CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

RONALD BROWN
OSMAN MUCUK

The monodromy groupoid of a Lie groupoid

Cahiers de topologie et géométrie différentielle catégoriques, tome 36, n^{o} 4 (1995), p. 345-369.

<http://www.numdam.org/item?id=CTGDC_1995__36_4_345_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XXXVI-4 (1995)

THE MONODROMY GROUPOID OF A LIE GROUPOID

by Ronald BROWN and Osman MUCUK

Résumé: Dans cet article, on montre que, sous des conditions générales, l'union disjointe des recouvrements universels des étoiles d'un groupoïde de Lie a la structure d'un groupoïde de Lie dans lequel la projection possède une propriété de monodromie pour les extensions des morphismes locaux réguliers. Ceci complète un rapport détaillé de résultats annoncés par J. Pradines.

Introduction

The notion of *monodromy groupoid* which we describe here arose from the grand scheme of J. Pradines in the early 1960s to generalise the standard construction of a simply connected Lie group from a Lie algebra to a corresponding construction of a Lie groupoid from a Lie algebroid, a notion first defined by Pradines. These results were published as [20, 21, 22, 23]. The recent survey by Mackenzie [17] puts these results in context.

The construction by Pradines involved several steps.

One was the passage from the infinitesimal Lie algebroid to a locally Lie groupoid. This we will not deal with here.

Next was the passage from the locally Lie groupoid to a Lie groupoid. In the case of groups, this is a simple, though not entirely trivial, step, and is part of classical theory. However, in the groupoid case, instead of the locally Lie structure extending directly, there is a groupoid lying over the original one and which is minimal with respect to the property that the Lie structure globalises to it. This groupoid may be called the *holonomy groupoid* of the locally Lie groupoid. This new result is the main content of Théorème 1 of the first note [20]. Its construction is given in detail (but in the topological case) in [2] (see section 5 below).

Finally, there is a need to obtain a maximal Lie groupoid analogous to the universal covering group in the group case, and in some sense locally isomorphic to any globalisation of the locally Lie structure. This groupoid may be called the *monodromy groupoid*, or star universal covering groupoid. Any globalisation of the locally Lie structure originally given is sandwiched between the holonomy and monodromy groupoids by star universal covering morphisms.

A feature of universal covering groups is the classical *Monodromy Principle*. This is an important tool for extending local morphisms on simply connected topological groups, and is formulated for example in Chevalley [8], p.46, so as to be useful also for constructing maps on simply connected topological spaces. The statement by Chevalley is more neatly expressed in the language of groupoids as follows:
if B is a simply connected space, then the topological groupoid $B \times B$, with multiplication $(y,z)(x,y) = (x,z)$, has the property that if $f: W \to H$ is any local morphism from an open connected subset W of $B \times B$ containing the diagonal, to a groupoid H, then f extends to a morphism on $B \times B$.

The word *monodromy* is also widely used for situations of parallel transport around loops, yielding morphisms from a fundamental group to a symmetric group whose image is called the *monodromy group* (see Encyclopaedic Dictionary of Mathematics, Iyanaga and Kawad [14]). It is notable that Poincaré's 1895 paper which defined the fundamental group is concerned at that point with the monodromy of complex functions of many variables.

Pradines required for his results the extendibility of local morphisms, and realised that these uses of monodromy had a common basis and general formulation, as a theory of extensions of local morphisms on differentiable groupoids, now called Lie groupoids. His result, which we call the *Monodromy Theorem*, see section 6 below, was stated as Théorème 2 in Pradines [20].

The note [20] gives no indications of the constructions of these groupoids or of the proofs. In the years 1981-85, Pradines outlined to the first author the constructions of the holonomy and monodromy groupoids and the proofs of the theorems, and an incomplete sketch was written up as Brown [3]. The results on holonomy were worked up in the topological case in the thesis of M.A.-F.E.-S. Aof, [1], and a further refined result, under still weaker assumptions, is published in Aof and Brown [2].

The main results of this paper can be summarised roughly as follows. Recall that the *stars* of a groupoid are the fibres of the source map.

Theorem A *Let G be a Lie groupoid of differentiability class $r \geq 1$ and whose space of objects is paracompact. Suppose the stars G_x at the vertices of G are path connected. Let ΠG denote the disjoint union of the universal covers of the stars G_x for all objects x of G, and let $p : \Pi G \to G$ denote the projection. Then there is a topology on ΠG making it a Lie groupoid and such that p restricts on each star to the universal covering map.*

Theorem B *Under the assumptions of Theorem A, let $f : V \to H$ be a local morphism from a neighbourhood V of the identities of G to a Lie groupoid H. Then V contains a neighbourhood W of the identities of G such that the restriction of f to W lifts to a morphism of Lie groupoids $\Pi G \to H$.*

We call ΠG the *monodromy groupoid* of the Lie groupoid G.

The two theorems are proved concurrently, and are given in more generality in section 6. The structure of the proof is explained in section 1.

Note that in the locally trivial case, Mackenzie [16] gives a non-trivial direct construction of the topology on ΠG and proves also that this ΠG satisfies a monodromy principle on the globalisability of continuous local morphisms on G.

In the case G is a connected topological group satisfying the usual local conditions of covering space theory, the monodromy groupoid ΠG is the universal covering group, while if G is the groupoid $X \times X$, for X a topological space, the monodromy groupoid is, again under suitable local conditions, the fundamental groupoid $\pi_1 X$. Thus part of the interest of these results is in giving a wider perspective to the fundamental groupoid of a space. However, the construction for the fundamental groupoid of a space is essentially contained in the locally trivial case dealt with in Mackenzie [16].

In a companion paper, Brown and Mucuk [7], we give the basis of one of Pradines' intended applications of holonomy by showing how a foliation on a paracompact manifold gives rise to a locally Lie groupoid, and how the existence of non-trivial holonomy of a foliation gives examples of non-extendibility of a locally Lie groupoid. It is notable how well the formal theory of the holonomy groupoid fits with the intuition.

Acknowledgements

We would like to thank: Jean Pradines for his generous sharing of ideas in this area, for permission to use his unpublished methods, and for acting as External Examiner of Aof's thesis; Kirill Mackenzie for conversations, and letters; and both of them for helpful comments on earlier drafts. The second author is grateful to the Ministry of Education of Turkey for support of his studies for this research, which formed a part of his PhD work at Bangor (Mucuk [18]), and to the examiners (Kirill Mackenzie and Mark Lawson), for helpful comments.

1 Outline of the proof

In the following statements, we omit certain local conditions which are in fact necessary for the validity of the proof.

