Kan complexes and multiple groupoid structures

By M. K. Dakin

Abstract: The purpose of this work is to define the notion of a T-*complex* and to give some of its basic properties. A T-complex is a simplicial set X with certain special elements in each dimension. These special elements are called *thin* and are required to satisfy the following three axioms:

(A1) all degenerate elements of X are thin,

(A2) any horn in X has a unique thin filler,

(A3) if all faces but one of a thin simplex of X are themselves thin, then so also is the last face.

Further a T-complex is said to be *of rank* n if above dimension n it consists only of thin simplices.

We show how the axioms for a T-complex X enable us to define n groupoid structures on the set X_n of n-simplicies. In particular we prove that the category of T-complexes of rank 1 is equivalent to the category of groupoids and that the category of rank 2 is equivalent to the category of *crossed modules over groupoids*. A crossed module over a groupoid is an extension of the idea of a crossed module as defined by Whitehead [11] where one has a morphism $d : A \to B$ of groups together with a group action of B on A written a^b and satisfying

(i) $d(a^b) = b^{-1}d(a)b$ and $a^{da'} = a'^{-1}aa'$.

The higher dimensional generalisation of a crossed module is a *crossed chain complex*, originally defined by Whitehead [11] and called by him a homotopy system, and we show how, by using relative homotopy groups, one can obtain a crossed chain complex from a T-complex.

Keywords: Simplicial sets, Simplicial T-complex, Kan complex, Crossed complex, Crossed modules, Crossed chain complex, Kan fibration, Homotopy, Holomolgy, Simplicial nerve, Simplicial subdivision.

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KAN COMPLEXES

AND

MULTIPLE GROUPOID STRUCTURES

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D E C L A R A T I O N

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

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CONTENTS

CHAPTER 5

T-Complexes of Rank 2

ABSTRACT

The purpose of this work is to define the notion of a T-complex and to give some of its basic properties. A T-complex is a simplicial set X with certain special elements in each dimension. These special elements are called thin and are required to satisfy the following three axioms :

(A1) all degenerate elements of X are thin,

(A2) any horn in X has a unique thin filler,

(A3) if all faces but one of a thin simplex of X are themselves thin, then so also is the last face.

Further a T-complex is said to be of rank n if above dimension n it consists only of thin simplices .

We show how the axioms for a T-complex X enable us to define n groupoid structures on the set X_n of n-simplices . In particular we prove that the category of T-complexes of rank 1 is equivalent to the category of groupoids and that the category of T-complexes of rank 2 is equivalent to the category of crossed modules over groupoids. A crossed module over a groupoid is an extension of the idea of a crossed module as defined by Whitehead [11] where one has a morphism $d: A \to B$ of groups together with a group action of B on A written a^b and satisfying (i) $d(a^b) = b^{-1}d(a)b$ and (ii) $a^{da'} = a'^{-1} a\, a'$.

The higher dimensional generalisation of a crossed module is a crossed chain complex, originally defined by Whitehead [11] and called by him a homotopy system, and we show how, by using relative homotopy groups, one can obtain a crossed chain complex from a T-complex.

INTRODUCTION

As part of a programme of work to obtain new information on the computation of second relative homotopy groups of topological pairs, R. Brown and C.B. Spencer [3] defined the notion of a double groupoid with connection. This was applied by R. Brown and P.J. Higgins [2] and new results on the computation of certain second relative homotopy groups were produced. The question of generalising these results to higher dimensions then arose, but it was not clear what the algebraic object generalising the idea of a double groupoid ought to be. In this thesis we put forward a suitable candidate and call it a T-complex.

A T-complex is a simplicial set X with certain special elements in each dimension. These special elements are called thin and are required to satisfy three simple axioms. These are

(A1) all degenerate elements of X are thin,

(A2) any horn in X has a unique thin filler,

(A3) if all faces but one of a thin simplex of X are themselves thin, then so also is the last face thin.

Chapter 1 of this thesis is devoted mainly to the definition of a T-complex and to showing how a T-complex has a natural filtration similar to the filtration of a simplicial set by its skeleta. We call this the T-filtration and we shall make use of it later.

A T-complex is said to be of rank n if above dimension n it consists only of thin simplices. In Chapter 2 we show that the category of

T-complexes of rank 1 is equivalent to the category of groupoids. This is done by showing that the simplicial nerve of a groupoid, as defined by Segal [9], is a T-complex and by showing how, conversely, the axioms for a T-complex enable us to define a canonical groupoid structure on the set of its 1-simplices.

More generally, we can obtain n groupoid structures on the set of n-simplicies of a T-complex and Chapter 3 shows how this is done. One might expect to obtain in this way a multiple category in the sense of Wyler [12], where we have an <u>interchange law.</u> In other words, we would have for each pair of groupoid compositions o_r and o_s, an identity of the form $(x\ o_r\ y)o_s(z\ o_r w) = (x\ o_s\ z)o_r(y\ o_s\ w)$. We have been unable to prove this for simplicial T-complexes although we note that P.J. Higgins has applied the axioms for a T-complex to a cubical set and has shown how one does then obtain an interchange law.

One of the essential parts of the work of Brown, Higgins and Spencer is that if we restrict ourselves to double groupoids with connection possessing only one vertex, then the category so formed is equivalent to the category of <u>crossed modules</u>. A crossed module (originally defined by Whitehead in [11]) is a triple (A, B, d) where $d: A \longrightarrow B$ is a morphism of groups such that there is a group action of B on A written a^b and satisfying (i) $d(a^b) = b^{-1}d(a)b$ and (ii) $a^{da'} = a'^{-1}a\,a'$. Brown and Higgins exploit the fact that if (X, A) is a topological pair then the homotopy groups $\pi_2(X, A)$ and $\pi_1(A)$ together with the boundary map between them constitute a crossed module. In Chapter 4 we follow

Lamotke [4] in showing that for a simplicial pair (X, A) one also obtains a crossed module $d : \pi_2(X, A) \rightarrow \pi_1(A)$. By taking the pair (X, A) to be the bottom two levels of the T-filtration of a T-complex, one obtains a particular crossed module associated to the T-complex.

The higher dimensional generalisation of a crossed module is the notion of a homotopy system as defined by Whitehead [11]. Brown and Higgins call this a crossed chain complex. A crossed chain complex C is essentially a chain complex $(C_n)_{n \geqslant 1}$ where each C_n for $n > 2$ is a C_1/dC_2-module and C_2 is a crossed C_1-module. Using Lamotke's methods we show further in Chapter 4 how one obtains a crossed chain complex from a T-complex by using the T-filtration. Crossed chain complexes were defined by Blakers in [1] but called group systems. Blakers went on to show how one could obtain a simplicial set from a group system.

Finally in Chapter 5 we prove an equivalence of categories. In order to avoid having to select one basepoint, we define the idea of a crossed module over a groupoid and prove that the category of T-complexes of rank 2 is equivalent to the category of crossed modules over groupoids In defining the simplicial nerve of a crossed module we show that there is a connection between this theory and the homotopy addition lemma since we require thin simplices to, in a suitable sense, have the sum of their faces zero.

An important conjecture is that the category of all T-complexes possessing only one vertex is equivalent to the category of crossed chain complexes. More generally one could define a crossed chain complex C over a groupoid where C_1 is a groupoid rather than a group and we have a crossed chain complex over each object of C_1. We conjecture that the category of these objects would be equivalent to the category of T-complexes.

The work of this thesis is by no means complete; its purpose is rather to define the notion of a T-complex and to suggest areas for future work. In particular the methods of proof need to be refined to avoid the necessity of using complicated diagrams as in Chapter 5. The application to topology is to construct a T-complex from a topological filtration by using maps of standard geometrical n-simplices into the filtration and then taking homotopy classes. By taking the associated crossed chain complex one then obtains relative homotopy groups of the filtered space. Brown and Higgins have done work on this using cubical T-complexes and they have shown that if one begins with a pushout of topological spaces then, under certain assumptions, one obtains a pushout in the category of T-complexes. By taking the associated crossed chain complex, they then obtain some new results on the relative homotopy groups.

Unless otherwise stated, all references to prior results obtained in this thesis refer to propositions, etc., given in the same chapter as that in which the reference is used.

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CHAPTER 1

T-Complexes

In our first chapter we define the notion of a T-complex and show how any T-complex has a natural filtration by dimension in a similar way that a simplicial set has a filtration by its n-skeletons. We also prove a general result (Theorem 3.1) on homomorphisms of T-complexes which we shall make use of later.

Before beginning, we establish some notation. For the face and degeneracy maps of a simplicial set we shall write d_i and s_i ; Δ^n will denote the standard n-simplex and $\Delta^{n,p}$ the subcomplex of Δ^n generated by the p-dimensional part of Δ^n. $\dot{\Delta}^n$ will denote the boundary of Δ^n. The p'th horn of Δ^n will be denoted by Λ^p_n and the p'th horn of an arbitrary simplex x by Λ^p_x. We shall also write $\Lambda^p x$ as the sequence $(x_0,\ldots,x_{p-1},-,x_{p+1},\ldots,x_n)$ where $x_i = d_i x$, the i'th face of x.

§1. The definition of a T-complex

DEFINITION 1.1 A T-complex consists of a pair (X,T) where X is a simplicial set and $T = (T_i)_{i \geqslant 1}$ is a graded subset of X with $T_i \subseteq X_i$. Elements of T are called thin and the following three axioms are satisfied:

(A1) all degenerate elements of X are thin,

(A2) any horn in X has a unique thin filler,

(A3) if all but one of the faces of a thin element of X are themselves thin, then so also is the remaining face thin.

Further, (X,T) is said to be <u>of rank n</u> if $X_i = T_i$ for all $i > n$ but $X_n \neq T_n$.

