General Topology and its Applications 8 (1978) 181–195.

SPACES OF PARTIAL MAPS, FIBRED MAPPING SPACES AND THE COMPACT-OPEN TOPOLOGY

Peter I. BOOTH and Ronald BROWN

Department of Mathematics, Memorial University of Newfoundland, St. John's Newfoundland A1C 5S7, Canada
School of Mathematics and Computer Science, University College of North Wales, Bangor LL57 2UW, Gwynedd, United Kingdom

Received 17 August 1976

The category $\mathbf{Top}_B$ of topological spaces over B is known to have a product, the fibred product. In order to determine exponential laws for this product, we construct first a compact-open topology on a function space of partial maps with closed domain, and from this construct in $\mathbf{Top}_B$ a fibred mapping space. Basic properties of this space are established, including conditions for exponential laws and for Hausdorffness.

AMS/MOS Subj. Class. (1970): Primary: 54C35
Secondary: 54D10, 54D50

compact-open topology	function space
exponential law	k-space
fibred mapping space	partial map

0. Introduction

Let **Top** denote the category of topological spaces and $\mathbf{Top}_B$ the category of spaces and maps over a fixed space B (spaces over B are maps $p: X \to B$, $q: Y \to B$, $r: Z \to B$, etc, and maps $p \to q$ are maps $f: X \to Y$ such that $qf = p$). It is well-known that $\mathbf{Top}_B$ has a product, the so called fibred product or pullback $p \sqcap q: X \sqcap Y \to B$ ($X \sqcap Y$ is often written $X \times_B Y$) of p and q. We consider the question: when does the obvious pullback functor for q

$$\mathbf{Top}_B \to \mathbf{Top}_B, \quad p \mapsto p \sqcap q,$$

have a right adjoint? In other words when is there a *fibred exponential law* giving a bijection

$$\phi: M(p \sqcap q, r) \to M(p, (qr)):$$

for a suitable map $(qr): (YZ) \to B$, where $M(p, q)$ denotes the set of maps $p \to q$?

From the set theoretic point of view, it is easy to see that (YZ) should be the disjoint union of the sets of maps $M(Y_b, Z_b)$ of fibres of q into corresponding fibres of r. The problem is to provide a topology for (YZ) and to obtain, at least in some circumstances, such a fibred exponential law.

Analogous laws (previously called exponential laws of maps) have already been considered in categories $\mathbf{C}_B$, where $\mathbf{C}$ is either the category $\mathbf{Q}$ of quasi-topological spaces [2], the category $\mathbf{K}$ of $\mathfrak{k}$-spaces [4], or (generalising the previous case) the category of A-generated spaces [12]; furthermore one of us has applied this construction to problems in homotopy theory [4, 5, 6, 7]. However, the main purpose of this paper is to obtain a topology on (YZ) by a simple modification of the compact-open topology, and so relate this space to the more usual function spaces of General Topology. For this reason, and in order to make clear which results are valid in $\mathbf{Top}_B$, we avoid working in a convenient category of topological spaces (except in Section 7).

The fibred exponential laws are themselves derived from exponential laws for *parc maps* (i.e. partial maps with closed domain) using a simple idea, the representability of parc maps. The function space of parc maps is likely to be of independent interest.

The structure of this paper is as follows. Section 1 discusses the *space of parc maps* (with the compact-open topology) and its exponential laws. Section 2 defines the *fibred mapping space* (YZ) for spaces Y, Z over a Hausdorff space B and Section 3 gives the fibred exponential laws. Section 4 gives some elementary examples of fibred mapping spaces (more interesting ones will be considered in [8]) and Section 5 shows that (YZ) is Hausdorff if Z is Hausdorff, B is locally compact and $q: Y \to B$ is a submersion. Sufficient conditions for maps to be submersions are discussed in Section 6.

Section 7 contains a few remarks describing the analogues of our arguments and results in convenient categories, in particular in the convenient category $\mathbf{K}$ of $\mathfrak{k}$-spaces.

In a sequel [8] to this paper we relate our construction to the well-known Hom construction for vector bundles and to other constructions, both new and old, in the theories of principal bundles, ex-spaces and relative lifting spaces. It is hoped that these two papers will give some idea of the breadth of applications of fibred mapping spaces.

1. Exponential laws for partial maps

If X, Y are spaces, then $M(X, Y)$ will denote the set of all maps (i.e. continuous functions) $X \to Y$, and $\mathrm{M}(X, Y)$ will denote the same set with the compact-open topology. (If $X = \emptyset$, then $M(X, Y)$ has a unique element, namely the empty map indexed by Y. If $X \neq \emptyset$, $Y = \emptyset$, then $M(X, Y) = \emptyset$.)

A *closed domain partial map* from X to Y is a map $A \dot{\to} Y$ for some closed subspace A of X—such a partial map is called a *parc* map and denoted by $X \rightarrowtail Y$. The set of all such maps is written $P(X, Y)$; note that we include in $P(X, Y)$ the empty parc map.