The topology on ΠG is constructed *via* a groupoid $M(G, W)$ which is defined to satisfy a monodromy principle, namely the globalisability of local morphisms defined on an open subset W of G such that $O_G \subseteq W \subseteq G$. At this first stage, $M(G, W)$ is not a Lie groupoid but has the weaker structure of a "star Lie groupoid". Also, the Monodromy Principle which it satisfies is weak, in that smoothness of the morphism is not involved. This we explain in section 3.

However, using an extra "locally sectionable" condition, we construct the structure of Lie groupoid on $M(G, W)$. This stage relies on methods of holonomy, which we outline in section 5. The method is described completely by Aof and Brown in [2].

Next, we relate this groupoid $M(G, W)$ to ΠG, which is defined, following Mackenzie [16], so that its stars are the universal covers of the stars of G. We give conditions under which there is an isomorphism of groupoids $M(G, W) \rightarrow \Pi G$ (theorem 4.2). This identifies the stars of

$M(G,W)$ as the universal covers of the stars of G. It also allows the imposition of a topology on ΠG from the topology on $M(G,W)$. The combination of these two approaches gives, in section 6, the general formulation and proofs of Theorems A and B of the Introduction.

2 Star Lie groupoids

Let G be a groupoid. We write O_G for the set of objects of G, and also identify O_G with the set of identities of G. An element of O_G may be written as x or 1_x as convenient. We write $\alpha, \beta: G \to O_G$ for the source and target maps, and, as usual, write $G(x,y)$ for $\alpha^{-1}(x) \cap \beta^{-1}(y)$, if $x, y \in O_G$. The product hg of two elements of G is defined if and only if $\alpha h = \beta g$, and so the product map $\gamma: (h,g) \mapsto hg$ is defined on the pullback $G_\alpha \times_\beta G$ of α and β. The *difference map* $\delta: G \times_\alpha G \to G$ is given by $\delta(g,h) = gh^{-1}$, and is defined on the double pullback of G by α.

If $x \in O_G$, and $W \subseteq G$, we write W_x for $W \cap \alpha^{-1}x$, and call W_x the *star* of W at x. The following definition is given in Pradines [20] but replacing the term "star" by "$\alpha-$"; a similar remark applies to Definition 1.4. The reason for our change of terminology is that notation other than α is often used for the source map.

In order to cover both the topological and differentiable cases, we use the term C^r manifold for $r \geq -1$, where the case $r = -1$ deals with the case of topological spaces and continuous maps, with no local assumptions, while the case $r \geq 0$ deals as usual with C^r manifolds and C^r maps. Of course, a C^0 map is just a continuous map. We then abbreviate C^r to *smooth.* The terms *Lie group* or *Lie groupoid* will then involve smoothness in this extended sense.

Definition 2.1 A *locally star Lie groupoid* is a pair (G, W) consisting of a groupoid G and a smooth manifold W such that:

(i) $O_G \subseteq W \subseteq G$;

(ii) W is the topological sum of the subspaces $W_x = W \cap \alpha^{-1}x$, $x \in O_G$;

(iii) if $g \in G$, then the set $W \cap Wg$ is open in W and the right translation $R_g: Wg^{-1} \cap W \to W \cap Wg$, $w \mapsto wg$, is a diffeomorphism.

Remark. In [2], the assumption is also made that W generates G. Since this assumption is not required in some of the following results, we omit it from the definition, although it will hold in the examples we study.

A *star Lie groupoid* is a locally star Lie groupoid of the form (G, G), and the notation (G, G) is then usually abbreviated to G. The stronger concept of a Lie groupoid will be defined later, and we will find that a Lie groupoid G is a star Lie groupoid if and only if G is topologically the sum of the spaces $G(x, y)$ for all $x, y \in O_G$. Also, a star Lie group is not necessarily a topological group, but is more analogous to what is called in the literature a semi-topological group, in which all that is assumed is that left and right translations are continuous. A Lie groupoid G may be retopologised as the topological sum of its stars to become a star Lie groupoid.

The following simple proposition shows that any locally star Lie groupoid is extendible to a star Lie groupoid. This is essentially Proposition 1 of section 1 of Pradines [20]. For the convenience of the reader, we repeat the proof as given in Proposition 5.2 of Aof and Brown [2].

Proposition 2.2 (star extendibility) *Let (G, W) be a locally star Lie groupoid. Then G may be given the structure of star Lie groupoid such that for all $x \in O_G$, W_x is an open subset of G_x.*

Proof We define charts for G to be the right translations

$$R_g : W_x \to W_y g$$

for $g \in G(x, y)$ and $x, y \in O_G$. Suppose that $h, g \in G_x$ and $W_y g$ meets $W_z h$. Then there are elements $u \in W_y$ and $v \in W_z$ such that $ug = vh$. So $(R_h)^{-1} R_g$ maps the open neighbourhood $Wv^{-1}u \cap W$ of u in W_y to the open neighbourhood $W \cap Wu^{-1}v$ of v in W_z. So these charts define a smooth structure as required. □

We shall use Proposition 2.2 in the situation of the following proposition, whose last statement, on covering maps, will be used several times later. This statement is, in the case G is a group, closely related to a result of section 1 of Douady-Lazard [10]. This result was a starting point for Pradines and us; however, the assumptions made there are different.

Proposition 2.3 *Let (G, W) be a locally star Lie groupoid, and let $i : W \to G$ be the inclusion. Let $p : M \to G$ be a morphism of groupoids which is the identity on objects, and suppose given $\tilde{\imath} : W \to M$ such that (a) $p\tilde{\imath} = i$, and (b) $\tilde{\imath}(ww') = (\tilde{\imath}w)(\tilde{\imath}w')$ whenever w, w', ww' belong to W. Let $\widetilde{W} = \tilde{\imath}(W)$ have the smooth structure induced by $\tilde{\imath}$ from that on W. Then $(M, \widetilde{W})$ is a locally star Lie groupoid. Further, if M is given the extended star Lie structure, then each map on stars $p_x : M_x \to G_x$ is a covering map.*

Proof We have to prove that the conditions for $(M, \widetilde{W})$ to be a locally star Lie groupoid are satisfied. This is trivial for the first two conditions. For the third, let $m \in M$. In order for $\widetilde{W} \cap \widetilde{W}m$ to be non-empty, we must have $m = \tilde{v}^{-1}\tilde{u}$ where $\tilde{v} = \tilde{\imath}v, \tilde{u} = \tilde{\imath}u$ for $u, v \in W$. Since the smooth structure on $\widetilde{W}$ is also induced from that on W by the inverse of the restriction of p mapping $\widetilde{W} \to W$, the required condition on $(M, \widetilde{W})$ follow from that on (G, W).

