ON SEQUENTIALLY PROPER MAPS AND A SEQUENTIAL COMPACTIFICATION

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Introduction

We study here the sequential versions of *proper maps* and of *one-point compactification*, thus continuing the programme suggested in [6] of re-examining general topology in a sequential light.

The main results on the sequential notion of proper maps give the equivalence of a number of conditions on a sequentially continuous map $f: X \to Y$ where X, Y have unique (sequential) limits. Of these I would like to pick out:

(a) *for any space Z, the map $f \times 1_Z : X \times Z \to Y \times Z$ is sequentially closed,*

(b) *if s is a sequence in X with no convergent subsequence in X then fs has no convergent subsequence in Y,*

(c) *f is sequentially closed and has sequentially compact fibres.*

The first condition is taken as the definition of a *sequentially proper map.*

The starting point of this investigation was the notion of *one-point sequential compactification* which defines for any space X a space $X^\wedge$ such that X is an open subspace of $X^\wedge$, $X^\wedge \setminus X$ consists of a single point ω_X, and $X^\wedge$ is sequentially compact. The intuitive idea is that if s is a sequence in X with no convergent subsequence, then s should converge to the " point at infinity " ω_X of $X^\wedge$.

I would like to thank P. Stefan for contributing the proof of the harder part of Theorem 2.8, and the referee for a number of references and helpful comments.

1. *Sequentially continuous functions*

The purpose of this section is to state some basic or technical results needed later.

The set of natural numbers is denoted by $\mathbf{N}$ and its one-point compactification by $\mathbf{N}^+$. If $s : \mathbf{N} \to X$ is a convergent sequence in a space X, then $\bar{s}$ denotes the union of $s(\mathbf{N})$ and its set of limits.

The basic sequential notions of [**6**, **8**] are assumed known. We abbreviate " X has unique sequential limits " to " X has unique limits ".

1.1 Proposition. *The space X has unique limits if and only if the diagonal of X is sequentially closed in $X \times X$.*

The proof is trivial.

1.2 Proposition. *A function $f: X \to Y$ is sequentially continuous if and only if $f^{-1}(B)$ is sequentially closed for every sequentially closed subset B of Y.*

Proof. If f is not sequentially continuous, then there is a point x and sequence s converging to x such that fs does not converge to $f(x)$, whence there is an open

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neighbourhood U of $f(x)$ such that s has a subsequence which is never in U. Let $B = Y \setminus U$. Then B is closed and hence sequentially closed, but $f^{-1}(B)$ is not sequentially closed.

The proof of the converse implication is trivial.

It is well known that the following conditions on a space X are equivalent. (i) Each point of X is closed. (ii) Each point of X is sequentially closed. (iii) X is T_1. Also, if X has unique limits then it is T_1 (by Theorem 1 of [**16**]).

We shall need later the following technical result.

1.3 PROPOSITION. *Let s be a sequence in X with no convergent subsequence. Let X be* T_1. *Then the set* $A = \{(s(n), n) : n \in \mathbf{N}\}$ *is sequentially closed in* $X \times \mathbf{N}^+$.

Proof. Suppose there is a sequence $t = (t_1, t_2)$ in A converging to a point (x, α) not in A.

If $\alpha = \omega$, then t_2 has a subsequence $t_2 j$ which is a strictly monotonic function $\mathbf{N} \to \mathbf{N}$. Then $t_1 j = s t_2 j$ is a subsequence of s converging to x, contradicting the assumption on s.

If $\alpha \neq \omega$, then t_2 is eventually constant at α, and so t_1 is eventually constant at $s(\alpha)$. Hence every neighbourhood of x contains $s(\alpha)$, and so X is not T_1.

We recall the basic properties of sequential compactness. (i) The product of two sequentially compact spaces is sequentially compact. (ii) A sequentially continuous image of a sequentially compact space is sequentially compact. (iii) A sequentially compact subset of a space with unique limits is sequentially closed. (iv) A sequentially closed subset of a sequentially compact space is sequentially compact. We shall need the consequence of (ii) and (iii) that if s is a convergent sequence in a space with unique limits, then $\bar{s}$ is sequentially closed. A consequence of (ii), (iii) and (iv) is that if X is sequentially compact, Y has unique limits, and $f: X \to Y$ is sequentially continuous, then $f: X \to Y$ is sequentially closed. It follows from this and 1.2 that if, further, f is bijective then f^{-1} is sequentially continuous.