A <u>morphism of T-complexes</u> is a simplicial map taking thin elements into thin elements and this gives us the category of T-complexes which we denote by $\underset{\sim}{T}$.

Given a horn h in a T-complex (X,T), we shall denote the thin filler of h by $T(h)$. This will be standard notation throughout this work.

For brevity, we shall generally refer to a T-complex (X,T) by the simplicial set X with the understanding that X has thin elements. Note that a T-complex X is certainly a Kan complex but of course a horn in X may have many other Kan fillers besides its unique thin filler.

It is clear from the axioms that T-complexes could equally well be defined using cubical sets rather than simplicial sets and P.J. Higgins has investigated some of the properties of cubical T-complexes.

Another alternative is to define the notation of a T-complex using <u>Δ-sets</u> rather than simplicial sets. A Δ-set, defined by Rourke and Sanderson [8], is a simplicial set for which no degenerate elements are defined. Using this method dispenses with axiom A1. Rourke and Sanderson have proved that a Δ-set satisfying the Kan extension condition admits a (non-canonical) set of degenerate elements and so becomes a Kan complex, but in our case we may simply define the degenerate simplices as thin simplices with suitable boundaries so that axiom A1 is automatically satisfied.

A further point about our axioms is that Levi [6] has described a method of representing the law of composition in a group G by means of a relation on $G \times G \times G$ written $[a,b,c]$ when $a \circ b \circ c = 1$. Levi gives a number of axioms which must be satisfied and one of these axioms is essentially our axiom A3 in dimension 3.

§2. The T-filtration of a T-complex

Let (X,T) be a T-complex. As a simplicial set, X has a natural filtration by the n-skeletons of X. However these are not T-complexes. We shall now show how the n-skeletons of X do generate T-complexes, thus giving a natural filtration of (X,T). This filtration will be called the T-filtration

For each $n \geqslant 0$, we define the graded set $\bar{X}^n$ recursively by

$$(\bar{X}^n)_k = \begin{cases} X_k & k \leqslant n \\ \{x \in T_k : d_i x \in (\bar{X}^n)_{k-1} \text{ for all } i\} & k > n \end{cases}$$

Thus $\bar{X}^n$ is a graded subset of X. We write $\bar{X}^n_k$ instead of $(\bar{X}^n)_k$. Notice that in fact, when $k > n$, $\bar{X}^n_k$ is just the set of all those thin elements of X_k whose i-dimensional faces, for all $i > n$, are themselves thin.

When $n \geqslant \operatorname{rank} X$, we have $\bar{X}^n = X$. Otherwise, we shall show that $\bar{X}^n$ has the structure of a T-complex of rank at most n.

LEMMA 2.1 <u>$\bar{X}^n$ is a simplicial subset of X.</u>

PROOF. We need to check that the faces and degeneracies of elements of $\bar{X}^n$ do themselves lie in $\bar{X}^n$.

Firstly, if $x \in \bar{X}^n_k$, then if $k \leq n+1$ we have, for each i, $d_i x \in X_{k-1}$ and if $k > n+1$ we have, by definition, $d_i x \in \bar{X}^n_{k-1}$.

Secondly, suppose $x \in \bar{X}^n_k$. If $k \leq n-1$, then for each i, $s_i x \in X_{k+1} = \bar{X}^n_{k+1}$. If, on the other hand, $k > n-1$ then we need to use induction to show that each $s_i x$ belongs to $\bar{X}^n_{k+1}$. Suppose as induction hypothesis that if $y \in \bar{X}^n_{k-1}$, then for all i, $s_i x \in \bar{X}^n_{k+1}$. Now each $s_i x$ is a thin element of X_{k+1} with each face being either x, which belongs to $\bar{X}^n_k$, or $s_p d_q x$ for some p and q. But we have already proved that $d_q x \in \bar{X}^n_{k-1}$ and so, by the induction hypothesis, $s_p d_q x \in \bar{X}^n_k$. Thus each $s_i x$ is a thin element of X_{k+1} with each face lying in $\bar{X}^n_k$, and so, by definition, each $s_i x$ belongs to $\bar{X}^n_{k+1}$. Finally, to start the induction, when $k = n$ we have $\bar{X}^n_k = X_k$, and so each $s_i x$ is certainly a thin element of X_{k+1} with all faces lying in $\bar{X}^n_k$. In other words each $s_i x$ is an element of $\bar{X}^n_{k+1}$. This completes the proof.

Next we show how each $\bar{X}^n$ may be given the structure of a T-complex. Define a graded subset $\bar{T}^n = \{\bar{T}^n_k\}_{k \geq 1}$ of $\bar{X}^n$ by

$$\bar{T}^n_k = \{x \in T_k : \text{for each } i,\ d_i x \in \bar{X}^n_{k-1}\}$$

$\bar{T}^n$ will be the set of thin elements of $\bar{X}^n$.

PROPOSITION 2.2 <u>For each $n \geq 0$, $(\bar{X}^n, \bar{T}^n)$ is a T-complex.</u>

PROOF. We must verify the three axioms A1, A2, and A3 of the definition of a T-complex.

A1 (degeneracies are thin) : let $x \in \bar{X}^n_k$ for some k then, for each i, $s_i x \in T_{k+1}$ by axiom A1 for (X,T) and, by Lemma 2.1, $d_j s_i x \in X^n_k$ for each j. Hence, by definition, $s_i x \in \bar{T}^n_{k+1}$.

A2 (every horn has a unique thin filler): let

$$h : \Lambda^p_k \longrightarrow X^n$$

be a k-horn in X^n. Then by axiom A2 for (X,T) there exists a unique

$$\bar{h} : \Delta^k \longrightarrow T \subseteq X$$

extending h. Writing x for the k-simplex defined by $\bar{h}$, we must show that $x \in \bar{T}^n$. We already know that $x \in T$ and so, by definition, it remains to show that, for each i, $d_i x \in \bar{X}^n$.

Case (i) $k \leq n+1$: we have $d_i x \in X_{k-1} = \bar{X}^n_{k-1}$.

Case (ii) $k > n+1$: for all $i \neq p$ we have

$$d_i x \in \bar{X}^n_{k-1} = \{ y \in T_{k-1} : d_j y \in \bar{X}^n_{k-2} \ \forall j \}$$

Thus, firstly, $d_i x \in T$ for all $i \neq p$, from which it follows by axiom A3 for (X,T) that $d_p x \in T$ also and, secondly, $d_j d_i x \in \bar{X}^n$ for all $i \neq p$ and all j, from which it follows that $d_j d_p x \in \bar{X}^n$ for all j also. Hence by definition, $d_i x \in \bar{X}^n$ for all i.

This verifies the two cases and so $x \in \bar{T}_n$. Thus h has a unique extension over Δ^k in T^n and this verifies axiom A2 for $(\bar{X}^n, \bar{T}^n)$.

A3 (if all faces but one of a thin element are themselves thin, then so also is the last face):

suppose x is an element of $\bar{T}^n$ such that $d_i x \in \bar{T}^n$ for all $i \neq p$ (say). Then by axiom A3 for (X,T), $d_p x \in T$. Further, by Lemma 2.1, $d_j d_p x \in \bar{X}^n$ for all j and so, by definition, $d_p x \in \bar{T}^n$. This verifies axiom A3.

THEOREM 2.3 Suppose (X,T) is a T-complex of rank q. If $n<q$ then the T-complex $(\bar{X}^n, \bar{T}^n)$ has rank at most n, and if $n \geq q$ then

$$(\bar{X}^n, \bar{T}^n) = (X,T)$$

PROOF. Firstly, for all n we recall that

$$X^n_k = \begin{cases} X_k & k \leq n \\ \{x \in T_k : d_i x \in \bar{X}^n_{k-1} \text{ for all } i\} & k > n \end{cases}$$

and

$$\bar{T}^n_k = \{x \in T_k : d_i x \quad X^n_{k-1} \text{ for all } i\}$$

Thus $\bar{T}^n_k = \bar{X}^n_k$ whenever $k > n$ and so $(\bar{X}^n, \bar{T}^n)$ cannot have rank greater than n.

Suppose now that $n \geq q$. For $k \leq n$ we have by definition $\bar{X}^n_k = X_k$ and $\bar{T}^n_k = T_k$. When $k > n$ we check this easily by induction. Suppose $\bar{X}^n_{k-1} = X_{k-1}$, then we have

$$\begin{aligned} \bar{X}^n_k &= \{x \in T_k : d_i x \in X_{k-1} \text{ for all } i\} \\ &= T_k \\ &= X_k \end{aligned}$$

since $k > q$. But we already know that $\bar{X}^n_n = X_n$ and so $\bar{X}^n_k = X_k$ for all $k > n$. Further, for $k > n$ we have

$$\bar{T}^n_k = \bar{X}^n_k = X_k = T_k$$

and so we have checked $(\bar{X}^n_k, \bar{T}^n_k) = (X_k, T_k)$ for all k.

PROPOSITION 2.4 Suppose (X,T) is a T-complex. For each $n \geq 1$, there is an inclusion of T-complexes

$$i_n : (\bar{X}^n, \bar{T}^n) \longrightarrow (\bar{X}^{n+1}, \bar{T}^{n+1})$$

PROOF. First we check that $\bar{X}_k^n \subseteq \bar{X}_k^{n+1}$ for all k. For $k \leq n$ we have $\bar{X}_k^n = X_k = \bar{X}_k^{n+1}$ and further we have $\bar{X}_{n+1}^n = T_{n+1} \subseteq X_{n+1} = \bar{X}_{n+1}^{n+1}$. When $k > n+1$, suppose as an induction hypothesis that $\bar{X}_{k-1}^n \subseteq \bar{X}_{k-1}^{n+1}$ (which we already know is true when $k = n+2$), then, since we have

$$\bar{X}_k^n = \{x \in T_k : d_i x \in \bar{X}_{k-1}^n \text{ for all } i\}$$

$$X_k^{n+1} = \{x \in T_k : d_i x \in \bar{X}_{k-1}^{n+1} \text{ for all } i\}$$

it follows that $\bar{X}_k^n \subseteq \bar{X}_k^{n+1}$. Hence by induction, $\bar{X}_k^n \subseteq \bar{X}_k^{n+1}$ for all $k > n+1$.