Our first simple observation is that $P(X, Y)$ can be given a *compact-open topology* which has as sub-basis the sets

$$W(K, U) = \{f \in P(X, Y) : f(K) \subseteq U\}$$

for all compact subsets K of X and open subsets U of Y, where of course $f(K) = \{f(x) : x \in K,\ f(x) \text{ is defined}\}$. The resulting space will be denoted by $\mathbf{P}(X, Y)$. Of course $\mathbf{P}(X, Y)$ is in general non-Hausdorff, since any non-empty open set contains the empty map $\emptyset$.

Our second simple observation is that the functor $X \mapsto \mathbf{P}(X, Y)$ is representable. To prove this, let $Y^{\sim}$ be the set $Y \cup \{\omega\}$ (where $\omega \notin Y$) with the topology in which C is closed in $Y^{\sim}$ if and only if $C = Y^{\sim}$ or C is closed in Y.

Proposition 1.1. *The function*

$$\lambda : \mathbf{M}(X, Y^{\sim}) \to \mathbf{P}(X, Y),$$
$$f \mapsto f \mid f^{-1}(Y)$$

is a homeomorphism.

The proof is simple.

This proposition enables facts on mapping spaces with the compact-open topology to be translated into new results on parc maps. We list the ones that we need, first of all discussing functoriality.

Proposition 1.2.(i) *Maps $f : W \to X$, $g : Y \to Z$ induce by composition maps*

$$g_* : \mathbf{P}(X, Y) \to \mathbf{P}(X, Z),$$
$$f^* : \mathbf{P}(X, Y) \to \mathbf{P}(W, Y).$$

(ii) *If X_0, Y_0 are subspaces of X, Y such that X_0 is closed, then $\mathbf{P}(X_0, Y_0)$ is a subspace of $\mathbf{P}(X, Y)$.*

We recall that the exponential function

$$\theta : M(X \times Y, Z) \to M(X, \mathbf{M}(Y, Z)),$$
$$\theta(f)(x)(y) = f(x, y), \quad x \in X,\ y \in Y,\ f \in M(X \times Y, Z),$$

is a well-defined injection.

Definition 1.3. The pair (X, Y) is called an *exponential pair* if for all spaces Z, the exponential function $\theta : M(X \times Y, Z) \to M(X, \mathbf{M}(Y, Z))$ is surjective.

Note that (X, Y) is an exponential pair if (i) Y is locally compact, i.e. each point of Y has a fundamental system of compact neighbourhoods ([10; p. 156] or [13, 16] for the Hausdorff case), or (ii) $X \times Y$ is a Hausdorff k-space [13]. (A k-space is one which has the weak topology with respect to its compact subspaces. For non-Hausdorff spaces, it is more useful to study $\mathfrak{k}$-spaces as defined in Section 7.)

Theorem 1.4 (Exponential law for parc maps). *The exponential function*

$$\theta : \mathbf{P}(X \times Y, Z) \to \mathbf{M}(X, \mathbf{P}(Y, Z)),$$

$$\theta(f)(x)(y) = f(x, y), \quad f \in \mathbf{P}(X \times Y, Z),\ x \in X,\ y \in Y,$$

is well-defined.

(i) *If* (X, Y) *is an exponential pair, then* θ *is surjective.*

(ii) *If* X *is Hausdorff, then* θ *is continuous.*

(iii) *If* X, Y *are Hausdorff, then* θ *is a homeomorphism into.*

Proof. These follow from Proposition 1.1 and standard results on exponential laws for maps [13, 16], on replacing Z by $Z^{\sim}$.

Remark 1.5. If Z is a singleton, then $\mathbf{P}(Y, Z)$ coincides with the space of closed sets of X with the topology considered in [1, §7].

Remark 1.6. Similar methods give exponential laws for *paro maps* (partial maps with open domains). The representing space for paro maps from X to Y is $Y' = Y \cup \{\omega\}$ with open sets U such that $U = Y'$ or U is open in Y.

2. Fibred mapping spaces

Let Y, Z, B be spaces and $q : Y \to B$, $r : Z \to B$ be maps. For each $b \in B$, let $Y_b = q^{-1}(b)$, $Z_b = r^{-1}(b)$. Let (YZ) be the set

$$\bigsqcup_{b \in B} M(Y_b, Z_b)$$

where $M(Y_b, Z_b)$ is as in Section 1. Let $(qr) : (YZ) \to B$ be the function with fibre over $b \in B$ the set $M(Y_b, Z_b)$. The conventional definition for $M(Y_b, Z_b)$ when Y_b or Z_b is empty implies that

$$(qr)(YZ) = (B \setminus q(Y)) \cup r(Z).$$

Assume from now on that B is T_1; then each fibre Y_b is closed. (The assumption that B is T_1 could be weakened in many places to the assumption that certain maps have closed fibres.) Let $i : (YZ) \to P(Y, Z)$ be the function which sends a map $Y_b \to Z_b$ to the parc map $Y \rightarrowtail Z$ with the same domain and same values. Note that i need not be injective, since whenever Y_b is empty, the unique map $Y_b \to Z_b$ is sent by i to the empty parc map $Y \rightarrowtail Z$.