2. *Sequentially proper maps*

We now show that the theory of proper maps, as given for example in [**1**], transfers completely into the sequential framework.

2.1 *Definition.* A sequentially continuous function $f: X \to Y$ is *sequentially proper* if $f \times 1_Z : X \times Z \to Y \times Z$ is sequentially closed for all spaces Z.

Clearly a sequentially proper map is itself sequentially closed, (take Z to be a point). The proofs of the following four results mimic almost exactly the proofs of corresponding results in [**1**], and are therefore omitted.

2.2 PROPOSITION. *The following conditions are equivalent for an injective sequentially continuous function* $f: X \to Y$:

(a) *f is sequentially proper,*

(b) *f is sequentially closed,*

(c) *f is a sequential homeomorphism to* $f(X)$, *which is sequentially closed in Y.*

2.3 PROPOSITION. *Let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ be sequentially proper. Then $f_1 \times f_2$ is sequentially proper. Conversely, if $f_1 \times f_2$ is sequentially proper and X_1, X_2 are non-empty, then f_1, f_2 are sequentially proper.*

2.4 COROLLARY. *If $f : X \to Y$ is sequentially proper and surjective, and X has unique limits, then Y has unique limits.*

2.5 PROPOSITION. *Let $f : X \to X'$, $g : X' \to X''$ be sequentially continuous.*

(a) *If f, g are sequentially proper then gf is sequentially proper.*

(b) *If gf is sequentially proper and f is surjective then g is sequentially proper.*

(c) *If gf is sequentially proper and g is injective, then f is sequentially proper.*

(d) *If gf is sequentially proper and X' has unique limits then f is sequentially proper.*

We now come to our main result.

2.6 THEOREM. *Let $f : X \to Y$ be sequentially continuous, and let Y have unique limits. Consider the following conditions:*

(a) *f is sequentially proper,*

(b) *$f \times 1 : X \times \mathbf{N}^+ \to Y \times \mathbf{N}^+$ is sequentially closed,*

(c) *if s is a sequence in X with no subsequence convergent in X, then fs has no subsequence convergent in Y,*

(d) *if B is a sequentially compact subset of Y, then $f^{-1}(B)$ is a sequentially compact subset of X,*

(e) *if $s : \mathbf{N} \to Y$ is a convergent sequence, then $f^{-1}(\bar{s})$ is sequentially compact.*

Then (c) ⇒ (d) ⇒ (e) ⇒ (a) ⇒ (b); *if further X is T_1, then* (b) ⇒ (c).

Proof. (c) ⇒ (d). Suppose s is a sequence in $f^{-1}(B)$. If s has no subsequence convergent in X, then fs has no subsequence convergent in Y (by (c)). Since fs is a sequence in B, this contradicts the sequential compactness of B.

So s has a subsequence sj convergent to x, say, and then fsj converges to $f(x)$. Since B is sequentially compact, and Y has unique limits, B is sequentially closed. Therefore $f(x) \in B$, and sj converges to the point x in $f^{-1}(B)$.

(d) ⇒ (e). This is clear since $\bar{s}$ is sequentially compact.

(e) ⇒ (a). Let A be sequentially closed in $X \times Z$ and let (s, t) be a sequence in $B = (f \times 1)(A)$ such that (s, t) converges to (y, z). We must prove that $(y, z) \in B$.

Choose a sequence s' in X such that (s', t) is a sequence in A and $fs' = s$. Then s' is a sequence in $f^{-1}(\bar{s})$ which is sequentially compact, and so s' has a subsequence $s'j$ converging to a point x of X. Then $(s'j, tj)$ converges to (x, z) which therefore belongs to A. But $sj = fs'j$ converges to $f(x)$ and also to y. Since Y has unique limits, $y = f(x)$ and so $(y, z) \in B$.

(a) ⇒ (b) This is trivial.