Let $g \in G(x, y)$. Then W_y, which is open in G_y, is taken by the right translation $R_g: G_y \to G_x$ to the open neighbourhood $W_y g$ of g in G_x. Further

$$\begin{aligned} p^{-1}(W_y g) &= p^{-1}(W_y)p^{-1}(g) \\ &= \bigcup\{\widetilde{W}_y h : h \in p^{-1}(g)\}. \end{aligned}$$

Since p is the identity on objects, $\beta h = y$. For each $h \in p^{-1}(g)$, the restriction $p \mid \widetilde{W}_y h, W_y g$ is a diffeomorphism since it is the composite of the diffeomorphisms $\widetilde{W}_y h \to W_y \to W_y g$.

Finally, the sets $\widetilde{W}_y h$ for all h such that $ph = g$ are disjoint. For suppose $m \in \widetilde{W}_y h \cap \widetilde{W}_y k$, where $ph = pk = g$. Then there are $u, v \in W$ such that $m = (\tilde{\imath}u)h = (\tilde{\imath}v)k$. Taking p of this gives $ug = vg$. Hence $u = v$ and so $h = k$.

The sum is hence topological because each of the sets $\widetilde{W}_y h$ are open, by definition of the topology in terms of charts. So $p \mid M_x, G_x$ is a covering map. □

For our purposes, the use of this result is that the easy Lie structure to obtain on the monodromy groupoid is that of star Lie groupoid.

We shall find the following result useful. It is due to Mackenzie (private communication, 1986). We first give a definition.

Definition 2.4 Let G be a star Lie groupoid. A subset W of G is called *star connected* (resp. *star simply connected*) if for each $x \in O_G$, the star W_x of W at x is connected (resp. simply connected). So a star Lie groupoid G is *star connected* (resp. *star simply connected*) if each star G_x of G is connected (resp. simply connected).

Proposition 2.5 *Let G be a star Lie groupoid which, as a groupoid, is generated by the subset W. Suppose that $O_G \subseteq W$ and that W is star connected. Then G is star connected. A similar result holds with connected replaced by path-connected.*

Proof Let $g \in G(x,y)$. Then $g = w_n \dots w_1$, where $w_r \in W \cap G(x_{r-1}, x_r)$, with $x_0 = x$ and $x_n = y$. Let $g_r = w_r \dots w_1$. By induction over r we have that g_r is in the same component as g_{r-1} because they both belong to $W_{x_r} g_{r-1}$ which is connected, because of the homeomorphism given by the right translation

$$\begin{array}{rccc} R_{g_{r-1}}\colon & G_{x_r} & \longrightarrow & G_x, \\ & h & \longmapsto & h g_{r-1} \end{array}$$

where we take $w_0 = 1_x$. Hence $g = g_n$ and g_1 are in the same component of G_x. But g_1 and 1_x are in W_x which is connected. So G_x is connected. That is, G is star connected. The proof for path-connectivity is similar. □

Corollary 2.6 *Suppose further that G is a Lie group, W is path connected, and $p : M \to G$, $\tilde{\imath} : W \to M$ are given as in Proposition 2.3, with M generated by $\tilde{\imath}(W)$. Then the group structure on M gives M the structure of Lie group such that p is a covering map and a morphism of Lie groups.*

Proof The assumptions imply that $M \to G$ is a covering map, by Proposition 2.3, and that M is connected, by Proposition 2.5. It is now standard that the group structure on G lifts to M with the given identity of M as the identity for this group structure. But the right multiplication on M with the first group structure is a lift of the right multiplication of G, and so coincides with that obtained from the new structure. Hence the two structures coincide, and the group M obtains the structure of Lie group as required. □

If G is a non-connected topological group, and $p: \widetilde{G} \to G$ is a universal covering map on each component of G, then there is an obstruction in $H^3(\pi_0 G, \pi_1(G, e))$ to the group structure on G being liftable to a topological group structure on $\widetilde{G}$. This result and generalisations of it are due to R.L. Taylor [25]. Proofs by modern groupoid methods are given in Mucuk [18] and Brown and Mucuk [6].

3 The weak monodromy principle

The construction here is a generalisation to the groupoid case of a construction for groups in Douady and Lazard [10]. Let G be a star Lie groupoid, and let W be any subset of G containing $X = O_G$, and such that $W = W^{-1}$. Then W obtains the structure of *pregroupoid*: this means that W is a reflexive graph, in the sense that it has the structure of maps $\alpha_W, \beta_W: W \to X$, $\iota: X \to W$ with $\alpha_W \iota = \beta_W \iota = 1_X$, and further there is a partial multiplication on W in which if vu is defined then $\beta_W u = \alpha_W v$, $(\iota\beta_W u)u = u(\iota\alpha_W u) = u$, and each $u \in W$ has an inverse u^{-1} such that $uu^{-1} = \iota\beta_W u$, $u^{-1}u = \iota\alpha_W u$. There is also an associativity condition. For further discussion of this, see for example Crowell and Smythe [9]. For our purposes, we do not need this, since we know already that W is embeddable in a groupoid.

There is a standard construction $M(W)$ associating to a pregroupoid W a morphism $\tilde{\imath}: W \to M(W)$ to a groupoid $M(W)$ and which is universal for pregroupoid morphisms to a groupoid. First form the free groupoid $F(W)$ on the graph W, and denote the inclusion $W \to F(W)$ by $u \mapsto [u]$. Let N be the normal subgroupoid (Higgins [13], Brown [4]) of $F(W)$ generated by the elements $[vu]^{-1}[v][u]$ for all $u, v \in W$ such that vu is defined and belongs to W. Then $M(W)$ is defined to be the quotient groupoid (loc. cit.) $F(W)/N$. The composition $W \to F(W) \to M(W)$ is written $\tilde{\imath}$, and is the required universal morphism.

In the case W is the pregroupoid arising from a subset W of a groupoid G, there is a unique morphism of groupoids $p: M(W) \to G$ such that $p\tilde{\imath}$ is the inclusion $i: W \to G$. It follows that $\tilde{\imath}$ is injective.

Clearly, p is surjective if and only if W generates G. In this case, we call $M(W)$ the *monodromy groupoid* of (G, W) and write it $M(G, W)$.

We now resume our assumption that G is a star Lie groupoid, and we assume W is open in G. Let $\widetilde{W} = \tilde{\imath}(W)$. It follows from Proposition 2.3 that the pair $(M, \widetilde{W})$, where $M = M(W)$, inherits the structure of locally star Lie groupoid. Further, this structure is extendible to make M into a star Lie groupoid, such that each map on stars $p_x : M_x \to G_x$ is a covering map.

