to (X, X_0)$, with partial addition induced by the usual composition of paths.

The main result we give was first proved in A. Razak's thesis [7]. The main technical work was to reduce connectivity conditions from four-fold to three-fold intersections. Here we achieve this by a simple application of Lebesgue covering dimension. Also, we only sketch the argument, following a similar outline in [3].

Other results on the fundamental group of a union of non-connected spaces are given in [1, 4, 8] but none seem as simple or as general as that given here.

2. Main Theorem. Let $\mathfrak{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ be a family of subsets of the space X such that the interiors of the sets U^λ, $\lambda \in \Lambda$, cover X. For each $\nu = (\nu_1, \ldots, \nu_n) \in \Lambda^n$, let $U^\nu = U^{\nu_1} \cap \ldots \cap U^{\nu_n}$; and if $X_0 \subset X$, let $U_0^\nu = U^\nu \cap X_0$. Let $\boldsymbol{X} = (X, X_0)$, $\boldsymbol{U}^\nu = (U^\nu, U_0^\nu)$. Then we have the π_1-*diagramm of the cover* $\mathfrak{U}$,

$$\bigsqcup_{\nu \in \Lambda^2} \pi_1 \boldsymbol{U}^\nu \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \pi_1 \boldsymbol{U}^\lambda \xrightarrow{c} \pi_1 \boldsymbol{X} \tag{2.1}$$

where $\bigsqcup$ denotes disjoint union in the category of groupoids; a, b are determined by the inclusions

$$a_\nu\colon U^\lambda \cap U^\mu \to U^\lambda, \quad b_\nu\colon U^\lambda \cap U^\mu \to U^\mu$$

for each $\nu = (\lambda, \mu) \in \Lambda^2$, and c is determined by the inclusions $c_\lambda\colon U^\lambda \to X$ for each $\lambda \in \Lambda$.

Theorem. *Suppose that X_0 meets each path-component of each two-fold and three-fold*

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intersection of distinct sets of $\mathfrak{U}$. *Then, in the* π_1*-diagram* (2.) *of the cover* $\mathfrak{U}$, c *is the coequaliser of* a, b *in the category of groupoids.*

The main lines of the proof of this theorem follows that given for the fundamental group in [5], but modified for the case of groupoids. The retraction argument used in [2] (in which the general case is deduced from that for $X_0 = X$) seems to work only for certain kinds of covers (for example, finite covers).

So we suppose given a morphism of groupoids

$$f'\colon \bigcup_{\lambda \in \Lambda} \pi_1 \mathbf{U}^\lambda \to G$$

to a groupoid G, such that $f' \circ a = f' \circ b$, and we prove that there is a unique morphism $f\colon \pi_1 X \to G$ of groupoids such that $f \circ c = f'$.

Let θ be a representative of an element of $\pi_1 X$, and suppose first that θ has a subdivision $\theta = \theta_1 + \ldots + \theta_n$ where each θ_i is a map $(I, \dot{I}) \to (X, X_0)$ and lies in some set $U^{\lambda(i)}$ of $\mathfrak{U}$. Then θ_i determines uniquely

$$\theta_i'\colon (I, \dot{I}) \to (U^{\lambda(i)}, U_0^{\lambda(i)}).$$

We write $F\theta_i$ for the element $f'[\theta_i']$ of G (where [ ] denotes homotopy class); these elements $F\theta_i$ are independent of the choice of $U^{\lambda(i)}$ in which θ_i lies, because of the condition $f' \circ a = f' \circ b$, and by the same condition are composable in G to give an element $F\theta_1 + \ldots + F\theta_n$ of G which we write as $F\theta$ although *a priori* it depends on the subdivision chosen.

In order to construct $F\alpha$ for an arbitrary path $\alpha\colon (I, \dot{I}) \to (X, X_0)$, we choose a subdivision $\alpha = \alpha_1 + \ldots + \alpha_n$ of α such that each α_i lies in some set $U^{\lambda(i)}$ of $\mathfrak{U}$ and such that $U^{\lambda(i)} \neq U^{\lambda(i+1)}$ for $i = 1, \ldots, n-1$. This is clearly possible. The condition that X_0 meets each path-component of each two-fold intersection of distinct sets of $\mathfrak{U}$ allows us to construct a homotopy $h\colon \alpha \simeq \theta$ rel $\dot{I}$ such that each restricted homotopy $h_i\colon \alpha_i \simeq \theta_i$ lies in $U^{\lambda(i)}$, θ_i has end points in X_0, and if α_i already has end points in X_0, then h_i is constant. We then define $F\alpha = F\theta$.