Finally the implication (b) $\Rightarrow$ (c) for the case X is T_1 follows easily from Proposition 1.3.

2.7 COROLLARY. *Let Y have unique limits. Then a sequence $s: \mathbf{N} \to Y$ is a sequentially proper map if and only if s has no convergent subsequence.*

Proof. Suppose that s has a convergent subsequence t. Then $s^{-1}(\bar{t})$ is not sequentially compact, and so s is not sequentially proper.

Suppose conversely that s has no convergent subsequence. Let B be sequentially compact in Y. If $s^{-1}(B)$ is infinite, then s has a subsequence in B and so a convergent subsequence. Hence $s^{-1}(B)$ is finite and so sequentially compact. Hence s is sequentially proper.

2.8 THEOREM. *Let $f: X \to Y$ be sequentially continuous and let X, Y have unique limits. Then f is sequentially proper if and only if f is sequentially closed and the fibres $f^{-1}(y)$ are sequentially compact for all $y \in Y$.*

Proof. The forward implication is immediate from 2.6. The reverse implication was conjectured in an earlier draft and first proved by P. Stefan.

We use the equivalence (e) $\Leftrightarrow$ (a) of Theorem 2.6. Let s be a sequence in Y converging to y. We prove that $f^{-1}(\bar{s})$ is sequentially compact.

Let t be a sequence in $f^{-1}(\bar{s})$. If t lies infinitely often in some fibre of f then t has a subsequence lying in this fibre and so by our assumption on the fibres, this subsequence has a convergent subsequence.

Otherwise, by replacing t by a subsequence if necessary, we may suppose that $ft = sj$, where sj is a subsequence of s never taking the value y. However, since s converges to y, so also does sj.

Let $A = t(\mathbf{N})$. If t has no convergent subsequence, then A is sequentially closed (this follows from 2.7). By our assumptions, $f(A)$ is sequentially closed, which is absurd since $y \notin f(A)$ but ft converges to y.

So t has a convergent subsequence, i.e. we may assume that t is convergent to x, say. Then $\bar{t}$ is sequentially closed (since X has unique limits) and so $f(\bar{t})$ is sequentially closed. Hence $y \in f(\bar{t})$. Since $y \notin f(A)$ we have $y = f(x)$, and so $x \in f^{-1}(\bar{s})$.

This completes the proof that $f^{-1}(\bar{s})$ is sequentially compact.

There are a number of results in the literature related to Theorems 2.6 and 2.8. A continuous mapping $f: X \to Y$ is *quasi-perfect* if f is closed and has countably compact fibres (this condition occurs for example in **[13, 15]**); and f is *quasi-sequential* **[11]** if f is continuous and $f^{-1}(\bar{s})$ is countably compact for each convergent sequence s in Y. It is in effect proved in **[15]** that, for Hausdorff spaces, quasi-perfect mappings onto sequential spaces are quasi-sequential. Now sequential compactness implies countable compactness, and for sequential Hausdorff spaces countable compactness implies sequential compactness **[8]**. So if X, Y are sequential Hausdorff spaces then the conditions sequentially proper, quasi-perfect and quasi-sequential on a mapping $f: X \to Y$ are all equivalent.

We conclude this section with some further results on sequential compactness.

2.9 PROPOSITION. *Consider the following conditions on a space* X.

(a) X *is sequentially compact,*

(b) *if* P *is a singleton space, then the unique map* $X \to P$ *is sequentially proper,*

(c) *for any space* Y *the projection* $X \times Y \to Y$ *is sequentially closed,*

(d) *the projection* $X \times \mathbf{N}^+ \to \mathbf{N}^+$ *is sequentially closed.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d), *and if* X *is* T_1 *then* (d) $\Rightarrow$ (a).

The proof is either trivial or a simple consequence of Theorem 2.6.

This proposition is related to results of [7] which gives the implication (a) $\Rightarrow$ (c) assuming Y to be Fréchet and which gives (effectively) the implication (d) $\Rightarrow$ (a) without assumption on X. In [15] it is also noted that a Hausdorff space X is countably compact if and only if the projection $X \times Y \to Y$ is closed for every sequential space Y—one part of this equivalence clearly follows from 2.9.