We can now state a Monodromy Principle, which we call "weak" because it involves no continuity or differentiability conditions on maps.

Theorem 3.1 (Weak Monodromy Principle) *Let G be a Lie groupoid and let W be an open subset of G containing O_G. Suppose that G is star simply connected and W is star connected. Let H be a groupoid over O_G and let $\phi\colon W \to H$ be a morphism of pregroupoids which is the identity on O_G. Then ϕ extends uniquely to a groupoid morphism $\tilde{\phi}\colon G \to H$.*

Proof By Proposition 2.5, $M(G, W)$ is star connected, and by Proposition 2.3, $p\colon M(G, W) \to G$ is, when restricted to stars, a covering map of connected spaces. Since G is star simply connected, it follows that p is an isomorphism. The result now follows from the universal property for $M(G, W)$. □

A useful special case of Theorem 3.1 is the following. Let $q\colon E \to O_G$ be a function and let the *symmetry groupoid* S_q of q be the groupoid over O_G of bijections $E_x \to E_y$ for all fibres $E_x = q^{-1}(x)$ of q, and all $x, y \in O_G$.

Corollary 3.2 *Let X be a connected and simply-connected space, let W be a connected neigbourhood of the diagonal of $X \times X$ such that each section $W_x = \{y \in X : (x, y) \in W\}$ is connected. Let $\phi\colon W \to S_q$ be a morphism of pregroupoids. Suppose $e_0 \in E$ is given. Then there is a unique function $\psi\colon X \to E$ such that $\psi q e_0 = e_0$ and $\psi y = \phi(x, y)x$ whenever $\phi(x, y)$ is defined.*

Proof This follows from the above Weak Monodromy Principle, by setting

$$\psi x = \tilde{\phi}(q e_0, x)(e_0),$$

for all $x \in X$. □

Corollary 3.2 is stated in Chevalley [8] as the Monodromy Principle. Note that in 3.1 and 3.2 there is no topology given on H or S_q and there are no assumptions of continuity or differentiability of ϕ.

4 The star universal cover of a star Lie groupoid

Let X be a topological space, and suppose that each path component of X admits a simply connected covering space. It is standard that if $\pi_1 X$ is the fundamental groupoid of X, topologised as in Brown and Danesh-Naruie [5], and $x \in X$, then the target map $\beta: (\pi_1 X)_x \to X$ is the universal covering map of X based at x (see also Brown [4], Chapter 9).

Let G be a star Lie groupoid. The groupoid ΠG is defined as follows. As a set, ΠG is the union of the stars $(\pi_1 G_x)_{1_x}$. The object set of ΠG is the same as that of G. The function $\alpha: \Pi G \to X$ maps all of $(\pi_1 G_x)_{1_x}$ to x, while $\beta: \Pi G \to X$ is on $(\pi_1 G_x)_{1_x}$ the composition of the two target maps

$$(\pi_1 G_x)_{1_x} \xrightarrow{\beta} G_x \xrightarrow{\beta} X.$$

As explained in Mackenzie [16], p.67, there is a multiplication on ΠG given by 'concatenation', i.e.

$$[b] \circ [a] = [ba(1) + a],$$

where the $+$ inside the bracket denotes the usual composition of paths. Here a is assumed to be a path in G_x from 1_x to $a(1)$, where $\beta(a(1)) = y$, say, so that b is a path in G_y, and for each $t \in [0,1]$, the product $b(t)a(1)$ is defined in G, yielding a path $b(a(1))$ from $a(1)$ to $b(1)a(1)$. It is straightforward to prove that in this way ΠG becomes a groupoid, and that the final map of paths induces a morphism of groupoids $p: \Pi G \to G$.

If each star G_x admits a simply connected cover at 1_x, then we may topologise each $(\Pi G)_x$ so that it is the universal cover of G_x based at 1_x, and then ΠG becomes a star Lie groupoid, as remarked above (see also Mackenzie [16]). We call ΠG the *star universal cover* of G. For example, if $G = X \times X$, then ΠG is the fundamental groupoid $\pi_1 X$. (In general, one could call ΠG the *fundamental groupoid* of G, but this might cause confusion.)

Let X be a topological space admitting a simply connected cover. A subset U of X is called *liftable* if U is open, path-connected and the inclusion $U \to X$ maps each fundamental group of U trivially. If U is liftable, and $q: Y \to X$ is a covering map, then for any $y \in Y$ and $x \in U$ such that $qy = x$, there is a unique map $\hat{\imath}: U \to Y$ such that $\hat{\imath}x = y$ and $q\hat{\imath}$ is the inclusion $U \to X$. This explains the term *liftable*.

We also need a result on covering maps and star Lie groupoids.

Proposition 4.1 *Let $q: H \to G$ be a morphism of star Lie groupoids which is the identity on objects and such that on each star, $q: H_x \to G_x$ is a covering map of spaces. Let W be a neighbourhood of O_G in G satisfying the following condition:*

($\star$) *W is star path-connected and W^2 is contained in a star path-connected neighbourhood V of O_G such that for all $x \in O_G$, V_x is liftable.*

Then the inclusion $i: W \to G$ lifts to a Lie pregroupoid morphism $W \to H$.

Proof Let $i: W \to G$ and $j: V \to G$ be the inclusions. For each $x \in G$, the inclusion $V_x \to G_x$ lifts uniquely to $\hat{\jmath}_x: V_x \to H_x$ mapping 1_x to 1_x, and this defines a lift $\hat{\imath}_x: W_x \to H_x$. We prove that the union of these maps $\hat{\jmath}: W \to H$ is a pregroupoid morphism.

Let $u, v \in W$ be such that $uv \in W$. Let $\alpha u = x$, $\alpha v = y$. Since W is star path-connected, there are paths a, b in W_x, W_y, from u, v to 1_x, 1_y, respectively. Since uv is defined, $\beta u = y$. The concatenation $b \circ a$ is then a path from uv to 1_x. Since $W^2 \subseteq V$, $b \circ a$ is a path in V_x. Hence $\hat{\jmath}(b \circ a)$ is a lift of $b \circ a$ to H. By uniqueness of path liftings, $\hat{\jmath}(b \circ a) = (\hat{\jmath}b) \circ (\hat{\jmath}a)$. Evaluating at 0, and using $\hat{\imath}(uv) = \hat{\imath}(uv)$, gives $\hat{\imath}(uv) = (\hat{\imath}u)(\hat{\imath}v)$. Hence $\hat{\imath}$ is a pregroupoid morphism. □