Definition 2.1. The *modified compact-open topology* on (YZ) is the initial topology with respect to the two functions

$$\begin{tikzcd}[row sep=small] & P(Y,Z) \\ (YZ) \arrow[ur, "i"] \arrow[dr, "(qr)"'] & \\ & B. \end{tikzcd}$$

This topology has as a sub-basis the sets $W(U) = (qr)^{-1}(U)$ for all U open in B and $W(A, V) = \{f \in (YZ): f(A) \subseteq V\}$ for all compact subsets A of Y and open subsets V of Z. (Notice that if $f: Y_b \to Z_b$ where Y_b is empty, then f belongs to any $W(A, V)$.) The space (YZ) will be called the *fibred mapping space of Y into Z over B*, and the map $(qr):(YZ)\to B$ the *fibred mapping projection or functional projection of* (YZ) to B.

Proposition 2.2. (i) *Maps $f: p \to q$, $g: r \to s$ induce by composition maps*

$$g_*: (qr) \to (qs),$$
$$f^*: (qr) \to (pr).$$

(ii) *If $q: Y \to B$, $r: Z \to B$ are maps and B_0 is a closed subspace of B, then (Y_0Z_0) is a subspace of (YZ), where Y_0 is $q^{-1}(B_0)$ and $Z_0 = r^{-1}(B_0)$.*

Proof. This follows easily from properties of parc maps and initial topologies.

3. The fibred exponential law

This result will be given in three parts, i.e., Theorems 3.1, 3.3 and 3.5. Let $p: X \to B$, $q: Y \to B$ be maps. The fibred product $p \sqcap q$ with the two projections $p \sqcap q \to p$, $p \sqcap q \to q$ is, of course, the product in the category $\mathbf{Top}_B$. In order to apply the methods of parc maps, we need to assume $X \sqcap Y$ is closed in $X \times Y$. *So for convenience we assume for the remainder of this section that B is Hausdorff* and leave the reader to see how the results apply under the weaker hypothesis that the pullback spaces which occur are closed.

Theorem 3.1. *Let $p: X \to B$, $q: Y \to B$, $r: Z \to B$ be maps. Then there is a fibred exponential function*

$$\phi: M(p \sqcap q, r) \to M(p, (qr)),$$

which for $f: X \sqcap Y \to Z$ a map over B, satisfies $\phi(f)(x)(y) = f(x, y)$ all $(x, y) \in X \sqcap Y$ and which is injective and natural in the obvious sense.

Proof. Let $f: X \sqcap Y \to Z$ be a map over B and let $x \in X$, $b = p(x)$. Then $\phi(f)(x)$ is given by $y \mapsto f(x, y)$ and hence maps $Y_b \to Z_b$ (if $Y_b = \emptyset$ we take $\phi(f)(x)$ to be the unique empty map $Y_b \to Z_b$). Thus $\phi(f): X \to (YZ)$ is a function over B. But $(qr)\circ\phi(f) = p$, which is continuous, and $i\circ\phi(f) = \theta(f')$, where $f': X \times Y \rightarrowtail Z$ is the parc map corresponding to f, and so $i\circ\phi(f)$ is continuous. Hence $\phi(f)$ is continuous.

The injectivity of ϕ is immediate, and details of naturality are left to the reader.

Remark 3.2. The above proof does not use the continuity of p, q, r except to ensure that $X \sqcap Y$ and the fibres of q are closed.

Let $\mathrm{M}(p,q)$ denote $M(p,q)$ with the compact-open topology.

Theorem 3.3. (i) *If X is Hausdorff, then $\phi: \mathrm{M}(p\sqcap q, r) \to \mathrm{M}(p,(qr))$ is continuous.*
(ii) *If X and Y are Hausdorff, then ϕ is a homeomorphism into.*

Proof. (i) It follows from [13, XII 5.1(a)] that a sub-basis for $\mathrm{M}(p,(qr))$ consists of the sets $W(C, W(D, V))$, $W(C, W(U))$ for C compact in X, D compact in Y, V open in Z and U open in B. But in this space $W(C, W(U))$ is either the whole space or the empty set according as $p(C)$ is or is not contained in U, and $\phi^{-1}(W(C, W(D, V))) = W(C \times D, V)$. Hence ϕ is continuous.

(ii) By [13, XII 5.1(b)] a sub-basis for the open sets of $\mathrm{M}(p\sqcap q, r)$ consists of sets of the form $W(C\sqcap D, V)$ for C, D compact in X, Y respectively and V open in Z. The result follows easily.

Definition 3.4. If $p: X \to B$ is a map and $A \subseteq B$ then $X \mid A$ will denote the space $p^{-1}(A)$ and $p \mid A$ the restriction of p to $X \mid A$. Then p is said to be *locally trivial with fibre F* if each $b \in B$ has an open neighbourhood U such that there is an associated homeomorphism $u_b: p \mid U \to t$, where $t: F \times U \to U$ is the projection.