It now follows by Lemma 2.1 that there is a simplicial inclusion

$$i_n : \bar{X}^n \longrightarrow \bar{X}^{n+1}$$

Now for all k we have by definition

$$\bar{T}_k^n = \{x \in T_k : d_i x \in \bar{X}_{k-1}^n \text{ for all } i\}$$

$$\bar{T}_k^{n+1} = \{x \in T_k : d_i x \in \bar{X}_{k-1}^{n+1} \text{ for all } i\}$$

and so by the above we have $\bar{T}_k^n \subseteq \bar{T}_k^{n+1}$. Hence i_n is an inclusion of T-complexes

COROLLARY 2.5 <u>For each n, the T-complex $(\bar{X}^n, \bar{T}^n)$ is a sub-T-complex of (X,T)</u>

PROOF. This follows either from Lemma 1.1 on noting that $\bar{T}_k^n \subseteq T_k$ for each k, or from Theorem 1.3 and proposition 1.4 together.

We have now shown how a T-complex X has a natural filtration by the T-complexes $\bar{X}^n$. We shall call this filtration the <u>T-filtration</u>

of X. Note that as simplicial sets, the $\bar{X}^n$ are of course not the same as the simplicial n-skeletons of X, since these latter consist only of degenerate elements above dimension n, whilst the $\bar{X}^n$ contain other thin elements. However it is easy to see that as a simplicial set, each $\bar{X}^n$ is just the Kan extension of the n-skeleton.

§3. A theorem on homomorphisms

Since a T-complex of rank n has only thin elements above level n, one might suspect that the maps comprising a homomorphism of T-complexes need only be specified up to level n. In order to prove this kind of result however, we have had to specify the maps at level n+1 also. We obtain the following theorem.

THEOREM 3.1 Suppose (X,S) and (Y,T) are T-complexes of rank n. Let

$$f = \{ f_i : (X_i, S_i) \longrightarrow (Y_i, T_i) \}_{i=0}^{n+1}$$

be a collection of maps of pairs satisfying $d_j f_i = f_{i-1} d_j$ for all $j \leq i \leq n+1$. Then f extends uniquely to a morphism $(X,S) \longrightarrow (Y,T)$ of T-complexes. If further each f_i $(0 \leq i \leq n+1)$ is a bijection of pairs, then the extension is an isomorphism.

PROOF. Denoting the postulated extension by $f = \{f_i\}_{i \geq 0}$ also, we shall

(i) define f recursively on each X_i $(i > n+1)$ in turn;

(ii) prove that f is a simplicial map and deduce that it is a morphism of T-complexes;

(iii) prove that f is unique as an extension;

(iv) show that if each f_i $(0 \leq i \leq n+1)$ is a bijection of pairs, then the extension is an isomorphism.

(i) Suppose that f is defined on X^{k-1} ($k > n+1$) and that $d_i f = fd_i$ for all $i \leq k-1$.

Let x be a point of X_k ($= S_k$) and let p be an integer satisfying $0 \leq p \leq k$. Then, since $d_i d_j x = d_{j-1} d_i x$ implies that $d_i fd_j x = d_{j-1} fd_i x$ (where $0 \leq i < j \leq k$), it follows that $f \wedge^p x$ is a horn in Y. Thus we may define

$$f_k x = T(f_{k-1} \wedge^p x)$$

However, it will not be sufficient for us to do this for a given <u>fixed</u> p and so we check that $T(f_{k-1} \wedge^p x)$ is in fact independent of the choice of p.

We already know that $d_i T(f \wedge^p x) = fd_i x$ for $i \neq p$ and we check further that $d_p T(f \wedge^p x) = fd_p x$. Now by the above remark, since $fd_i x$ is thin for all i ($d_i x$ is thin), it follows by axiom A3 that $d_p T(f \wedge^p x)$ is thin also. Thus, since a thin q-simplex is entirely determined by any q of its faces, it is sufficient for us to check that

$$\wedge^p d_p T(f \wedge^p x) = \wedge^p fd_p x$$

Let $t = T(f \wedge^p x)$ then, using the simplicial identities, we have

$$\wedge^p d_p t = (d_o d_p t, \ldots, \underline{p}, \ldots, d_{k-1} d_p t)$$

$$= (d_{p-1} d_o t, \ldots, d_{p-1} d_{p-1} t, -, d_p d_{p+2} t, \ldots, d_p d_k t)$$

$$= f(d_{p-1}d_o x, \ldots, d_{p-1}d_{p-1}x, -, d_p d_{p+2}x, \ldots, d_p d_k x)$$

$$= f(d_o d_p x, \ldots, \underline{p}, \ldots, d_{k-1}d_p x)$$

$$= (d_o f d_p x, \ldots, \underline{p}, \ldots, d_{k-1} f d_p x)$$

$$= \Lambda^p f d_p x$$

as required. Thus

$$d_i T(f \Lambda^p x) = f d_i x$$

for all $i \leqslant k$ and so, by the uniqueness in axiom A2, $T(f \Lambda^p x)$ is independent of the choice of p. Our definition of $f_k x$ may therefore be written

$$f_k x = T(f_{k-1} \Lambda^p x)$$

where p is any arbitrary number less than k, $f_k x$ being invariant under the choice of p. From this we have $d_i f_k = f_{k-1} d_i$ for all $i \leqslant k$ and so f is defined on X^k and satisfies $d_i f = f d_i$ for all $i \leqslant k$.

Finally, by assumption, f is defined on X^{n+1} with $d_i f = f d_i$ and so $f = \{f_i\}_{i \geqslant o}$ is defined recursively over the whole of X and satisfies $d_i f = f d_i$ for all $i \geqslant 0$. Note that, by definition, f preserves thin elements.

(ii) We already know that f satisfies $d_i f = f d_i$ for all $i \geqslant 0$ and so in order to show that f is simplicial we must simply check that $s_i f = f s_i$ for all $i \geqslant 0$. For this we use induction. Suppose $s_i f = f s_i$ on X^{k-1}, that is, for all $i \leqslant j \leqslant k-1$, $s_i f_j = f_{j+1} s_i$ and let x be a point of X_k. Then, for all $i \leq k$, by axiom A1 and using the simplicial identities we have

$$s_i x = S(s_{i-1}d_o x,\ldots,s_{i-1}d_{i-1}x,x,\overset{i+1}{-},s_i d_{i+1}x,\ldots,s_i d_k x)$$

and so, as f preserves thin elements,

$$fs_i x = T(fs_{i-1}d_o x,\ldots,fs_{i-1}d_{i-1}x,fx,\overset{i+1}{-},\ fs_i d_{i+1}x,\ldots,fs_i d_k x)$$

$$= T(s_{i-1}d_o fx,\ldots,s_{i-1}d_{i-1}fx,fx,\overset{i+1}{-},s_i d_{i+1}fx,\ldots,s_i d_k fx)$$

by assumption

$$= s_i fx$$

Thus $s_i f = fs_i$ for all $i \leq k$ on X^k. But we know that for all points x of X_o, $s_o fx = fs_o x$ since f preserves thin elements and, by axioms A1 and A2, the only thin elements of X_1 are the degeneracies. Hence, by induction, we have $s_i f = fs_i$ (for all i) over the whole of X and so f is a simplicial map. Moreover, since f preserves thin elements, it is a morphism of T-complexes.

(iii) Suppose that f and f' are two extensions so that $f_i = f'_i$ for $i \leq n+1$. If $f \neq f'$ then there exists $k > n+1$ such that $f_k \neq f'_k$. Let x be a point of X_k such that $f_k x \neq f'_k x$, that is

$$T(f_{k-1} \wedge P_x) \neq T(f'_{k-1} \wedge P_x)$$

Hence $f_{k-1} \wedge^p x \neq f'_{k-1} \wedge^p x$ and so $f_{k-1} \neq f'_{k-1}$. Continuing the process we obtain by induction that $f_{n+1} \neq f'_{n+1}$. But this is false and so $f = f'$.

(iv) Now suppose we are given that each f_i for $0 \leq i \leq n+1$ is a bijection of pairs. First we show that the remaining f_i's as defined in (i) are bijective. Again we use induction. Firstly suppose $f_{k-1} (k > n+1)$ is injective, then, if $f_k x' = f_k x$, that is

$$T(f_{k-1} \wedge^p x) = T(f_{k-1} \wedge^p x^1)$$

where x and x' are points of $X_{k-1} = S_{k-1}$, it follows that $\wedge^p x = \wedge^p x'$ and so, since x and x' are thin, $x = x'$. Thus f_k is injective. But we know f_{n+1} is injective and so by induction it follows that f_i is injective for all $i \geq 0$. Secondly, suppose that f_{k-1} $(k > n+1)$ is surjective and let y be a point of $Y_k = T_k$. Then $y = T(\wedge^p y)$ $(0 \leq p \leq k)$. Now, since f_{k-1} is surjective, there exist points x_j for $j = 0,\ldots,k$ such that $d_j y = f_{k-1} x_j$ and, using the simplicial identities, we have

$$f_{k-2} d_i x_j = d_i d_j y = d_{j-1} d_i y = f_{k-2} d_{j-1} x_i$$

for all $i < j \leq k$. But f_{k-2} (in particular) is injective, and so $d_i x_j = d_{j-1} x_i$ for all $i < j \leq k$. It follows that, given p

$$h = (x_0,\ldots,x_{p-1},-,x_{p+1},\ldots,x_k)$$

is a horn in X_{k-1} and we define $x = S(h)$. Then we have

$$\begin{aligned} f_k x &= T(f_{k-1}\Lambda^p x) \\ &= T(f_{k-1}h) \\ &= T(f_{k-1}x_o,\ldots,p,\ldots,f_{k-1}x_k) \\ &= T(d_o y,\ldots,\underline{p},\ldots,d_k y) \\ &= y \end{aligned}$$

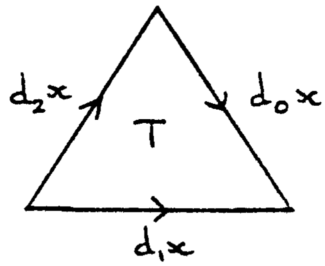
and it follows that f_k is surjective. But we know that f_{n+1} is surjective and so, by induction, f_i is surjective for all $i \geqslant 0$. Thus, since for $i > n+1$, $X_i = S_i$ and $Y_i = T_i$, it follows that $f_i : (X_i, S_i) \longrightarrow (Y_i T_i)$ is a bijection of pairs for all $i \geqslant 0$.