Theorem 4.2 *Suppose that G is a star connected star Lie groupoid and W is an open neighbourhood of O_G satisfying the condition ($\star$). Then there is an isomorphism over G of star Lie groupoids $M(G, W) \to \Pi G$, and hence the morphism $M(G, W) \to G$ is a star universal covering map.*

Proof By Proposition 4.1, the inclusion $i: W \to G$ lifts uniquely to a morphism of star Lie pregroupoids $i': W \to \Pi G$ such that $pi' = i$. The universal property yields a morphism $\phi: M(G, W) \to \Pi G$ of star Lie groupoids. Hence for each $x \in X$ we have a commutative diagram

$$\begin{tikzcd} M(G,W)_x \arrow[rr, "\phi_x"] \arrow[dr, "p_x"'] & & (\Pi G)_x \arrow[dl, "q_x"] \\ & G_x & \end{tikzcd}$$

But p_x and q_x are covering maps, q_x is universal and $M(G, W)_x$ is connected. Hence $\phi_x: M(G, W)_x \to (\Pi G)_x$ is an isomorphism. Hence ϕ is an isomorphism. □

This theorem has a useful corollary in the group case .

Corollary 4.3 *Let G be a path-connected Lie group and let W be a path-connected neighbourhood of the identity e of G such that W^2 is contained in a liftable neighbourhood of e. Then the morphism $M(G, W) \to G$ is a universal covering map, and hence $M(G, W)$ obtains the structure of Lie group.*

Corollary 4.4 (Weak monodromy principle) *Let $f: W \to H$ be a morphism from the star Lie pregroupoid W to the star Lie groupoid H, where W satisfies the condition $(\star)$. Then f determines uniquely a morphism $f': \Pi G \to H$ of star Lie groupoids such that $f'j' = f$.*

5 Locally Lie groupoids and holonomy groupoids

The aim now is to obtain a similar type of result for Lie groupoids and Lie morphisms. We recall that the locally trivial case is handled in Mackenzie [16].

Our aim is the locally sectionable case (see Definition 5.1). The technique of using here the holonomy groupoid construction was outlined by Pradines to the first author in 1982 (see Brown [4]), and for

Pradines [20] was a happy application of the holonomy construction which he had already found.

We need a Lie version of the globalisability theorem proved in Aof-Brown [2] for a locally topological groupoid. The aim of this section is to give the modifications necessary for the Lie case. No essentially new ideas are involved in the proofs, but we do make a simple but crucial deduction which clarifies the relation between extendibility and holonomy, and in particular gives a useable condition for extendibility.

One of the key differences between the cases $r = -1$ or 0 and $r \geq 1$ is that for $r \geq 1$, the pullback of C^r maps need not be a smooth submanifold of the product, and so differentiability of maps on the pullback cannot always be defined. We therefore adopt the following definition of Lie groupoid. Mackenzie [16] (pp. 84-86) discusses the utility of various definitions of differentiable groupoid.

A *Lie groupoid* is a topological groupoid G such that
(i) the space of arrows is a smooth manifold, and the space of objects is a smooth submanifold of G,
(ii) the source and target maps α, β, are smooth maps and are submersions,
(iii) the domain $G \times_\alpha G$ of the difference map is a smooth submanifold of $G \times G$, and
(iv) the difference map δ is a smooth map.
The term *locally Lie groupoid* (G, W) is defined later.

The following definition is due to Ehresmann [11].

Definition 5.1 Let G be a groupoid and let $X = O_G$ be a smooth manifold. An *admissible local section* of G is a function $s : U \to G$ from an open set in X such that

1. $\alpha s(x) = x$ for all $x \in U$;
2. $\beta s(U)$ is open in X, and
3. βs maps U diffeomorphically to $\beta s(U)$.

Let W be a subset of G and let W have the structure of a smooth manifold such that X is a submanifold. We say that (α, β, W) is *locally sectionable* if for each $w \in W$ there is an admissible local section $s : U \to G$ of G such that (i) $s\alpha(w) = w$, (ii) $s(U) \subseteq W$ and (iii) s is smooth as a function from U to W. Such an s is called a smooth *admissible local section.*

The following definition is due to Pradines [20] under the name "*morceau de groupoide différentiables*". Recall that if G is a groupoid then the difference map δ is $\delta : G \times_\alpha G \to G, (g,h) \mapsto gh^{-1}$.

Definition 5.2 A *locally Lie groupoid* is a pair (G, W) consisting of a groupoid G and a smooth manifold W such that:
$G_1)$ $O_G \subseteq W \subseteq G$;
$G_2)$ $W = W^{-1}$;
$G_3)$ the set $W(\delta) = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction of δ to $W(\delta)$ is smooth;
$G_4)$ the restrictions to W of the source and target maps α and β are smooth and the triple (α, β, W) is locally sectionable;
$G_5)$ W generates G as a groupoid.

Note that, in this definition, G is a groupoid but does not need to have a topology. The locally Lie groupoid (G, W) is said to be *extendible* if there can be found a topology on G making it a Lie groupoid and for which W is an open submanifold. The main result of Aof and Brown [2] is a version of Théorème 1 of Pradines [20] and was stated in the topological case. In the smooth case it states:

Theorem 5.3 (Pradines [20], Aof and Brown [2]) (Globalisability theorem) *Let (G, W) be a locally Lie groupoid. Then there is a Lie groupoid H, a morphism $\phi : H \to G$ of groupoids and an embedding $i : W \to H$ of W to an open neighborhood of O_H such that the following conditions are satisfied.*
i) ϕ is the identity on objects, $\phi i = id_W, \phi^{-1}(W)$ is open in H,and the restriction $\phi_W : \phi^{-1}(W) \to W$ of ϕ is smooth;
ii) if A is a Lie groupoid and $\xi : A \to G$ is a morphism of groupoids such that:
a) ξ is the identity on objects;
b) the restriction $\xi_W : \xi^{-1}(W) \to W$ of ξ is smooth and $\xi^{-1}(W)$ is open in A and generates A;
c) the triple (α_A, β_A, A) is locally sectionable,
then there is a unique morphism $\xi' : A \to H$ of Lie groupoids such that $\phi\xi' = \xi$ and $\xi' a = i\xi a$ for $a \in \xi^{-1}(W)$.

The groupoid H is called the *holonomy groupoid* $Hol(G, W)$ of the locally Lie groupoid (G, W).

Outline proof: Some details of part of the construction are needed for Proposition 5.5.