Theorem 3.5. *The exponential function $\phi: M(p\sqcap q, r) \to M(p,(qr))$ is a surjection (and hence a bijection) if any one of the following list of conditions is satisfied.*

(a) *(X, Y) is an exponential pair.*

(b) *X, Y are Hausdorff and $X\sqcap Y$, Y are k-spaces.*

(c) *$X = F \times A$ where F is Hausdorff, $A \subset B$, $p: X \to B$ is the projection, Y is a Hausdorff k-space and $F \times (Y \mid A)$ is a k-space.*

(d) *$Y = G \times A$ where X, G are Hausdorff, $A \subset B$, $q: Y \to B$ is the projection, and $G \times A$ and $G \times (X \mid A)$ are k-spaces.*

(e) *p is locally trivial with Hausdorff fibre F, Y is a Hausdorff k-space and $F \times Y$ is a k-space.*

(f) *B is regular, X is Hausdorff, q is locally trivial with Hausdorff fibre G such that $G \times B$ and $G \times X$ are k-spaces.*

The proof is removed to Section 8.

Definition 3.6. If $p: X \to B$ is a map, then a section of p is a map $s: B \to X$ such that $ps = 1_B$. We write $\mathbf{sec}\, p$ for the set $\sec p$ of sections to p topologised with the compact-open topology.

Corollary 3.7. *If $q : Y \to B$, $r : Z \to B$ are maps, then there is a natural injection*

$$\phi : M(q, r) \to \sec(qr), \qquad \phi(f)(b) = f \mid Y_b : Y_b \to Z_b, \quad f \in M(q, r), \; b \in B.$$

If Y is a Hausdorff k-space, then $\phi : \mathbf{M}(q, r) \to \mathbf{sec}(qr)$ is a homeomorphism.

Proof. This follows easily from Theorem 3.3 and Theorem 3.5(c), putting $A = B$, $p = 1_B$ and noticing that $1_B \sqcap q \cong q$ and $\mathbf{sec}(qr) = \mathbf{M}(1_B, (qr))$.

There are two more corollaries covering the case where r is a trivial fibration; they require the following preliminary result.

Lemma 3.8. (i) *Given a map $q : Y \to B$ and a space W; if $t = t(W)$ denotes the projection $W \times B \to B$, then*

$$\xi : M(Y, W) \to M(q, t), \qquad \xi(f) = (f, q), \quad f \in M(Y, W)$$

is a bijection. In other words the functor

$$\mathbf{Top}_B \to \mathbf{Top}, \qquad (q : Y \to B) \mapsto Y, \qquad (f : p \to q) \mapsto (f : X \to Y)$$

is left adjoint to the functor

$$\mathbf{Top} \to \mathbf{Top}_B, \qquad W \mapsto t(W), \qquad (f : W \to W') \mapsto (f \times 1_B).$$

(ii) *If Y is Hausdorff, then $\xi : \mathbf{M}(Y, W) \to \mathbf{M}(q, t)$ is a homeomorphism.*

Proof. (i) ξ is clearly a well defined function. If $s : W \times B \to W$ denotes the projection, then we define

$$\xi' : M(q, t) \to M(Y, W), \qquad \xi'(g) = sg, \quad g \in M(q, t).$$

Now $\xi\xi'$ and $\xi'\xi$ are identity functions and so ξ is a bijection.

We notice that ξ' is continuous, for it is the composite of the inclusion $\mathbf{M}(q, t) \to \mathbf{M}(Y, W \times B)$ and the map $\mathbf{M}(Y, W \times B) \to \mathbf{M}(Y, W)$ induced by the projection $W \times B \to W$.

(ii) The composite of ξ and the inclusion $\mathbf{M}(q, t) \to \mathbf{M}(Y, W \times B)$ is also the composite

$$\mathbf{M}(Y, W) \xrightarrow[\delta]{} \mathbf{M}(Y, W) \times \mathbf{M}(Y, B) \xrightarrow[\eta]{} \mathbf{M}(Y, W \times B),$$

where δ is $f \mapsto (f, q)$ and η is the standard bijection. Certainly δ is continuous, and η is continuous since Y is Hausdorff, so ξ is continuous. Hence ξ is a homeomorphism.

The next result will be applied elsewhere to the problem of computing the cohomology of pullback spaces; its $\mathfrak{k}$-space version is (implicitly) the central concept in the argument of [7].

Corollary 3.9. *If $p:X\to B$, $q:Y\to B$ are maps, W is a space and $t=t(W): W\times B\to B$, then there is a natural injection*

$$\psi : M(X\sqcap Y, W)\to M(p,(qt)), \qquad \psi(f)(x)(y) = (f(x,y), q(y)),$$
$$f\in M(X\sqcap Y, W), \quad (x,y)\in X\sqcap Y.$$

(i) If p, q satisfy any one of the conditions (a)$\cdots$(f) *of Theorem* 3.5 *then ψ is a surjection.*

(ii) If X and Y are Hausdorff then $\psi : \mathbf{M}(X\sqcap Y, W)\to \mathbf{M}(p,(qt))$ is a homeomorphism into.

Proof. ψ is the composite of ϕ of Theorem 3.3, and ξ of Lemma 3.8.

The next result is relevant to the problem of computing the cohomology of the total space of a fibration (a relative version of the corresponding result in the category **K** has already been used to study $H^n(E,F;\pi)$ for a fibration $F\to E\xrightarrow[p]{} B$ [5, pp. 334–336] and [6, p. 21]).

