Compositions and convolutions in a double groupoid, by Ronald Brown

The definition of double groupoid is given in the references.

On matrix notation.

There is a matrix notation for the compositions in a double groupoid. We write:

$$u +_1 w = \begin{bmatrix} u \\ w \end{bmatrix} \qquad u +_2 v = [u, v].$$

With this notation we can represent all the rules in the definition of double categories. For instance, we have

$$\begin{bmatrix} u \\ |\;| \end{bmatrix} = \begin{bmatrix} u, & = \end{bmatrix} = u.$$

Choosing the matrix description, the 'interchange law' may be written:

$$\begin{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} & \begin{bmatrix} v \\ x \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u & v \end{bmatrix} \\ \begin{bmatrix} w & x \end{bmatrix} \end{bmatrix}$$

and this common value is also written

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix}.$$

Here is a caution about using this interchange law. Let u, v be squares in a double category such that

$$w = \begin{bmatrix} u & v \end{bmatrix} = u +_2 v$$

is defined. Suppose further that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 +_1 v_1 \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_2 +_1 v_2.$$

Then we can assert

$$w = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

only when $u_1 +_2 v_1$, and $u_2 +_2 v_2$ are defined. Thus care is needed in 2-dimensional rewriting.

Thus if we have four functions f, g, h, k on a finite double groupoid we can define a "matrix convolution" by:

$$\begin{bmatrix} f & g \\ h & k \end{bmatrix}(w) = \sum_{\begin{bmatrix} x & y \\ u & v \end{bmatrix} = w} f(x)g(y)h(u)k(v)$$

This has a generalisation which uses a more general matrix notation for multiple compositions.

Definition 0.1 Let D be a double category. A *composable array* (u_{ij}) in D, is given by elements $u_{ij} \in D_2$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ satisfying

$$\begin{cases} \partial_2^+ u_{i,j-1} = \partial_2^- u_{i,j} & (1 \leqslant i \leqslant m, 2 \leqslant j \leqslant n), \\ \partial_1^+ u_{i-1,j} = \partial_1^- u_{i,j} & (2 \leqslant i \leqslant m, 1 \leqslant j \leqslant n). \end{cases}$$

It follows from the interchange law that a composable array (u_{ij}) in D can be composed both ways, getting the same result which is denoted by $[u_{ij}]$.

If $u \in D_2$, and (u_{ij}) is a composable array in D satisfying $[u_{ij}] = u$, we say that the array (u_{ij}) is a *subdivision* of u. We also say that u is the *composite* of the array (u_{ij}). □

Subdivisions and their use. The interchange law implies that if in the composable array (u_{ij}) we partition the rows and columns into blocks which produce another composable array and compute the composite v_{kl} of each block, then $[u_{ij}] = [v_{kl}]$. We call the (u_{ij}) a *refinement* of (v_{kl}) in this case.

So now we can define general matrix convolutions by:

$$\left[f_{ij}\right](w) = \sum_{[w_{ij}]=w} \prod f_{ij}(w_{ij}).$$

The problem is to relate all these, since when we write $[w_{ij}] = w$ we are assuming that all the internal compositions are defined so in particular when we look at one row of $[w_{ij}]$ then that does not consist all of possible rows but only those which are defined with those above and below.

References

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