Next, denoting the inverse function of f by f^{-1}, since f is a simplicial map it follows that f^{-1} is also a simplicial map. Further f^{-1} must preserve thin elements and so it is a morphism of T-complexes. Thus f is an isomorphism of T-complexes and this completes the proof of the theorem.

The Groupoid Structure in Dimension 1 and T-complexes of Rank 1

In this chapter we show how the axioms for a T-complex X enable one to define a law of composition on the set X_1 of 1-simplices of X so that X_1 becomes a groupoid. We then show that the simplicial nerve of a groupoid, as defined by Segal [9], is in fact a T-complex and we deduce that the category of T-complexes of rank 1 is equivalent to the category of groupoids.

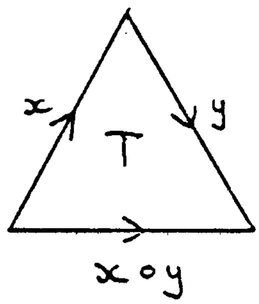
Let (X,T) be a T-complex. Given a thin 2-simplex x of X_1, x is entirely determined by any two of its faces and so we may represent x by the diagram



where the letter T denotes that the diagram represents the thin simplex determined by the given faces.

We define a partial law of composition on X_1 as follows : suppose $x,y \in X_1$ are such that $d_0x = d_1y$. We define the composite $x \circ y$ by

$$x \circ y = d_1T(y,-,x)$$



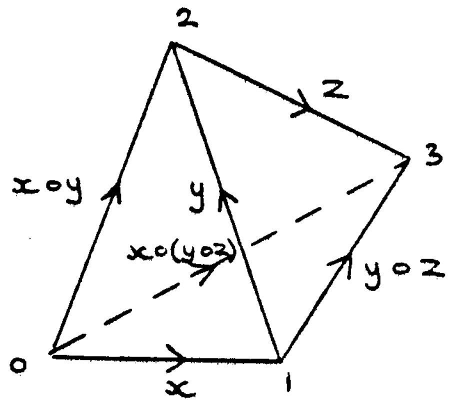
PROPOSITION 1 <u>With the above law of composition, X_1 is a groupoid over X_0.</u>

PROOF. We check that (i) the law of composition o is associative, (ii) the degeneracy map $s_0 : X_0 \to X_1$ gives identities and (iii) inverses exist.

(i) Let x,y,z be elements of X_1 satisfying $d_0 x = d_1 y$ and $d_0 y = d_1 z$ so that the composites (x o y) o z and x o (y o z) exist.

Construct the thin 3-simplex

$$t = T(T(z,-,\ y),-,\ T(y \circ z,\ -,\ x),\ T(y,-,x)\ )$$



By axiom A3 of definition 2.1, $d_1 t$ is thin and so $x \circ (y \circ z) = d_1 T(z,-,x \circ y)$, that is $x \circ (y \circ z) = (x \circ y) \circ z$.

(ii) Suppose that $x \in X_1$ then, by axiom A1 together with the uniqueness in axiom A2,

$$x \circ (s_0 d_0 x) = d_1 T(s_0 d_0 x,-,x)$$

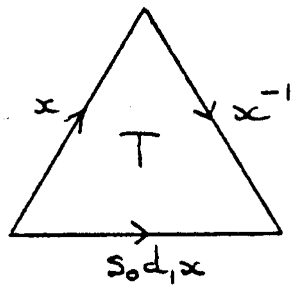
$$= d_1 s_1 x$$

$$= x$$

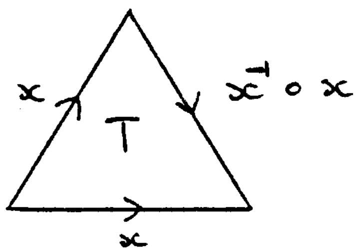
Similarly $(s_o d_1 x) \circ x = x$ and we write $s_o a = 1_a$ for each vertex a of X_o.

(iii) Let $x \in X_1$ and define $x^{-1} \in X_1$ by

$$x^{-1} = d_o T(-, s_o d_1 x, x)$$



so that $x \circ x^{-1} = s_o d_1 x$ (where we have made use of the uniqueness in axiom A2) Further, by associativity, we then have $x = (x \circ x^{-1}) \circ x = x \circ (x^{-1} \circ x) = d_1 T(x^{-1} \circ x, -, x)$ and so $x^{-1} \circ x = d_o T(-, x, x)$. But by axiom A1, $T(-, x, x)$ is degenerate and so we have $x^{-1} \circ x = s_o d_o x$



This completes the proof.

We denote the groupoid obtained in the above fashion by G(X) and it is clear that we have a functor G from the category of T-complexes to the category of groupoids.

Now let N be the simplicial nerve functor defined by Segal [9]. Then, if C is a category, the 1-simplices of NC are the elements of C, the 2-simplices are commutative triangles of elements of C and so on.

PROPOSITION 2 <u>N determines a functor from the category of groupoids to the category of T-complexes of rank 1.</u>

PROOF. Suppose Γ is a groupoid. Define a set of T of thin elements for $N\Gamma$ by letting T_1 be the degenerate 1-simplices of $N\Gamma$ and letting $T_i = N\Gamma_i$ for $i \geqslant 2$. The axioms are trivially verified.

THEOREM 3 <u>The pair of functors G and N give an equivalence between the category of T-complexes of rank 1 and the category of groupoids.</u>

PROOF. We must construct a pair of natural transformations $NG \simeq 1$ and $GN \simeq 1$. This is quite trivial for firstly if (X,T) is a T-complex of rank 1 then $NG(X)$ has rank 1 and

$$NG(X)_o = \mathrm{Ob}G(X) = X_o$$

$$NG(X)_1 = \mathrm{Arr}G(X) = X_1$$

$$\begin{aligned} NG(X)_2 &= \{\text{commutative triangles in } G(X)\} \\ &\cong T_2 \\ &= X_2 \end{aligned}$$

this last isomorphism being canonical. It is easy to see that, using Theorem 3.1 of Chapter I, a natural transformation $NG \simeq 1$ is canonically defined, the naturality being immediate.

Secondly, if Γ is a groupoid, then

$$\mathrm{Ob}\,GN(\Gamma) = N\Gamma_o = \mathrm{Ob}\,\Gamma$$

$$\mathrm{Arr}\ GN(\Gamma) = N\Gamma_1 = \mathrm{Arr}\,\Gamma$$

Composition in both Γ and $GN(\Gamma)$ corresponds to commutative triangles in $N\Gamma$ and so a natural transformation $GN \simeq 1$ is canonically defined. Naturality is again immediate.

As a corollary we have the following result of Lee [5]:

COROLLARY 4 <u>Let C be a category, then NC is a Kan complex if and only if C is a groupoid.</u>

PROOF. If C is a groupoid then NC is a T-complex and so certainly a Kan complex. On the other hand, suppose that NC is a Kan complex and let $x \in C(p,q)$. We need to show that there is an element $x^{-1} \in C(q,p)$ such that $x \circ x^{-1} = 1_p$. Consider the horn $(-,1_p,x)$ in NC_1. Since NC is a Kan complex, this horn has a filler u and we let $x^{-1} = d_o u$. Now the 2-simplices of NC are commutative triangles in C_1 and so we have $x \circ x^{-1} = 1_p$ as required.

The Groupoid Structures in Higher Dimensions

In our last chapter we showed how a T-complex possesses a canonical groupoid structure on the set of its 1-simplices . We now demonstrate that a T-complex admits canonical groupoid structures in all dimensions. The aim of this chapter is to prove the following theorem.

THEOREM Let X be a T-complex with face and degeneracy maps d_i and s_i. For each $n \geq 1$, there exist n canonical groupoid structures o_r for $1 \leq r \leq n$ with X_n as the set of arrows, X_{n-1} as the set of objects and initial, final and identity maps being d_r, d_{r-1} and s_{r-1} respectively. Furthermore, these structures satisfy

$$d_i(x \ o_r y) = \begin{cases} d_i x \ o_{r-1} \ d_i y & i < r-1 \\ d_i x \ \ o_r \ \ d_i y & i > r \end{cases}$$

The chapter is divided into three sections. In §1 we define the laws of composition o_r and show that the faces of $x \ o_r y$ are as stated in the theorem. §2 is devoted to the proof of associativity and in §3 we deduce the existence of identities and inverses.

§1. The laws of composition

Let (X,T) be a T-complex with face and degeneracy maps d_i and s_i respectively. We first show how, for a given $n \geq 1$, two elements x and y

of X_n may, under suitable conditions, be composed.