Let $\Gamma(G)$ be the set of all admissible local sections of G. Define a product on $\Gamma(G)$ by

$$(ts)x = (t\beta sx)(sx)$$

for two admissible local sections s and t. If s is an admissible local section then write s^{-1} for the admissible local section $\beta s\mathcal{D}(s) \to G, \beta sx \mapsto (sx)^{-1}$. With this product $\Gamma(G)$ becomes an inverse semigroup. Let $\Gamma^r(W)$ be the subset of $\Gamma(G)$ consisting of admissible local sections which have values in W and are smooth. Let $\Gamma^r(G,W)$ be the subsemigroup of $\Gamma(G)$ generated by $\Gamma^r(W)$. Then $\Gamma^r(G,W)$ is again an inverse semigroup. Intuitively, it contains information on the iteration of local procedures.

Let $J(G)$ be the sheaf of germs of admissible local sections of G. Thus the elements of $J(G)$ are the equivalence classes of pairs (x,s) such that $s \in \Gamma(G), x \in \mathcal{D}(s)$, and (x,s) is equivalent to (y,t) if and only if $x = y$ and s and t agree on a neighbourhood of x. The equivalence class of (x,s) is written $[s]_x$. The product structure on $\Gamma(G)$ induces a groupoid structure on $J(G)$ with X as the set of objects, and source and target maps $[s]_x \mapsto x, [s]_x \mapsto \beta sx$. Let $J^r(G,W)$ be the subsheaf of $J(G)$ of germs of elements of $\Gamma^r(G,W)$. Then $J^r(G,W)$ is generated as a subgroupoid of $J(G)$ by the sheaf $J^r(W)$ of germs of elements of $\Gamma^r(W)$. Thus an element of $J^r(G,W)$ is of the form

$$[s]_x = [s_n]_{x_n} \dots [s_1]_{x_1}$$

where $s = s_n \dots s_1$ with $[s_i]_{x_i} \in J^r(W), x_{i+1} = \beta s_i x_i, i = 1,\dots,n$ and $x_1 = x \in \mathcal{D}(s)$.

Let $\psi : J(G) \to G$ be the final map defined by $\psi([s]_x) = s(x)$, where s is an admissible local section. Then $\psi(J^r(G,W)) = G$. Let $J_0 = J^r(W) \cap \ker\psi$. Then J_0 is a normal subgroupoid of $J^r(G,W)$; the proof is the same as in [2] Lemma 2.2. The holonomy groupoid $H = Hol(G,W)$ is defined to be the quotient $J^r(G,W)/J_0$. Let $p : J^r(G,W) \to H$ be the quotient morphism and let $p([s]_x)$ be denoted by $< s >_x$. Since $J_0 \subseteq \ker\psi$ there is a surjective morphism $\phi : H \to G$ such that $\phi p = \psi$.

The topology on the holonomy groupoid H such that H with this topology is a Lie groupoid is constructed as follows. Let $s \in \Gamma^r(G,W)$. A partial function $\sigma_s : W \to H$ is defined as follows. The domain of σ_s is the set of $w \in W$ such that $\beta w \in \mathcal{D}(s)$. A smooth admissible local section f through w is chosen and the value $\sigma_s w$ is defined to be

$$\sigma_s w = < s >_{\beta w} < f >_{\alpha w} = < sf >_{\alpha w} .$$

It is proven that $\sigma_s w$ is independent of the choice of the local section f and that these σ_s form a set of charts. Then the initial topology with respect to the charts σ_s is imposed on H. With this topology H becomes a Lie groupoid. Again the proof is essentially the same as in Aof-Brown [2].

Remark 5.4 The above construction shows that the holonomy groupoid $Hol(G,W)$ depends on the class C^r chosen, and so should strictly be written $Hol^r(G,W)$. An example of this dependence is given in Aof-Brown [2].

From the construction of the holonomy groupoid we easily obtain the following extendibility condition.

Proposition 5.5 *The locally Lie groupoid (G,W) is extendible to a Lie groupoid structure on G if and only if the following condition holds:*
(1): *if $x \in O_G$, and s is a product $s_n \dots s_1$ of local sections about x such that each s_i lies in $\Gamma^r(W)$ and $s(x) = 1_x$, then there is a restriction s' of s to a neighbourhood of x such that s' has image in W and is smooth, i.e. $s' \in \Gamma^r(W)$.*

Proof The canonical morphism $\phi : H \to G$ is an isomorphism if and only if $\ker\psi \cap J^r(W) = \ker\psi$. This is equivalent to $\ker\psi \subseteq J^r(W)$. We now show that $\ker\psi \subseteq J^r(W)$ if and only if the condition (1) is satisfied.

Suppose $\ker\psi \subseteq J^r(W)$. Let $s = s_n \dots s_1$ be a product of admissible local sections about $x \in O_G$ with $s_i \in \Gamma^r(W)$ and $x \in \mathcal{D}_s$ such that $s(x) = 1_x$. Then $[s]_x \in J^r(G,W)$ and $\psi([s]_x) = s(x) = 1_x$. So $[s]_x \in \ker\psi$, so that $[s]_x \in J^r(W)$. So there is a neighbourhood U of x such that the restriction $s \mid U \in \Gamma^r(W)$.

Suppose the condition (1) is satisfied. Let $[s]_x \in \ker \psi$. Since $[s]_x \in J^r(G,W)$, then $[s]_x = [s_n]_{x_n} \ldots [s_1]_{x_1}$ where $s = s_n \ldots s_1$ and $[s_i]_{x_i} \in J^r(W)$, $x_{i+1} = \beta s_i x_i, i = 1, \ldots, n$ and $x_1 = x \in \mathcal{D}(s)$. Since $s(x) = 1_x$, then by (1), $[s]_x \in J^r(W)$. □

In effect, Proposition 5.5 states that the non-extendibility of (G, W) arises from the *holonomically non trivial* elements of $J^r(G,W)$. Intuitively, such an element h is an iteration of local procedures (i.e. of elements of $J^r(W)$) such that h returns to the starting point (i.e. $\alpha h = \beta h$) but h does not return to the starting value (i.e. $\psi h \neq 1$).

The following gives a circumstance in which this extendibility condition is easily seen to apply.

Corollary 5.6 *Let G be a Lie groupoid and let $p : M \to G$ be a morphism of groupoids such that $p : O_M \to O_G$ is the identity. Let W be an open subset of G such that*

a) $O_G \subseteq W$;
b) $W = W^{-1}$;
c) W generates G;
d) (α_W, β_W, W) is smoothly locally sectionable;

and suppose that $\tilde{\imath} : W \to M$ is given such that $p\tilde{\imath} = i : W \to G$ is the inclusion and $W' = \tilde{\imath}(W)$ generates M.

Then M admits a unique structure of Lie groupoid such that W' is an open subset and $p : M \to G$ is a morphism of Lie groupoids mapping W' diffeomorphically to W.