3. *A one-point sequential compactification*

Let X be any space. Our object in this section is to define a space $X^\wedge = X \cup \{\omega\}$ by adding to X a point ω not in X (ω is called the point at infinity) such that $X^\wedge$ is sequentially compact and X is an open subspace of $X^\wedge$.

If X is already sequentially compact, then $X^\wedge$ is to be simply the topological sum of X and $\{\omega\}$. Otherwise, let $S(X)$ be the set of all sequences in X which have no convergent subsequences. We let the open sets of $X^\wedge$ be those of X and also those sets U in X which contain ω and satisfy (i) $U \setminus \{\omega\}$ is open in X, (ii) every element of $S(X)$ is eventually in U. The axioms for a topology are easily verified.

3.1 THEOREM. *All the elements of* $S(X)$ *converge to* ω, *and to no other point of* $X^\wedge$; X *is open in* $X^\wedge$; *and* $X^\wedge$ *is sequentially compact.*

The proof is trivial.

Since sequential compactness implies countable compactness, $X^\wedge$ is also a one-point countable-compactification of X.

3.2 THEOREM. *If* X *is sequential, then* $X^\wedge$ *is sequential. If, further,* X *has unique limits, then so also does* $X^\wedge$.

Proof. Let X be sequential. Let U be a sequentially open subset of $X^\wedge$. Then $U \cap X$ is sequentially open in X, and hence open in X. If $\omega \notin U$, this completes the proof that U is open. If $\omega \in U$, then every element of $S(X)$ is eventually in U because these elements converge to ω, and so U is open.

Suppose now X has unique limits. Since X is open, if a sequence s in $X^\wedge$ converges to both x and y in X, then $x = y$. So we suppose s converges to x in X and prove s does not converge to ω.

Since X is open, we may assume that s takes all its values in X. Since X has unique limits, $\bar{s}$ is sequentially closed in X. Since X is sequential, $\bar{s}$ is closed in X and so $U = X \setminus \bar{s}$ is open in X. If $t \in S(X)$, then t is eventually in U (for otherwise t would have a subsequence in the sequentially compact set $\bar{s}$ and so would have a convergent subsequence). Hence $U \cup \{\omega\}$ is open, and so s does not converge to ω.

Let X^+ be the usual one-point compactification of X. The referee has pointed out that Theorems 4 and 5 of **[16]**, which show that X^+ has unique limits if X is KC (i.e. if each compact subset of X is closed), and that if X is KC, then X^+ is KC if and only if X is a k-space, help to explain the appearance of a good separation property on $X^\wedge$ even without local compactness conditions on X.

3.3 *Example.* Let X be a countable, sequential space with unique limits. Then $X^\wedge$ is sequentially compact and countable and hence compact. If $X^\wedge$ is also Hausdorff, then $X^\wedge = X^+$, and so X must be locally compact. This shows that if X is the rationals, then $X^\wedge$ is a compact, countable, sequential space with unique limits which is not Hausdorff. (Other examples of sequential spaces with unique limits which are not Hausdorff are given in **[8]**).

It would be interesting to have an example of a non-sequential space X such that $X^\wedge$ has unique limits (compare p. 265 of **[16]**).

We now investigate the uniqueness of one-point sequential compactifications.

3.4 THEOREM. *Let Y be any space with unique limits containing X as an open subspace, and let $f: Y \to X^\wedge$ be defined by*

$$f(x) = \begin{cases} x & \text{if } x \in X \\ \omega & \text{if } x \notin X. \end{cases}$$

Then f is sequentially continuous.

Proof. Let s be a sequence in Y converging to y. If $y \in X$, then s is eventually in the open set X and so fs converges to $f(y)$. So we suppose $y \in Y \setminus X$.

If s has only a finite number of terms in X then fs is eventually constant with value ω so that fs converges to ω. If s has only a finite number of terms in $Y \setminus X$ then fs converges to ω because s has no subsequence convergent in X (since Y has unique limits). The remaining case is that s has infinitely many terms in both X and $Y \setminus X$. Then s is obtained by gluing together two subsequences si and sj, one lying in X and the other lying in $Y \setminus X$. The above arguments show that fsi, fsj both converge to ω, whence fs converges to ω.