LEMMA 1.1 Given $n \geq 1$, suppose x and y are n-simplices of X satisfying $d_q x = d_{p-1} y$ where $0 \leq q < p-1 \leq n$. Then there can be assigned to x and y a unique thin simplex $T[x,y]_{p,q}$ of dimension $n+1$ such that

$$d_p T[x,y]_{p,q} = x$$

$$d_q T[x,y]_{p,q} = y$$

$$d_i T[x,y]_{p,q} = \begin{cases} T[d_i x, d_i y]_{p-1,q-1} & i < q \\ T[d_i x, d_{i-1} y]_{p-1,q} & q+1 < i < p \\ T[d_{i-1} x, d_{i-1} y]_{p,q} & i > p \end{cases}$$

Using this lemma we may define laws of composition o_r on X_n for $1 \leq r \leq n$ by

$$x \; o_r y = d_r \; T[x,y]_{r+1,r-1}$$

where $d_{r-1} x = d_r y$. We shall show later that these are in fact groupoid structures.

PROOF OF 1.1 The proof is by induction. Suppose that the lemma is true for dimension $n-1$ and suppose that, for all $(n-1)$-simplices u and v satisfying $d_q u = d_{p-1} v$ for some p and q with $0 \leq q < p-1 \leq n-1$, the thin simplices $T[u,v]_{p,q}$ have been assigned.

Now let x and $y \in X_n$ be such that $d_q x = d_{p-1} y$ for some p and q with $o \leq q < p-1 \leq n$. To prove the lemma, we must check that the postulated faces of $T[x,y]_{p,q}$ do actually fit together to form a horn.

First, if $i < q$,

$$d_{q-1}\, d_i x = d_i d_q x = d_i d_{p-1} y = d_{p-2} d_i x$$

and so, by the induction hypothesis, $T[d_i x,\ d_i y]_{p-1,q-1}$ is defined.

Similarly, if $q+1 < i < p$,

$$d_q d_i x = d_{i-1} d_q x = d_{i-1} d_{p-1} y = d_{p-2} d_{i-1} y$$

and so $T[d_i x,\ d_{i-1} y]_{p-1,q}$ is defined.

Finally, if $i > p$,

$$d_q d_{i-1} x = d_{i-2} d_q x = d_{i-2} d_{p-1} y = d_{p-1} d_{i-1} y$$

and so $T[d_{i-1} x,\ d_{i-1} y]_{p,q}$ is defined.

Thus all the postulated faces of $T[x,y]_{p,q}$ certainly do exist and we now check that they form a horn. Denoting these faces by h_i $(i \neq q+1)$, we have to check that $d_i\, h_j = d_{j-1}\, h_i$ for all i,j with o i j n and $i,j, \neq q+1$. There are a number of cases :

<u>Case 1</u> : $i < j < q$

$$d_i h_j = d_i T[d_j x,\ d_j y]_{p-1,q-1}$$

$$= T[d_i d_j x,\ d_i d_j y]_{p-2,q-2}$$

$$= T[d_{j-1} d_i x,\ d_{j-1}\, d_i y]_{p-2,q-2}$$

$$= d_{j-1}\, T\left[d_i x,\ d_i y\right]_{p-1,\ q-1}$$

$$= d_{j-1}\, h_i$$

<u>Case 2</u> : $i < j = q$

$$d_i h_j = d_i\, y$$

$$= d_{q-1}\, T\left[d_i x,\ d_i y\right]_{p-1,\ q-1}$$

$$= d_{j-1}\, h_i$$

<u>Case 3</u> : $i < q,\ q+1 < j < p$

$$d_i h_j = d_i\, T\left[d_j x,\ d_{j-1} y\right]_{p-1,q}$$

$$= T\left[d_i d_j x,\ d_i d_{j-1} y\right]_{p-2,\ q-1}$$

$$= T\left[d_{j-1}\ d_i x,\ d_{j-2} d_i y\right]_{p-2,q-1}$$

$$= d_{j-1}\ T\left[d_i x,\ d_i y\right]_{p-1,q-1}$$

$$= d_{j-1}\, h_i$$

<u>Case 4</u> : $i = q,\ q+1 < j < p$

$$d_i h_j = d_q\ T\left[d_j x, d_{j-1} y\right]_{p-1,q}$$

$$= d_{j-1}y$$

$$= d_{j-1}\, d_q\, T[x,y]_{p,q}$$

$$= d_{j-1}\, h_i$$

Case 5 : $i < q,\ j = p$

$$d_i h_j = d_i x$$

$$= d_{p-1}\, T[d_i x, d_i y]_{p-1,q-1}$$

$$= d_{j-1}\, h_i$$

Case 6 : $i = q,\ j = p$

$$d_i h_j = d_q\, x$$

$$= d_{p-1}y$$

$$= d_{j-1}\, h_i$$

Case 7 : $q+1 < i < p,\ j = p$

$$d_i h_j = d_i x$$

$$= d_{p-1}\ T\left[d_i x,\ d_{i-1} y\right]_{p-1,q}$$

$$= d_{j-1}\ h_i$$

Case 8 : $i < q,\quad j > p$

$$d_i h_j = d_i\ T\left[d_{j-1} x,\ d_{j-1} y\right]_{p,q}$$

$$= T\left[d_i d_{j-1} x,\quad d_i d_{j-1} y\right]_{p-1,q-1}$$

$$= T\left[d_{j-2} d_i x,\ d_{j-2} d_i y\right]_{p-1,q-1}$$

$$= d_{j-1}\ T\left[d_i x,\ d_i y\right]_{p-1,q-1}$$

$$= d_{j-1}\ h_i$$

Case 9 : $i = q,\ j > p$

$$d_i h_j = d_q\ T\left[d_{j-1} x,\ d_{j-1} y\right]_{p,q}$$

$$= d_{j-1} y$$

$$= d_{j-1}\ h_i$$

Case 10 : $q+1 < i < p, \quad j > p$

$$d_i h_j = d_i \; T\left[d_{j-1}x, d_{j-1}y \right]_{p,q}$$

$$= T\left[d_i d_{j-1}x, \; d_i d_{j-1} \, y \right]_{p-1,q}$$

$$= T\left[d_{j-2} d_i x, \; d_{j-2} d_i y \right]_{p-1,q}$$

$$= d_{j-1} \; T\left[d_i x, \; d_i y \right]_{p-1,q}$$

$$= d_{j-1} \; h_i$$

Case 11 : $i = p, \; j > p$

$$d_i h_j = d_p \; T\left[d_{j-1}x, d_{j-1}y \right]_{p,q}$$

$$= d_{j-1}x$$

$$= d_{j-1} \; h_i$$

Case 12 : $j > i > p$

$$d_i h_j = d_i \; T\left[d_{j-1}x, \; d_{j-1}y \right]_{p,q}$$

$$= T\left[d_{i-1} d_{j-1}x, \; d_{i-1} d_{j-1}y \right]_{p,q}$$

$$= T\left[d_{j-2}d_{i-1}x,\ d_{j-2}d_{i-1}y\right]_{p,q}$$

$$= d_{j-1}\ T\left[d_{i-1}x,\ d_{i-1}y\right]_{p,q}$$

$$= d_{j-1}\ h_i$$

Thus we have shown that the postulated faces do constitute a horn in X and we take its unique thin filler to be $T[x,y]_{p,q}$. This then has the required faces and, as such, is unique.

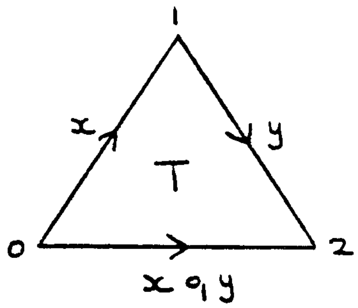
Finally, to start the induction, if $n = 1$ the only possibility is $q = 0$ and $p = 2$. We define $T[x,y]_{2,0}$ to be the unique thin filler $T(y,-,x)$ and this satisfies the required conditions. This completes the proof of the lemma.

Using Lemma 1.1, we are now in a position to define laws of composition on the T-complex (X,T). Suppose, for some $n \geqslant 1$, x and y are two n-simplices of X satisfying $d_{r-1}x = d_r y$ for some r, then, by Lemma 1.1, we may define the composite simplex $x \circ_r y$ by

$$x \circ_r y = d_r\ T[x,y]_{r+1,\ r-1}$$

Notice that, by virtue of the uniqueness of the thin element $T[x,y]_{r+1,r-1}$, the law of composition $\circ_r$ is both well-defined and canonical. Further, if x and y belong to X_1, then $x \circ_1 y$ is precisely the composite $x \circ y$ described in Chapter 2, namely

$$x \circ_1 y = d_1\ T\ (y,-,x)$$



LEMMA 1.2 The laws of composition o_r are preserved by the face operators d_i. Explicitly,

$$d_i(x\ o_r y) = \begin{cases} d_i x\ o_{r-1}\ d_i y & i < r-1 \\ d_i x\ o_r\ d_i y & i > r \end{cases}$$

whenever x and y are composable n-simplices for some $n \geqslant 2$.

Furthermore

$$d_r(x\ o_r y) = d_r x$$

$$d_{r-1}(x\ o_r\ y) = d_{r-1} y$$

PROOF. Using Lemma 1.1 we have

Case 1 $(i < r-1)$:

$$d_i(x\ o_r y) = d_i d_r T[x,y]_{r+1,r-1}$$

$$= d_{r-1} d_i T[x,y]_{r+1,r-1}$$

$$= d_{r-1} T[d_i x, d_i y]_{r,r-2}$$

$$= d_i x \; o_{r-1} \; d_i y$$

Case 2 $(i > r)$:

$$d_i(x \; o_r \; y) = d_i d_r T[x,y]_{r+1,r-1}$$

$$= d_r \; d_{i+1} T \, [x,y]_{r+1,r-1}$$

$$= d_r \; T[d_i x, \; d_i y]_{r+1,r-1}$$

$$= d_i x \; o_r \; d_i y$$

Also

$$d_r(x \; o_r \; y) = d_r d_r T \, [x,y]_{r+1,r-1}$$

$$= d_r d_{r+1} T[x,y]_{r+1,r-1}$$

$$= d_r x$$

and

$$d_{r-1}(x \; o_r \; y) = d_{r-1} \; d_r T \, [x,y]_{r+1,r-1}$$

$$= d_{r-1} \; d_{r-1} \; T[x,y]_{r+1,r-1}$$

$$= d_{r-1} \; y$$

Thus we have shown that the face operators of the T-complex (X,T) behave "correctly" with respect to the laws of composition o_r. One might ask whether a similar result holds good for the degeneracy operators. In fact a similar result for degeneracy operators does hold if one sets up similar machinery using cubical, rather than simplicial, T-complexes.