Proof It is easy to check that (M, W') is a locally Lie groupoid. We prove that condition (1) in Proposition 5.5 is satisfied (with (G, W) replaced by (M, W')).

Suppose given the data of (1). Clearly, $ps = ps_n \ldots ps_1$, and so ps is smooth, since G is a Lie groupoid. Since $s(x) = 1_x$, there is a restriction s' of s to a neighbourhood of x such that $Im(ps) \subseteq W$. Since p maps W' diffeomorphically to W, then s' is smooth and has image contained in W. So (1) holds, and by Proposition 5.5, the topology on W' is extendible to make M a Lie groupoid. □

Remark 5.7 It may seem unnecessary to construct the holonomy groupoid in order to verify extendibility under condition (1) of Proposition 5.5. However the construction of the smooth structure on M in the last corollary, and the proof that this yields a Lie groupoid, would have to follow more or less the steps given in Aof and Brown [2] as sketched above. Thus it is more sensible to rely on the general result. As Corollary 5.6 shows, the utility of (1) is that it is a checkable condition, both positively or negatively, and so gives clear proofs of the non-existence or existence of non-trivial holonomy. We apply Corollary 5.6 in the next section to obtain a Lie structure on the monodromy groupoid of a Lie groupoid.

We need one more result which refines Proposition 5.3 of [2], and which will be useful in section 6 to relate the monodromy and holonomy groupoids.

Proposition 5.8 *Let (G, W) be a locally Lie groupoid. Let W' be the space with underlying set W, retopologised as the sum of the stars $W_x, x \in O_G$. Then (G, W') has the structure of locally star Lie groupoid. Further, the associated star Lie groupoid structure on G makes the associated holonomy morphism $\phi : Hol(G, W) \to G$ a star covering map.*

Proof The verification of the first two properties of a locally star Lie groupoid is trivial. For the third, let $g \in G$ and suppose $Wg^{-1} \cap W$ is non-empty, containing an element u say. Then there is an element $v \in W$ such that $u = vg^{-1}$, and so $g = u^{-1}v$. Hence $R_g = R_v R_{u^{-1}}$. Now $R_{u^{-1}}$ is smooth at u and maps u to 1, while R_v maps 1 to v and is smooth at 1. Hence R_g is smooth at u, and similarly so is its inverse. Hence R_g is a diffeomorphism.

The final part of the proposition now follows from Proposition 2.3, with $(M, \widetilde{W})$ replaced by $(Hol(G, W), i(W))$. □

6 Lie groupoid structure on the monodromy groupoid: the Strong Monodromy Principle

We can now state and prove our main results. These imply Theorem 6.3 which we call the *Strong Monodromy Principle*, namely the globalisation of smooth local morphisms to smooth morphisms on the monodromy groupoid.

Theorem 6.1 *Let G be a locally sectionable Lie groupoid and let W be an open subset of G containing O_G, such that $W = W^{-1}$, and W generates G. Then the monodromy groupoid $M = M(G, W)$ admits the structure of Lie groupoid such that $\tilde{\imath}(W)$ is an open subspace of M and any smooth local morphism on W globalises to a smooth morphism on M.*

Proof The Lie groupoid structure on M follows from Corollary 5.6. Let $f: W \to H$ be a smooth local morphism to a Lie groupoid H. By construction of M, f determines uniquely a morphism $f': M \to H$ of groupoids such that $f'\tilde{\imath} = f$. To prove that f' is C^r, it is enough, by local sectionability, to prove f' is smooth at the identities of M. This follows since $p: M \to G$ maps $\tilde{\imath}(W)$ diffeomorphically to W. □

Theorem 6.2 *Suppose further to the assumptions of Theorem 6.1 that G is star path-conn-ected, that each of its stars has a simply connected covering, and that W^2 is contained in an open neighbourhood V of O_G such that each star V_x in G_x is liftable. Then the projection $p: M(G, W) \to G$ is the universal covering map on each star, and so $M(G, W)$ is isomorphic to the star universal cover ΠG of G.*

Proof The assumptions allow us to construct the star Lie groupoid ΠG. They also imply that we have an isomorphism of underlying star Lie groupoids $M(G, W) \to \Pi G$. This gives the result. □

This result is new even in the group case considered in Douady-Lazard [10].

As a corollary of the previous results we obtain the following:

Theorem 6.3 (Strong monodromy principle) *Let G be a locally sectionable star path-connected Lie groupoid and let W be a neighbourhood of O_G in G such that W satisfies the condition:*
($\star$) W is path-connected and W^2 is contained in a star path-connected neighbourhood V of O_G such that for all $x \in O_G$, V_x is liftable.
Let $f : W \to H$ be a smooth pregroupoid morphism from W to the Lie groupoid H. Then f determines uniquely a morphism $f' : \Pi G \to H$ of Lie groupoids such that $f'j' = f$. □

We end this section by giving a result of Pradines which is taken from the Appendix to [1], and which enables the construction of the set W as in Theorem 6.2.

Theorem 6.4 *Let (G, V) be a locally Lie groupoid whose space X of objects is paracompact. Then there is an open subset W of G containing X and such that $W = W^{-1}$ and $W^2 \subseteq V$.*

Proof A subset $F \subseteq G$ is said to be *anchored* (at $Y \subseteq X$) if $F \cap X = \alpha(F) = \beta(F)\ (= Y)$.

Lemma 6.5 *There exist families $(P_i), (Q_i)$ $(i \in I)$ such that :*

(P_i) is a point finite open covering of X;

Q_i is open in V and anchored at P_i;

$\delta(Q_i \times_\alpha Q_i) \subseteq V$.

Proof For every $x \in X$, there exists, by continuity of δ at (x, x), an open neighbourhood $R(x)$ of x in V such that $\delta(R(x) \times_\alpha R(x)) \subseteq V$. Set $S(x) = R(x) \cap X$.

By paracompactness (more precisely metacompactness) of X, there exists an open covering $(P_i)_{i \in I}$ of X refining $(S(x))_{x \in X}$ and point finite.

Take $Q_i = (\alpha, \beta)^{-1}(P_i \times P_i) \cap R(x(i))$ with $x(i)$ chosen in order to have $P_i \subseteq S(x(i))$, and where $(\alpha, \beta) : G \to X \times X$. □

Now let us start with (P_i), (Q_i) as in the lemma, and denote by I_x (for $x \in X$) the (finite) subset of I consisting of those i's such that $x \in P_i$.

Set $N(x) = \bigcap_{i \in I_x} Q_i$, which is anchored at $S(x) = \bigcap_{i \in I_x} P_i$.