3.5 COROLLARY. *Let Y be a sequentially compact space with unique limits containing the sequential space X as an open subspace and such that $Y \setminus X$ has exactly one point. Then the unique bijection $f: Y \to X^\wedge$ which is the identity on X is a sequential homeomorphism.*

The proof follows from 3.4 and the final sentence of §1.

The referee has raised the question of giving classes of spaces X for which $X^\wedge = X^+$. From 3.2 and 3.5 we have $X^\wedge = X^+$ if both X and X^+ are sequential with unique limits. For X^+ to have unique limits it is sufficient that X be KC, by **[16**; Theorem 4]. We can also ensure that X^+ is first countable (and hence sequential) by requiring that X be both first countable and the union of a countable number of closed compact sets K_i such that any compact set is contained in some K_i.

Problem: find more general conditions for X^+ to be sequential.

We now relate sequentially proper maps and the one-point sequential compactification. If $f: X \to Y$ is a function, then $f^\wedge : X^\wedge \to Y^\wedge$ will denote the obvious extension of f which takes ω_X, the point at infinity of X, to ω_Y, the point at infinity of Y.

3.6 THEOREM. *Let X, Y be sequential spaces with unique limits, and let $f: X \to Y$ be continuous. Then $f^\wedge$ is continuous if and only if f is sequentially proper.*

Proof. Suppose first that f is sequentially proper. It is sufficient to prove that $f^\wedge$ is sequentially continuous at ω_X. Let s be a sequence in $X^\wedge$ converging to ω_X. It is no loss of generality to assume that s does not take the value ω_X. Since $X^\wedge$ has unique limits, s is a sequence in X with no subsequence converging in X. Then fs has no subsequence converging in Y, and fs converges to ω_Y.

Conversely, suppose $f^\wedge$ is continuous at ω_X. Then a sequence s in X with no subsequence convergent in X must converge to ω_X, so that $fs = f^\wedge s$ converges to ω_Y. Since $Y^\wedge$ has unique limits, fs has no subsequence convergent in Y.

4. *Some remarks on sequential topology*

One pathological aspect of sequential space theory is that finite products of sequential spaces are not necessarily sequential. This fact has been held to damage the case for the sequential theory, as has the fact that some weak* topologies are not sequential [**12**].

There is, however, a well-known reflection σ from topological spaces to sequential spaces which may be expressed as follows. We take $\mathbf{N}^+$ as universal example [**3**; p.306] and for any space X let $\Sigma(X)$ consist of the sets $s(\mathbf{N}^+)$ for all continuous functions $s: \mathbf{N}^+ \to X$. Then Σ is a natural cover ([**3**; p.306] and [**9**]) and $\sigma: X \to X_\Sigma$ is a reflection functor. So, instead of the usual product, we can use the product $(X \times Y)_\Sigma$, and this is a categorical product for sequential spaces and continuous functions. (The use of this product rather than the usual one would not affect the results of our §2).

It should also be noted that a cs-open topology on the space of continuous functions, in which convergent sequence is used instead of compact set, has been developed in [**10**]. Thus one is led to construct a Cartesian closed category of Hausdorff sequential spaces which is analogous to and has many of the advantages of the Cartesian closed catagory of compactly generated Hausdorff spaces which has been discussed in, for example, [**2, 14**] for applications to topology and in [**5**] for applications to analysis.

The reader is invited to try and extend the results of [3, **4**] to a theory including the sequential case, by starting from a class $\mathcal{N}$ of compact Hausdorff spaces, and replacing for all other Hausdorff spaces the words " compact subspace " by " continuous image of a space in $\mathcal{N}$ ". Two extreme cases are when $\mathcal{N}$ is all compact Hausdorff spaces, and when $\mathcal{N}$ consists of a singleton space. An interesting intermediate case is when $\mathcal{N} = \{\mathbf{N}^+\}$, and this is the sequential theory.

I do not know, however, of a general setting which includes the theories of proper and of sequentially proper maps.

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