This has been proved by P.J. Higgins. However, in our case there is no corresponding result since composites of degenerate elements do not exist except in certain very special cases.

§2. Associativity of o_r

In this section we prove

LEMMA 2.1 The laws of composition o_r are associative.

The proof of this lemma is lengthy but notice that it consists only of applying axiom A3 of the definition of a T-complex.

PROOF. Let x,y and z be three n-simplices of the T-complex (X,T) satisfying $d_{r-1}x = d_r y$ and $d_{r-1}y = d_r z$. Then, by Lemma 1.2, the composites $x \; o_r \; (y \; o_r \; z)$ and $(x \; o_r \; y) \; o_r \; z$ are defined. Let

$$a = T[x,y]_{r+1,r-1} \qquad\qquad b = T[y,z]_{r+1,r-1}$$

so that $d_{r-1}a = y = d_{r+1}b$. Applying Lemma 1.1, we have the thin (n+2)-simplex

$$u = T[a,b]_{r+2,r-1}$$

Now by the simplicial identities for X we know that $d_r d_r u = d_r d_{r+1} u$ and so it will be sufficient to prove that

(i) $d_r d_{r+1} u = x \; o_r \; (y \; o_r \; z)$

(ii) $d_r d_r u = (x \; o_r \; y) \; o_r \; z$

(i) This is trivial for, using Lemma 2.1, we have

$$d_r d_{r+1} u = d_r \, T\left[d_{r+1}a, \; d_r b\right]_{r+1,r-1}$$

$$= d_r \, T\left[x, \; y \; o_r \; z\right]_{r+1,r-1}$$

$$= x \; o_r \; (y \; o_r \; z)$$

(ii) In order to prove this, we resort to induction to show that

$$d_r u = T\left[x \; o_r \; y, \; z\right]_{r+1,r-1}$$

that is

$$d_r \, T\left[a,b\right]_{r+2,r-1} = T\left[x \; o_r \; y, \; z\right]_{r+1,r-1} \qquad (*)$$

where

$$a = T\left[x,y\right]_{r+1,r-1} \qquad\qquad b = T\left[y,z\right]_{r+1,r-1}$$

Let $c = d_r u$ and assume that (*) is true for all values of r in all dimensions less than n, that is, given $m < n$ and r with $1 \leqslant r \leqslant m$, we assume that (*) holds good whenever x, y and z are replaced by suitable elements of dimension m.

Since all faces $d_i u$ of u, except when $i = r$, are thin, it follows by axiom A3 that $c = d_r u$ must also be thin. Hence, in order to check (*), by axiom A2 it will be sufficient to show that

$$d_i c = d_i \; T\left[x \; o_r \; y, \; z\right]_{r+1,r-1}$$

for all values of i except $i = r$, since it will then follow that

$$c = T\left[x \; o_r \; y, \; z\right]_{r+1,r-1}$$

as both these elements are thin.

Now we have

$$
\begin{aligned}
d_{r-1}c &= d_{r-1}\, d_r u \\
&= d_{r-1}\, d_{r-1} u \\
&= d_{r-1}\, d_{r-1}\, T\left[a,b\right]_{r+2,r-1} \\
&= d_{r-1}\, b \\
&= z \\
&= d_{r-1}\, T\left[x\ o_r\ y,\ z\right]_{r+1,r-1}
\end{aligned}
$$

and

$$
\begin{aligned}
d_{r+1}\, c &= d_{r+1}\, d_r u \\
&= d_r d_{r+2}\, T\left[a,b\right]_{r+2,r-1} \\
&= d_r a \\
&= x\ o_r\ y \\
&= d_{r+1}\, T\left[x\ o_r\ y,\ z\right]_{r+1,r-1}
\end{aligned}
$$

Immediately, this verifies (*) in the case when $n = 1$, as in this case $r = 1$ and the only possible values of i are 0, 1 and 2. This begins the induction.

When $n > 1$ we have extra faces of c to calculate, namely $d_i c$ for $i < r-1$ and $i > r+1$. First suppose $i < r-1$, then

$$
\begin{aligned}
d_i c &= d_i\, d_r\, u \\
&= d_{r-1}\, d_i\, u
\end{aligned}
$$

$$= d_{r-1}\ d_i\ T\left[a,b\right]_{r+2,r-1}$$

$$= d_{r-1}\ T\left[d_i a,\ d_i b\right]_{r+1,r-2}$$

by Lemma 1.1. Now also by Lemma 1.1 we have

$$d_i a = d_i\ T\left[x,\ y\right]_{r+1,r-1} = T\left[d_i x,\ d_i y\right]_{r,r-2}$$

$$d_i b = d_i\ T\left[y,\ z\right]_{r+1,r-1} = T\left[d_i y,\ d_i z\right]_{r,r-1}$$

and hence, if we replace x, y, z and r in (*) by $d_i x$, $d_i y$, $d_i z$ and r-1 respectively, a and b are replaced by $d_i a$ and $d_i b$. But, by the induction hypothesis, (*) is true in this case and so, using Lemmas 1.1 and 1.2, we deduce from the above that

$$d_i c = T\left[d_i x\ o_{r-1}\ d_i y,\ d_i z\right]_{r,r-2}$$

$$= T\left[d_i(x\ o_r\ y),\ d_i z\right]_{r,r-2}$$

$$= d_i\ T\left[x\ o_r\ y,\ z\right]_{r+1,r-1}$$

Secondly we have the case $i > r+1$ to check. Here we proceed in an exactly similar way. We have

$$d_i c = d_i d_r u$$

$$= d_r d_{i+1}\ u$$

$$= d_r d_{i+1}\ T\left[a,\ b\right]_{r+2,r-1}$$

$$= d_r\ T\left[d_i a,\ d_i b\right]_{r+2,r-1}$$

Now

$$d_i a = d_i\, T\left[x, y\right]_{r+1,r-1} = T\left[d_{i-1}\, x, d_{i-1}\, y\right]_{r+1,r-1}$$

$$d_i b = d_i\, T\left[y, z\right]_{r+1,r-1} = T\left[d_{i-1}\, y, d_{i-1}\, z\right]_{r+1,r-1}$$

and so this time, replacing x, y, z and r in (*) by $d_{i-1}\, x$, $d_{i-1}\, y$, $d_{i-1}\, z$ and r, a and b are replaced by $d_i a$ and $d_i b$. But then (*) is true by the induction hypothesis and so using Lemmas 1.1 and 1.2 we have

$$d_i c = T\left[d_{i-1}\, x \circ_r d_{i-1}\, y, d_{i-1}\, z\right]_{r+1,r-1}$$

$$= T\left[d_{i-1}(x \circ_r y), d_{i-1}\, z\right]_{r+1,r-1}$$

$$= d_i\, T\left[x \circ_r y, z\right]_{r+1,r-1}$$

Thus we have proved that all faces of c except the r'th are in accordance with those of $T\left[x \circ_r y, z\right]_{r+1,r-1}$ and so, since both these elements are thin, it follows by axiom A2 that

$$c = T\left[x \circ_r y, z\right]_{r+1,r-1}$$

But then we have

$$d_r d_r u = d_r c$$

$$= d_r\, T\left[x \circ_r y, z\right]_{r+1,r-1}$$

$$= (x \circ_r y) \circ_r z$$

and so (ii) is proved. It now follows that

$$x \circ_r (y \circ_r z) = (x \circ_r y) \circ_r z$$

as required.

§3. The Groupoid Structures

We now show that there exist identities and inverses for the laws of composition o_r on (X, T).

Let x be a member of X_n for some n and suppose $d_{r-1}x = a$. Write $1_a^r = s_{r-1}a$.

LEMMA 3.1 1_a^r is a right identity for x with respect to the composition o_r, that is

$$x \; o_r \; 1_a^r = x$$

PROOF. Since $d_r 1_a^r = d_r \; s_{r-1} \; a = a = d_{r-1}x$, $x \; o_r \; 1_a^r$ is defined and we have by definition

$$x \; o_r \; 1_a^r = d_r \; T\left[x, \; s_{r-1} \; d_{r-1} \; x\right]_{r+1,r-1}$$

We shall show that

$$T\left[x, \; s_{r-1} \; d_{r-1} \; x\right]_{r+1,r-1} = s_r x \qquad (*)$$

for then it will follow that $x \; o_r \; s_{r-1}a = d_r s_r x = x$. Since degenerate simplices are thin, in order to check (*) it will, by Axiom A2, be sufficient to check that

$$d_i T\left[x, \; s_{r-1} \; d_{r-1} \; x\right]_{r+1,r-1} = d_i \; s_r \; x$$

for all $i \neq r$.

Suppose, as an induction hypothesis, that (*) is true for all r whenever x is replaced by an element of X of dimension less than n. Then we have

(i) $d_{r-1} \; T\left[x, \; s_{r-1} \; d_{r-1} \; x\right]_{r+1,r-1} = s_{r-1}d_{r-1}x = d_{r-1} \; s_r \; x$

(ii) $d_{r+1} T[x, s_{r-1} d_{r-1} x]_{r+1,r-1} = x = d_{r+1} s_r x$

Immediately this verifies (*) for the case n = 1 as then the only possible value of r is 1 and there are no other faces to check.
This begins the induction.