Corollary 3.10. *If $q : Y\to B$ is a map, W is a space and $t = t(W): W\times B\to B$ then there is a natural injection*

$$\eta : M(Y, W)\to \sec(qt), \qquad \eta(f)(b)(y) = (f(y), b),$$
$$f\in M(Y, W),\ y\in Y,\ b = q(y).$$

If Y is a Hausdorff k-space, then $\eta : \mathbf{M}(Y, W)\to \mathbf{sec}(q,t)$ is a homeomorphism.

Proof. η is the composite of ϕ of Corollary 3.7 and ξ of Lemma 3.8. If X and Y are spaces then

$$e : \mathbf{M}(X, Y)\times X\to Y, \qquad (f,x)\mapsto f(x), \quad f\in M(X, Y),\ x\in X,$$

is called the *evaluation function for* $\mathbf{M}(X, Y)$; it is well-known that if X is locally compact then e is continuous. We conclude this section by examining the fibred mapping space analogue of e; the result will prove useful in Section 4 below. If $q : Y\to B$, $r : Z\to B$ are maps, then

$$e : (qr)\sqcap q\to r, \qquad e(f,x) = f(x), \quad (f,x)\in (YZ)\sqcap Y$$

will be called the *evaluation function for* (qr).

Proposition 3.11. *If Y is locally compact, then $e : (qr)\sqcap q\to r$ is continuous.*

Proof. This follows by applying Theorem 3.5(a) to the identity map $(qr)\to(qr)$.

4. Elementary examples of fibred mapping spaces

Example 4.1. If Y, Z are spaces and $q : Y \to *$, $r : Z \to *$ are maps to the singleton space $*$, then (YZ) may be identified with the usual function space $\mathrm{M}(Y, Z)$. From this and Proposition 2.2(ii) we deduce:

Corollary 4.2. *If $q : Y \to B$, $r : Z \to B$ are maps, then for each b in B the fibre of (qr) over b is the space $\mathrm{M}(Y_b, Z_b)$.*

Notice also that if $B = *$, then the fibred exponential laws of Section 3 reduce to the usual exponential laws for topological spaces.

Example 4.3. Suppose now that $q : Y \to B$ is a map and $r : Z \to B$ is the inclusion of the subspace Z of B. Then there is a continuous bijection $g : (YZ) \to Z \cup (B \setminus q(Y))$ which is a homeomorphism if B is Hausdorff.

Proof. The projection $(qr) : (YZ) \to B$ is continuous, and (since $r : Z \to B$ is an inclusion) is injective and has range $A = Z \cup (B \setminus q(Y))$. So (qr) restricts to a continuous bijection $g : (YZ) \to A$. However, the projection $A \sqcap Y \to Z$ is continuous, and hence, by Theorem 3.1 and the Hausdorff condition on B, the adjoint map $A \to (YZ)$ is continuous. Since this map is g^{-1}, the result follows.

Example 4.4. If $q : Y \to B$, $r : Z \to B$ are both inclusions of subspaces and B is Hausdorff, then (YZ) is homeomorphic to $(B \setminus Y) \cup (Y \cap Z)$. (This follows from Example 4.3.)

Example 4.5. If $q : Y \to B$ is a map, and B is Hausdorff, then $(q 1_B) : (YB) \to B$ is a homeomorphism. (This follows from Example 4.3.)

Example 4.6. If $r : Z \to B$ is a map, then there is a natural bijection $g : Z \to (BZ)$ over B. If B is Hausdorff then g is continuous. If also B is locally compact, then g is a homeomorphism.

Proof. The function g is defined by the rule that if $z \in Z_b$, then $g(z) : \{b\} \to Z_b$ has value z. Thus g is adjoint to the projection $Z \sqcap B \to Z$ and so is continuous if B is Hausdorff. Suppose B is locally compact. Then the evaluation function $e : (BZ) \sqcap B \to Z$ is continuous (Proposition 3.11); since g^{-1} is the composite of e and the canonical identification $(BZ) \to (BZ) \sqcap B$, it follows that g^{-1} is continuous.

5. Hausdorffness of (YZ)

We obtain both sufficient conditions for (YZ) to be Hausdorff and an example of a situation in which (YZ) is non-Hausdorff.

Theorem 5.1. *Assume Z is Hausdorff and $q : Y \to B$ has the following property: if $y \in Y$, then $q(y)$ has a neighbourhood C which is the image by q of a compact subset A of Y such that $A \cap q^{-1}q(y) = \{y\}$. Then (YZ) is Hausdorff.*

Proof. Let $f : Y_b \to Z_b$, $g : Y_b \to Z_{b'}$, belong to (YZ) and suppose $f \neq g$. If $b \neq b'$, then we can choose disjoint neighbourhoods V, V' of b, b' so that $(qr)^{-1}(V)$, $(qr)^{-1}(V')$ are disjoint neighbourhoods of f, g. Suppose $b = b'$. Since $f \neq g$, Y_b is non-empty and so there is a y in Y_b such that $f(y) \neq g(y)$. Let U, U' be disjoint neighbourhoods of $f(y), g(y)$. Let C, A be as in the assumptions of the theorem. Then $f \in W(A, U)$, $g \in W(A, U')$ and so

$$W(A, U) \cap (qr)^{-1}(C),$$

$$W(A, U') \cap (qr)^{-1}(C)$$

are disjoint neighbourhoods of f, g respectively. This completes the proof.