To prove $F\alpha$ is independent of the choices involved, it is necessary to prove that if θ, θ' are two representatives of an element of $\pi_1 X$, and $h\colon \theta \simeq \theta'$ is a homotopy rel $\dot{I}$, and if elements $F\theta$, $F\theta'$ of G are determined as above by subdivisions $\theta = \theta_1 + \ldots + \theta_n$, $\theta' = \theta' + \ldots + \theta'_{n'}$ then $F\theta = F\theta'$.

Now h is a map $I^2 \to X$. The key step which improves the argument of [3] Lemma 2, is as follows.

Let $h^{-1}\mathfrak{U}$ be the cover $\{h^{-1}U^\lambda\}_{\lambda \in \Lambda}$ of I^2. Since I^2 has Lebesgue covering dimension 2, the cover $h^{-1}\mathfrak{U}$ has a refinement $\mathfrak{V}$ such that $\mathfrak{V}$ has order 2 (hence any four-fold intersection of distinct sets of $\mathfrak{V}$ is empty). Subdivide I^2, by lines parallel to the axes, into a set S of small squares s each of which is contained in a set V^s, say, of $\mathfrak{V}$. For each set V^s choose a set U^s in $\mathfrak{U}$ such that $h(V^s) \subset U^s$.

Let v be a vertex of the subdivision of I^2, and consider the set S_v of squares s in S for which $v \in s$. Then v belongs to the intersection of the V^s for $s \in S_v$, and so at most three of these V^s are distinct. Hence at most three of these U^s are distinct.

We require that $h(v)$ can be joined to X_0 by a path in the intersection of the U^s for $s \in S_v$. This is guaranteed by our assumptions if two or three of these U^s are distinct. Suppose, however, that they all coincide with a set U of $\mathfrak{U}$. Join v to $(0, 0)$ by a path a in I^2. If $h(I^2) \subset U$, then ha joins $h(v)$ in U to $h(0, 0) \in X_0$. If $h(I^2) \not\subset U$, then a portion of ha joins v to a point y in $U \cap U'$ for some U' of distinct from U. From the given conditions, we can join y to X_0 in $U \cap U'$. So $h(v)$ can be joined to X_0 in U. The remainder of the proof is now accomplished as described in [3].

3. Remarks.

3.1. This proof uses neither the existence of arbitrary colimits in the category of groupoids, nor any specific description of such colimits. All that is done is to verify the required universal property of a particular diagram. This feature of the proof is important for extensions of these methods to higher dimensions (see [3] and the references there).

3.2. This theorem gives the transition from topology to algebra. The methods required to construct a presentation of the vertex groups of a groupoid given by such a coequaliser diagram are described in [6], and use graph theory (compare with [1, 4]).

3.3. The advantages of using groupoids rather than groups in this theorem are several. First, coproduct of groupoids is simply disjoint union, and this leads to an easy statement of the theorem. Second, one obtains a more general result, without any extra difficulty of proof. Even the case when $\mathfrak{U}$ has two sets U, V with intersection W is difficult to give in groups alone if, say, U, V have each 15 path-components and W has 147! Further, colimits of groupoids include such constructions as HNN-extensions of groups.

3.4. The condition on three-fold intersections cannot be weakened to a condition on two-fold intersections. For example the disc $D = \{(x, y): x^2 + y^2 \leqq 1\}$ is the union of $U^1 = \{(x, y) \in D: \frac{1}{2} < x^2 + y^2 \leqq 1\}$, $U^2 = \{(x, y) \in D: y > -\frac{1}{4}\}$, and $U^3 = \{(x, y) \in D: y < \frac{1}{4}\}$. If $x_0 = (1, 0)$, then x_0 meets each component of each two-fold intersection of U^1, U^2 and U^3, but the π_1-diagram of the cover is not a coequaliser. (This example contradicts Comment B of [8].) Of course one obtains a coequaliser diagram of groupoids when the point $(-1, 0)$ is added to the object sets.

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