When $n > 1$, by the induction hypothesis we have also

(iii) if $i < r-1$, then

$$\begin{aligned} d_i T[x, s_{r-1} d_{r-1} x]_{r+1,r-1} &= T[d_i x, d_i s_{r-1} d_{r-1} x]_{r,r-2} \\ &= T[d_i x, s_{r-2} d_{r-2} d_i x]_{r,r-2} \\ &= s_{r-1} d_i x \\ &= d_i s_r x \end{aligned}$$

and

(iv) if $i > r+1$, then

$$\begin{aligned} d_i T[x, s_{r-1} d_{r-1} x]_{r+1,r-1} &= T[d_{i-1} x, d_{i-1} s_{r-1} d_{r-1} x]_{r+1,r-1} \\ &= T[d_{i-1} x, s_{r-1} d_{r-1} d_{i-1} x]_{r+1,r-1} \\ &= s_r d_{i-1} x \\ &= d_i s_r x \end{aligned}$$

It now follows, by axiom A2, that (*) is true and this completes the proof of the lemma.

This proves the existence of right identities; the existence of left identities is deduced in Corollary 3.3.

LEMMA 3.2 <u>Suppose, for some $n \geq 1$, x and y are n-simplices of the T-complex (X,T) satisfying $d_r x = d_r y$ for some r with $1 \leq r \leq n$. Then there exists a unique element a of X_n such that</u>

$$x \circ_r a = y$$

NOTE. It is more convenient to prove this general lemma in order to demonstrate the existence of inverses, rather than simply attempt to prove directly that each element has an inverse.

PROOF. We first prove the uniqueness part of the lemma by means of induction. As induction hypothesis, suppose that, given members u, v and w of X_m for some $m < n$, such that $u \circ_r v = u \circ_r w$ for some r, then $v = w$.

Now suppose that $x \circ_r a = x \circ_r b$. We prove that

$$T[x,a]_{r+1,r-1} = T[x,b]_{r+1,r-1}$$

by checking that all corresponding faces except the $(r-1)$'th are equal.

Firstly we have

$$\begin{aligned} d_r T[x,a]_{r+1,r-1} &= x \circ_r a \\ &= x \circ_r b \\ &= d_r T[x, b]_{r+1,r-1} \end{aligned}$$

and

$$d_{r+1}\, T[x, a]_{r+1,r-1} = x = d_{r+1}\, T[x, b]_{r+1,r-1}$$

When $n = 1$, these are the only faces to check and it follows by the uniqueness of thin fillers (axiom A2) that, in dimension 1,

$$T[x, a]_{r+1,r-1} = T[x, b]_{r+1,r-1}$$

Hence, taking the $(r-1)$'th face, we have $a = b$ and this begins the induction.

If $n>1$ then for $i<r-1$ we have

$$d_i x \circ_{r-1} d_i a = d_i(x \circ_r a)$$

$$= d_i(x \circ_r b)$$

$$= d_i x \circ_{r-1} d_i b$$

and so, by the induction hypothesis, $d_i a = d_i b$. It then follows that

$$d_i\ T[x,\ a]_{r+1,r-1} = T[d_i x,\ d_i a]_{r,r-2}$$

$$= T[d_i x,\ d_i b]_{r,r-2}$$

$$= d_i\ T[x,\ b]$$

Similarly, for $i>r+1$, we have

$$d_{i-1}x \circ_r d_{i-1}a = d_{i-1}(x \circ_r a)$$

$$= d_{i-1}(x \circ_r b)$$

$$= d_{i-1}x \circ_r d_{i-1}b$$

from which it follows, by the induction hypothesis, that $d_{i-1}a = d_{i-1}b$. We then have

$$d_i\ T[x,\ a]_{r+1,r-1} = T[d_{i-1}x,\ d_{i-1}a]_{r+1,r-1}$$

$$= T[d_{i-1}x,\ d_{i-1}b]_{r+1,r-1}$$

$$= d_i\ T[x,\ b]_{r+1,r-1}$$

Now, using axiom A2, it follows that

$$T\left[x,\ a\right]_{r+1,r-1} = T\left[x,\ b\right]_{r+1,r-1}$$

and so, taking the (r-1)'th face, we have a = b. This completes the induction and thus we have proved uniqueness.

Next we show the existence of a as shown in the lemma. Again we use induction. Suppose that the lemma is true whenever x and y are of dimension less than n. Then there exist (n-1)-simplices a_i for $0 \leq i \leq n$ with $i \neq r, r-1$ such that

$$d_i x \circ_{r-1} a_i = d_i y \qquad i < r-1$$

$$d_i x \circ_r a_i = d_i y \qquad i > r$$

We check that

$$d_i\ a_j = d_{j-1}\ a_i \tag{*}$$

for all i and j with $i < j$ and $i, j \neq r-1, r, r+1$.

Firstly, if $i < j < r-1$, we have

$$\begin{aligned} d_i(d_j x \circ_{r-1} a_j) &= d_i d_j y \\ &= d_{j-1}\ d_i y \\ &= d_{j-1}(d_i x \circ_{r-1} a_i) \end{aligned}$$

which, by Lemma 1.2, is equivalent to

$$d_i d_j x \circ_{r-2} d_i a_j = d_{j-1} d_i x \circ_{r-2} d_{j-1}\ a_i$$

But $d_i d_j x = d_{j-1}\, d_i x$ and so, by the uniqueness part of the lemma which we have already proved, it follows that

$$d_i a_j = d_{j-1}\, a_i$$

Secondly, in the case $i < r-1$, $j > r+1$, we have in a similar fashion

$$\begin{aligned} d_i d_j x \circ_{r-1} d_i a_j &= d_i(d_j x \circ_r a_j) \\ &= d_i\, d_j\, y \\ &= d_{j-1}\, d_i\, y \\ &= d_{j-1}(d_i x \circ_{r-1} a_i) \\ &= d_{j-1} d_i x \circ_{r-1} d_{j-1} a_i \end{aligned}$$

Again $d_i d_j x = d_{j-1} d_i x$ and so, by the uniqueness part of the lemma, we have

$$d_i a_j = d_{j-1}\, a_i$$

Finally, if $j > i > r+1$, then we have

$$\begin{aligned} d_i d_j x \circ_r d_i a_j &= d_i(d_j x \circ_r a_j) \\ &= d_i d_j y \\ &= d_{j-1}\, d_i\, y \\ &= d_{j-1}(d_i x \circ_r a_i) \\ &= d_{j-1} d_i x \circ_r d_{j-1} a_i \end{aligned}$$

and since $d_i d_j x = d_{j-1} d_i x$, it follows that

$$d_i a_j = d_{j-1} a_i$$

This completes the check that (*) is true.

Now set

$$T_i = \begin{cases} T\left[d_i x,\ a_i\right]_{r,r-2} & i < r-1 \\ y & i = r \\ x & i = r+1 \\ T\left[d_{i-1} x,\ a_{i-1}\right]_{r+1,r-1} & i > r+1 \end{cases}$$

We check that the T_i form a horn in X, that is $d_i T_j = d_{j-1} T_i$ for $i < j$ and $i, j \neq r-1$. There are a number of cases.

<u>Case 1</u> : $i < j < r-1$

$$\begin{aligned} d_i T_j &= d_i T\left[d_j x,\ a_j\right]_{r,r-2} \\ &= T\left[d_i d_j x,\ d_i a_j\right]_{r-1,r-3} \\ &= T\left[d_{j-1} d_i x,\ d_{j-1} a_i\right]_{r-1,r-3} \\ &= d_{j-1} T\left[d_i x,\ a_i\right]_{r,r-2} \\ &= d_{j-1}\ T_i \end{aligned}$$

<u>Case 2</u> : $i < r-1,\ j = r$

$$d_i T_j = d_i y$$

$$= d_i x \circ_{r-1} a_i$$

$$= d_{r-1}\, T\left[d_i x,\ a_i\right]_{r,r-2}$$

$$= d_{j-1}\, T_i$$

<u>Case 3</u> : $i < r-1,\ j = r+1$

$$d_i T_j = d_i x$$

$$= d_r T\left[d_i x,\ a_i\right]_{r,r-2}$$

$$= d_{j-1}\, T_i$$

<u>Case 4</u> : $i = r,\ j = r+1$

$$d_i T_j = d_r x$$

$$= d_r y$$

$$= d_{j-1}\, T_i$$

<u>Case 5</u> : $i < r-1,\ j > r+1$

$$d_i T_j = d_i\, T\left[d_{j-1} x,\ a_{j-1}\right]_{r+1,r-1}$$

$$= T\left[d_i d_{j-1} x,\ d_i a_{j-1}\right]_{r,r-2}$$

$$= T\left[d_{j-2} d_i x,\ d_{j-2} a_i\right]_{r,r-2}$$

$$= d_{j-1}\, T\left[d_i x,\ a_i\right]_{r,r-2}$$

$$= d_{j-1}\, T_i$$

Case 6 : $i = r,\ j > r+1$

$$d_i T_j = d_r\ T\left[d_{j-1}x,\ a_{j-1}\right]_{r+1,r-1}$$

$$= d_{j-1}x \circ_r a_{j-1}$$

$$= d_{j-1}y$$

$$= d_{j-1}\ T_i$$

Case 7 : $i = r+1,\ j > r+1$

$$d_i T_j = d_{r+1} T\left[d_{j-1}x,\ a_{j-1}\right]_{r+1,r-1}$$

$$= d_{j-1}x$$

$$= d_{j-1}T_i$$

Case 8 : $j > i > r+1$

$$d_i T_j = d_i T\left[d_{j-1}x,\ a_{j-1}\right]_{r+1,r-1}$$

$$= T\left[d_{i-1}\ d_{j-1}x,\ d_{i-1}a_{j-1}\right]_{r+1,r-1}$$

$$= T\left[d_{j-2}\ d_{i-1}x,\ d_{j-2}a_{i-1}\right]_{r+1,r-1}$$

$$= d_{j-1}\ T\left[d_{i-1}x,\ a_{i-1}\right]_{r+1,r-1}$$

$$= d_{j-1}\ T_i$$

This completes the check that the T_i form a horn in X. Let T be the unique thin filler of this horn and set