Recall that a map $q : Y \to B$ is a *submersion* if for each y in Y there is a neighbourhood V of $q(y)$ and map $\lambda : V \to Y$ such that $\lambda q(y) = y$ and $q\lambda = 1_V$. Various sufficient conditions for maps to be submersions are given in the next section.

Corollary 5.2. *If Z is Hausdorff, B is locally compact and $q : Y \to B$ is a submersion, then (YZ) is Hausdorff.*

Proposition 5.3. *Let $G \times B \to B$ be the projection where G is a space with more than one element and B is not locally compact. Then $(G \times B \quad G \times B)$ is not Hausdorff.*

Let b be a point of B which has no compact neighbourhood. Let $f, g : G \times \{b\} \to G \times \{b\}$ be two distinct elements of $(G \times B \quad G \times B)$ (e.g. constant maps to different points of G) and suppose

$$M = W(U) \cap W(C_1, U_1) \cap \cdots \cap W(C_m, U_m)$$

is a basic neighbourhood of f and

$$N = W(V) \cap W(D_1, V_1) \cap \cdots \cap W(D_n, V_n)$$

is a basic neighbourhood of g. Then

$$E = C_1 \cup \cdots \cup C_m \cup D_1 \cup \cdots \cup D_n$$

is compact in $G \times B$ and so its projection $p(E)$ in B is not a neighbourhood of b. Hence there is an element x in $(U \cap V) \setminus p(E)$. Let $h : G \times \{x\} \to G \times \{x\}$ be the identity function. Then $h(C_i)$, $h(D_j)$ are empty and so $h \in M \cap N$. Thus $(G \times B \quad G \times B)$ is not Hausdorff.

6. Submersions

It is clear that any locally trivial map, either in our sense or in the sense of [13, p. 404], is a submersion.

Theorem 6.1. *If $q: Y \to B$ is a Hurewicz fibration and B is a CW-complex then q is a submersion.*

Proof. This follows from the result [13, p. 406, Corollary 4.4], that q is locally trivial in the sense of [13, p. 404].

A stronger result can be obtained using an (as yet) unpublished result of P.R. Heath, that Hurewicz fibrations with LEC base spaces are regular.

Theorem 6.2. (R. Brown and P.R. Heath). *If $q: Y \to B$ is a Hurewicz fibration and B is an LEC space, then q is a submersion.*

This generalises 6.1 because CW-complexes are LEC spaces [14].

Proof. LEC spaces are locally contractible [14]. Let V be a neighbourhood of $q(y)$ in B and $\phi_t: V \to B$ be a homotopy such that ϕ_0 is an inclusion and ϕ_1 is the constant map relative to $q(Y)$.

The constant map ϕ_1 lifts to $V \to \{y\}$ and ϕ_t lifts regularly to ψ_t. Then ψ_t is a section to q over V taking $q(y)$ to y, hence q is a submersion.

If X is any pointed space then $q: PX \to X$ will denote the path fibration over X.

Theorem 6.3. *If X is a space such that all of the inclusions $\{x\} \to X$ of its points are closed cofibrations, i.e. X is a T_1-space with no degenerate points, then $q: PX \to X$ is a submersion.*

Proof. Let x_0 be the base point of X and $\gamma: I \to X$ be a path in X with $\gamma(0) = x_0$, $\gamma(1) = x_1$. By [10, 7.3.6(C)] there is a map $u: X \to [0, \infty[$ and a map $\phi: X_u \to X$, where

$$X_u = \{(x, t) \in X \times I : 0 \leq t \leq \min\{u(x), 1\}\},$$

such that $u(x_1) = 0$, $\phi(x, 0) = x$ for all $x \in X$, and $\phi(x, u(x)) = x$ if $u(x) \leq 1$. Let $U = u^{-1}([0, 1[)$ and $v(y) = (1 + u(y))^{-1}$, $y \in U$. Define a map $\psi: U \times I \to X$ by

$$\psi(y, t) = \begin{cases} \lambda(tv(y)), & 0 \leq t \leq v(y), \\ \phi(y, (1-t)(1+u(y))), & v(y) \leq t \leq 1. \end{cases}$$

Then ψ is continuous and its adjoint $\psi': U \to PX$ satisfies $p\psi'(y) = y (y \in U)$, $\psi'(x_1) = \lambda$.

Theorem 6.4. *If $p : X \to B$ is a submersion and $f : A \to B$ is a map, then the induced map $p_f : X \sqcap A \to A$ is a submersion.*

Proof. This is easy and left to the reader.

It follows, from Corollary 5.2 and the results of this section, that some of the fibred mapping spaces most likely to arise in applications to homotopy theory are Hausdorff.

7. Convenient categories

Because the usual topological category **Top** does not have a function space and product satusfying an exponential law (i.e. **Top** is not a cartesian closed category) various other related categories have been proposed as replacements. In this section we discuss their significance for fibred exponential laws.