The paracompactness of X implies the existence of a star refinement $(T_j)_{j \in J}$ of (P_i), i.e. an open covering such that the union of the T_k's meeting T_j is contained in some $P_{i(j)}$.

For every $x \in X$, choose $j(x)$ such that x belongs to $Y(x) = T(j(x))$.

Set $U(x) = N(x) \cap (\alpha, \beta)^{-1}(Y(x) \times Y(x))$, which is anchored at $S(x) \cap Y(x)$. Finally, take $U = \bigcup_{x \in X} U(x)$. We claim that $\delta(U \times_\alpha U) \subseteq V$ (then taking $W = U \cap U^{-1}$, one gets $W^2 \subseteq V$ as desired).

To prove this, take $u, u' \in U$ with the same source $y \in X$. One has $u \in U(x)$, $u' \in U(x')$, and $y \in Y(x) \cap Y(x')$, so that there exists an i such that P_i contains both $Y(x)$ and $Y(x')$. Therefore i belongs to both I_x and $I_{x'}$, and Q_i contains both $U(x)$ and $U(x')$. Then one has

$$\delta(u, u') \in \delta(U(x) \times_\alpha U(x')) \subseteq \delta(Q_i \times_\alpha Q_i) \subseteq V$$

and we are done. □

7 Relations between monodromy and holonomy

Let (G, W) be a locally Lie groupoid. Form the holonomy groupoid $H = Hol(G, W)$, as explained in Aof and Brown [2] and section 5. So we have a projection $\phi : H \to G$, and imbedding $i: W \to H$. Then the Lie structure on H is determined by the copy $W' = i(W)$ of W and the smooth local admissible sections of W'.

Now we can form the monodromy groupoid $M = M(H, W')$, and a morphism

$$\xi : M(H, W') \to Hol(G, W)$$

which is the identity on objects. This shows that the holonomy groupoid is a quotient of the monodromy groupoid. Our earlier results show how to construct this monodromy groupoid as a star Lie groupoid, namely by giving G the structure of star Lie groupoid induced from the star topology on W, and then forming ΠG. Under assumptions given earlier, the composition of ξ with $\phi : Hol(G, W) \to G$ is a star universal covering map, and hence ξ is a star universal covering map. This suggests that a direct construction of the holonomy groupoid might be possible as a quotient of ΠG, in the spirit of the method of [19] for the foliation case.

We deal with the relationship of these results to the standard holonomy groupoid of a foliation in the companion paper [7]. Holonomy and monodromy groupoids of the more general local equivalence relations are studied in [15], by topos theoretic methods.

References

[1] AOF, M. E.-S.A.-F. *Topological aspects of holonomy groupoids*, PhD Thesis, University of Wales, Bangor, 1988.

[2] AOF, M.E.-S.A.-F. AND BROWN, R., *The holonomy groupoid of a locally topological groupoid*, Top. Appl., 47 (1992) 97-113.

[3] BROWN, R., *Holonomy and monodromy groupoids*, U.C.N.W. Maths Preprint 82.02. 1982.

[4] BROWN, R., *Topology: A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Ellis Horwood, Chichester; Prentice Hall, New York, 1988.

[5] BROWN R., AND DANESH-NARUIE, G., *The fundamental groupoid as a topological groupoid*, Proc. Edinburgh Math. Soc. 19 (2), 237-244, 1975.

[6] BROWN, R. AND MUCUK, O., *Covering groups of non-connected topological groups revisited*, Math. Proc. Camb. Phil. Soc. 115 (1994) 97-110.

[7] BROWN, R. AND MUCUK, O., *Foliations, locally Lie groupoids and holonomy*, Cah. Top. Géom. Diff. Cat., (1996) (to appear).

[8] CHEVALLEY, C., *Theory of Lie groups*, Princeton University Press, 1946.

[9] CROWELL, R. AND SMYTHE, N., *Congruences and geometric amalgamations of groupoids*, Houston J. Math., 13 (1987) 499-525.

[10] DOUADY, L. AND LAZARD, M., *Espaces fibrés en algèbres de Lie et en groupes*, Invent. Math. 1 (1966) 133-151.

[11] EHRESMANN, C., *Structures feuillétées*, Proc. 5th Conf. Canadian Math. Congress Montreal 1961, "Oeuvres Complètes et Commentées", Amiens 1980-84, II-2, 563-626.

[12] HAEFLIGER, A., *Groupoïde d'holonomie et classifiants*, Astérisques, 116, (1984) 70-97.

[13] HIGGINS, P.J., *Categories and groupoids*, Van Nostrand, New York, 1971.

[14] IYANAGA, S. AND KAWADA. Y., (editors), *Encyclopaedic Dictionary of Mathematics*, Mathematical Society of Japan, and MIT Press, Cambridge, Mass., and London, England, Third edition (1987).

[15] KOCK,A. AND MOERDIJK, I., *Local equivalence relations and their sheaves*, Aarhus Preprint (1991) No. 19, 59pp.

[16] MACKENZIE, K.C.H., *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Note Series 124, Cambridge University Press, 1987.

[17] MACKENZIE, K.C.H., *Lie algebroids and Lie pseudoalgebras*, Bull. London Math. Soc. 27 (1995) 97-147.

[18] MUCUK, O., *Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid*, PhD Thesis, University of Wales, 1993.

[19] PHILLIPS, J., *The holonomic imperative and the homotopy groupoid of a foliated manifold*, Rocky Mountain J. Math., 17 (1987) 151-165.

[20] PRADINES, J., *Théorie de Lie pour les groupoïdes différentiables, relation entre propriétés locales et globales*, Comptes Rendus Acad. Sci. Paris, Sér A, 263 (1966), 907-910.

[21] PRADINES,J., *Théorie de Lie pour les groupoïdes différentiables, Calcul différentiel dans la catégorie des groupoïdes infinitesimaux*, ibid, 264 (1967) 245-248.

[22] PRADINES,J. *Géométrie différentielle au-dessus d'un groupoïde*, ibid, 266 (1968) 1194-1196.

[23] PRADINES, J. *Troisième théorème de Lie pour les groupoïdes différentiables*, ibid, 267 (1968) 21-23.

[24] PRADINES, J., *Private communication*, 1982.

[25] TAYLOR, R.L., *Covering groups of non-connected topological groups*, Proc. Amer. Math. Soc., 5 (1954) 753-768.

Ronald Brown, School of Mathematics, University of Wales, Bangor, Gwynedd LL57 1UT, United Kingdom
email: R.Brown@bangor.ac.uk
Osman Mucuk, Erciyes University, Faculty of Science, Department of Mathematics, Kayseri, TURKEY.
email: mucuk@trerun.earn