7.1. *The category of Hausdorff k-spaces*

This was one of the first convenient categories proposed [9], and has since been considered by many writers. The function space is the k-ification of the usual function space $\mathbf{M}(X, Y)$. However the difficulty which arises here is that many fibred mapping spaces, and presumably also their k-ifications, are not Hausdorff.

7.2. *The category* $\mathbf{K}$ *of* $\mathfrak{k}$*-spaces*

This category is considered in [3, pp. 276–279], and also in [18, p. 554] (where it is denoted by **HG**). We outline how the results of Sections 1, 2, 3 extend to this category.

If X is any space, then $\mathfrak{k}X$ is defined to be X re-topologised with the final topology with respect to all maps of compact Hausdorff spaces into X. We say X is a $\mathfrak{k}$-*space* if $X = \mathfrak{k}X$. The category of $\mathfrak{k}$-spaces and continuous maps is written $\mathbf{K}$; clearly $X \mapsto \mathfrak{k}X$ defines a reflection $\mathfrak{k} : \mathbf{Top} \to \mathbf{K}$, so that if X is a $\mathfrak{k}$-space and Y is any space, then $M(X, Y) = M(X, \mathfrak{k}Y)$ (c.f. [3, Proposition 4.3]).

The basic constructions in $\mathbf{K}$ are then obtained by applying $\mathfrak{k}$ to the corresponding constructions in **Top**. This gives a product space $X \times_{\mathfrak{k}} Y = \mathfrak{k}(X \times Y)$ [3, p. 278], subspace [3, p. 277], function space $\mathbf{K}(X, Y) = \mathfrak{k}\mathbf{M}(X, Y)$ (see [3, p. 279]).

A $\mathfrak{k}$-space X is said to be $\mathfrak{k}$-*Hausdorff* if the diagonal subset Δ is closed in $X \times_{\mathfrak{k}} X$. Clearly any Hausdorff $\mathfrak{k}$-space is $\mathfrak{k}$-Hausdorff. Also if X is $\mathfrak{k}$-Hausdorff then X is T_1, since if $y \in X$ then $\{y\} = j^{-1}(\Delta)$ where $j : X \to X \times_{\mathfrak{k}} X$ is the map $x \mapsto (y, x)$.

If X, Y are $\mathfrak{k}$-spaces, then $\mathbf{PK}(X, Y)$ will denote the set of all parc maps $X \to Y$ topologised in such a way that the bijection

$$\lambda : \mathbf{K}(X, \mathfrak{k}(Y^{\sim})) \to \mathbf{PK}(X, Y),$$

defined as in Proposition 1.1, is a homeomorphism.

Theorem 7.1 (Exponential law for parc maps in $\mathbf{K}$). *Given* $\mathfrak{k}$-*spaces* X, Y, Z *the exponential function*

$$\theta : \mathbf{PK}(X \times_{\mathfrak{k}} Y, Z) \to \mathbf{K}(X, \mathbf{PK}(Y, Z))$$

is a well-defined homeomorphism.

Proof. That θ is a well-defined bijection follows from the exponential law in $\mathbf{K}$ (see [4, Theorem 1.1]). That θ is a homeomorphism now follows from a standard argument using the associativity of the product and the fact that all the spaces concerned lie in $\mathbf{K}$.

Now let B be a $\mathfrak{k}$-space and let $\mathbf{K}_B$ denote the category of $\mathfrak{k}$-spaces over B. This category has a product, namely the pull-back space $X \sqcap_{\mathfrak{k}} Y$ [4, p. 312] with projection $p \sqcap_{\mathfrak{k}} q : X \sqcap_{\mathfrak{k}} Y \to B$. If $p : X \to B$, $q : Y \to B$ are $\mathfrak{k}$-spaces over B, then $\mathbf{K}(p, q) = \mathfrak{k}\mathbf{M}(p, q)$ is the *space of maps* in $\mathbf{K}_B$.

Definition 7.2. Let $q : Y \to B$, $r : Z \to B$ be maps in $\mathbf{K}$, where B is T_1. Let $(YZ)^{\dagger}$ have the same underlying set as (YZ) but with the initial topology with respect to the two functions

$$\begin{tikzcd} & \mathbf{PK}(Y,Z) \\ (YZ) \arrow[ur, "i"] \arrow[dr, "(qr)_\cdot"'] & \\ & B \end{tikzcd}$$

Let $(YZ)_{\mathfrak{k}} = \mathfrak{k}((YZ)^{\dagger})$, and let $(qr)_{\mathfrak{k}}$ denote the projection $(YZ)_{\mathfrak{k}} \to B$.

Theorem 7.3. (Fibred exponential law for $\mathfrak{k}$-spaces). *If* $p : X \to B$, $q : Y \to B$, $r : Z \to B$ *are maps in* $\mathbf{K}$ *and* B *is* $\mathfrak{k}$-*Hausdorff, then the exponential function*

$$\phi : \mathbf{K}(p \sqcap_{\mathfrak{k}} q, r) \to \mathbf{K}(p, (qr)_{\mathfrak{k}})$$

is well-defined and is a homeomorphism.

Proof. The proof that ϕ is a bijection is analogous to that of Theorems 3.1, 3.5, but using Theorem 7.1 instead of Theorem 1.4. We also use [4, Lemma 5.1] to ensure that $X \sqcap_{\mathfrak{k}} Y$ is a subspace of $X \times_{\mathfrak{k}} Y$, and the $\mathfrak{k}$-Hausdorff condition to ensure that $X \sqcap_{\mathfrak{k}} Y$ is closed in $X \times_{\mathfrak{k}} Y$. That ϕ is a homeomorphism now follows by a standard argument.

Remark 7.4. (1) Since right-adjoints are unique, Theorem 7.3 shows that $(qr)_{\mathfrak{k}}$ coincides with the fibred mapping space projection of [3, p. 280]. Further, our parc map approach explains the requirement for B to be $\mathfrak{k}$-Hausdorff in [3, p. 280] and in [4, 6, 7].

(2) We know little about conditions for $(YZ)_{\mathfrak{k}}$ to be $\mathfrak{k}$-Hausdorff other than those

which imply it is Hausdorff (as in Section 5) or when B is a point (for then $(YZ)_t = K(Y, Z)$ which is $\mathfrak{k}$-Hausdorff if Z is $\mathfrak{k}$-Hausdorff [11, p. 27]).

(3) The remarks of this section extend to other convenient categories ([18, 19, 12, Theorem 3.4]).

8. Proof of Theorem 3.5

Let $f: X \to (YZ)$ be a map over B. Then $g = \phi^{-1}(f)$ is a function $X \sqcap Y \to Z$ over B, and the only problem is to prove g continuous.

(a) Let $f' = i \circ f: X \to \mathrm{P}(Y, Z)$. Since (X, Y) is an exponential pair, $\theta^{-1}(f')$ is a continuous parc map $X \times Y \rightarrowtail Z$. But $g = \theta^{-1}(f')$ as functions, so g is continuous.

(b) Let C be a compact subset of $X \sqcap Y$. We have to prove $g \mid C$ continuous. Let A be image of C under the projection $X \sqcap Y \to X$. Then A is compact. So (A, Y) is an exponential pair, since Y and hence $A \times Y$ is a Hausdorff k-space. So $\phi^{-1}(f \mid A): A \sqcap Y \to Z$ is continuous. Since C is contained if $A \sqcap Y$, it follows that $g \mid C$, and hence also g, is continuous.

(c) $X \sqcap Y$ is homeomorphic to $F \times (Y \mid A)$ and the result follows from (b).

(d) $X \sqcap Y$ is homeomorphic to $G \times (X \mid A)$ and the result follows from (b).

(e) It is sufficient to prove that if $p \mid U$ is trivial then, $\phi^{-1}(f \mid U) = g \mid U$ is continuous on its domain $(X \mid U) \sqcap Y = (X \sqcap Y) \mid U$. Since U is open $F \times (Y \mid U)$ is an open subspace of the k-space $F \times Y$ and hence is a k-space [14, 1.5.3, p. 10]. Also $p \mid U$ is essentially the projection $F \times U \to U$ followed by the inclusion $U \to B$, so the result follows from (c).

(f) The proof of this depends on a result of [3], and is given there (Corollary 1.6).

References

[1] R. Arens and J. Dugundji, Topologies for function spaces, Pacific J. Math. 1 (1951) 5–32.

[2] P.I. Booth, The exponential law of maps I, Proc. London Math. Soc. 20 (3) (1970) 179–192.

[3] P.I. Booth, The section problem and the lifting problem, Math. Z. 121 (1971) 273–287.

[4] P.I. Booth, The exponential law of maps II, Math. Z. 121 (1971) 311–319.

[5] P.I. Booth, A unified treatment of some basic problems in homotopy theory, Bull. Am. Math. Soc. 79 (1973) 331–336.

[6] P.I. Booth, Fibrations, mapping spaces and generalised cohomology, Univ. of Warwick preprint (1974).

[7] P.I. Booth, Fibrations and cofibred pairs, Math. Scand. 35 (1974) 145–148.

[8] P.I. Booth and R. Brown, On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps. General Topology and Appl. 8 (2) 165–179.

[9] R. Brown, Some problems in algebraic topology: function spaces, function complexes and FD-complexes, Thesis, Oxford (1961).

[10] R. Brown, Elements of Modern Topology (McGraw-Hill, New York, 1968).

[11] A. Clark, Quasi-topology and compactly generated spaces, Mimeographed Notes, Brown University.

[12] B. Day, A reflection theorem for closed categories, J. Pure Appl. Algebra 2 (1972) 1–11.

[13] J. Dugundji, Topology (Allyn and Bacon, Boston, MA, 1966).
[14] E. Dyer and S. Eilenberg, An adjunction theorem for locally equi-connected spaces, Pacific J. Math. 41 (1972) 669–685.
[15] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory (Springer, Berlin, 1967).
[16] S.T. Hu, Elements of General Topology (Holden-Day, ·····, 1964).
[17] A. Lundell and S. Weingram, The Topology of CW-complexes (Van Nostrand Reinhold, New York, 1969).
[18] R.M. Vogt, Convenient categories of topological spaces for homotopy theory, Arch. Math. (Basel) XXII (1971) 546-555.
[19] O. Wyler, Convenient categories for topology, General Topology and Appl. 3 (1973) 225–242.