$$a = d_{r-1}T$$

Then we have for $i < r-1$

$$d_i a = d_i d_{r-1}T$$

$$= d_{r-2}d_i T$$

$$= d_{r-2}T_i$$

$$= d_{r-2}\, T\left[d_i x,\ a_i\right]_{r,r-2}$$

$$= a_i$$

and for $i > r+1$

$$d_i a = d_i d_{r-1}\, T$$

$$= d_{r-1}\, d_{i+1}\, T$$

$$= d_{r-1}\, T_{i+1}$$

$$= d_{r-1}\, T\left[d_i x,\ a_i\right]_{r+1,r\ 1}$$

$$= a_i$$

Hence the faces of T are given by

$$T_i = \begin{cases} T\left[d_i x,\ d_i a\right]_{r,r-2} & i < r-1 \\ a & i = r-1 \\ y & i = r \\ x & i = r+1 \\ T\left[d_{i-1} x,\ d_{i-1} a\right]_{r+1,r-1} & i > r+1 \end{cases}$$

But it now follows that T must be the unique thin element with these faces (for $i \neq r$) whose existence is asserted by Lemma 1.1, that is

$$T = T\left[x, a\right]_{r+1,r-1}$$

and so we have

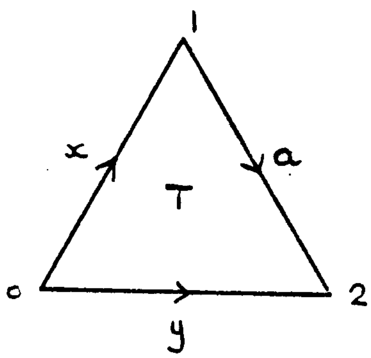
$$x \circ_r a = d_r T[x, a]_{r+1, r-1}$$

$$= d_r T$$

$$= y$$

To complete the induction, we need to check the existence of a in the case n = 1. This is trivial, for r must be equal to 1 and we set

$$a = d_0 T(-, y, x)$$



Then by the uniqueness of thin fillers, it follows that $x \circ_1 a = y$. This completes the proof of the existence of a and of the lemma.

We now deduce the existence of left identities for $\circ_r$. Let x be a member of X_n for some n and suppose $d_r x = b$.

COROLLARY 3.3 <u>$1_b^r = s_{r-1} b$ is a left identity for x with respect to $\circ_r$, that is</u>

$$1_b^r \circ_r x = x$$

PROOF. By Lemma 3.1, $1_b^r \circ_r 1_b^r = 1_b^r$ and so, using Lemma 2.1 (associativity), we have

$$1_b^r \circ_r (1_b^r \circ_r x) = (1_b^r \circ_r 1_b^r) \circ_r x$$
$$= 1_b^r \circ_r x$$

But now, by the uniqueness part of Lemma 3.2, it follows that $1_b^r \circ_r x = x$

COROLLARY 3.4 <u>Let x be a member of X_n and suppose that for some r, $a = d_r x$ and $b = d_{r-1} x$. Then there exists a unique element x^{-1} of X_n such that</u>

$$x \circ_r x^{-1} = 1_a^r$$

$$x^{-1} \circ_r x = 1_b^r$$

Note, of course that x^{-1} is dependent on r.

PROOF. The existence of a unique x^{-1} such that $x \circ_r x^{-1} = 1_a^r$ is given by Lemma 3.2. Using Lemma 2.1, we then have

$$x \circ_r (x^{-1} \circ_r x) = (x \circ_r x^{-1}) \circ_r x$$
$$= 1_a^r \circ_r x$$
$$= x$$
$$= x \circ_r 1_b^r$$

But then, by the uniqueness part of Lemma 3.2, it follows that $x^{-1} \circ_r x = 1_b^r$ as required.

Collecting together all these results, we have now shown the existence of the canonical groupoid structures $\circ_r$ on each X_n and we have proved the theorem stated at the beginning of the chapter.

CHAPTER 4

Homotopy Groups of T-Complexes and the Crossed Chain Complex

In this chapter we give some results on the homotopy groups of a T-complex and the associated T-filtration. We show how certain relative homotopy groups of the T-filtration give a Crossed Chain Complex as originally defined by Whitehead [11] and called by him a Homotopy System. A crossed chain complex is a higher dimensional extension of the crossed modules used by Brown and Higgins [2] to obtain results on the second relative homotopy groups $\pi_2(X, A)$. The reason for setting up this machinery is that we suggest it may be possible to use these methods in order to obtain results on higher dimensional relative homotopy groups.

§1. Some results on the homotopy groups of a T-complex

Suppose that X is a T-complex. By the homotopy groups of X, we mean the homotopy groups of X as a simplicial set and we assume that these are constructed as in May [7]. Thus if * is a base point for X then we let * denote also the simplicial subset generated by the base point and define $X_n(*)$, for each $n > 0$, to be the set of all $x \in X_n$ satisfying $d_i x = *$ for all i. Then $\pi_n(X, *) = X_n(*)/\sim$ where $x \sim y$ if there is a homotopy, as described by May, from x to y. Similarly, if A is a subcomplex of X, then $X_n(A,*)$ denotes the set of n-simplices x of X satisfying $d_i x = *, i \geq 1$, and $d_0 x \in A_{n-1}$. The relative homotopy group $\pi_n(X, A, *)$ is then defined to be $X_n(A, *)/\sim$ where $x \sim y$ if there is a homotopy rel A (see May) from x to y. For brevity we shall suppose in this section that the base-point * is fixed and write $\pi_n(X)$ instead of

$\pi_n(X,*)$ and $\pi_n(X, A)$ instead of $\pi_n(X, A, *)$.

PROPOSITION 1.1 Suppose that for some $r \geq 1$, every r-simplex of the T-complex X is thin. Then

$$\pi_r(X) = 0$$

PROOF. By definition, $\pi_r(X) = X_r(*)/\sim$ and we have

$$X_r(*) = \{x \in X_r : d_i x = * \text{ for all } i\}$$
$$= \{x \in T_r : d_i x = * \text{ for all } i\}$$

But there is only one thin simplex of X_r with all faces *, namely the degenerate simplex * belonging to X_r. Thus $X_r(*) = \{*\}$ and so $\pi_r(X) = 0$.

COROLLARY 1.2 Suppose the T-complex X has rank n. Then $\pi_i(X)$ is zero for all $i > n$.

PROOF. By definition, $T_i = X_i$ for all $i > n$ and so the result follows from Proposition 1.1 above.

In certain special cases, we obtain $K(\pi, n)$-complexes [7], the simplicial analogue of the CW $K(\pi, n)$-spaces of Eilenberg and Maclane, where all homotopy groups except the n'th are zero.

COROLLARY 1.3 Suppose that the only non-thin elements of the T-complex X lie in dimension n. Then X is a $K(\pi, n)$-complex.

PROOF. Simply apply Proposition 1.1 in all dimensions except n.

We now give a result on the homotopy groups of the filtration $\{\bar{X}^n\}$ of the T-complex X analagous to a similar result in topology on the homotopy groups of the n-skeletons of a CW-complex (see [10], Theorem 6.11). Let $i : X^n \longrightarrow X$ be the inclusion of the T-complex (X^n, T^n) into the T-complex (X, T) (see Corollary 2.5 of Chapter 1), then we know that for each $r \geq 1$, i induces a morphism of groups $i_* : \pi_r(\bar{X}^n) \longrightarrow \pi_r(X)$ (see [4] or [7]).

THEOREM 1.4 The induced morphism

$$i_* : \pi_r(\bar{X}^n) \longrightarrow \pi_r(X)$$

is an isomorphism for $r < n$ and an epimorphism for $r = n$.

PROOF. By definition we have

$$\pi_r(\bar{X}^n) = \bar{X}^n_r\,(*)/\sim$$

$$\pi_r(X) = X_r\,(*)/\sim$$

where the relations $\sim$ are as stated earlier. Now for $r < n$, the restriction of i gives an equality $\bar{X}^n_r(*) \to X_r(*)$, since by definition $\bar{X}^n_r = X_r$ when $r \leq n$. It follows that i_* is surjective and hence an epimorphism whenever $r \leq n$. Further, by definition the relations $\sim$ on the r-simplices of $\bar{X}^n$ and X depend only on the existence of (r+1)-simplices. Hence, if $r < n$, since we then have $\bar{X}^n_{r+1} = X_{r+1}$, it follows that the relations $\sim$ on $\bar{X}^n_r$ and X_r are identical. Thus for $r < n$, i_* is a bijection and hence an isomorphism.

Since, by Corollary 3.2, $\pi_r(\bar{X}^n)$ is zero for $r > n$, we have shown that $\bar{X}^n$ has the homotopy groups we would expect it to have.

§2. The Crossed Chain Complex associated to a T-Complex

We now show how a T-complex gives rise to a Crossed Chain Complex and in particular to a Crossed Module (we shall define these concepts later). First we need to state some further results on the homotopy groups of a semi-simplicial set.

Suppose X is a Kan complex and A is a sub Kan complex of X. Suppose further that * is a base point lying within A. We follow Lamotke [4] in describing how the group $\pi_1(X, *)$ acts on each homotopy group $\pi_n(X, *)$ for $n \geq 1$ and how the group $\pi_1(A, *)$ acts on each relative homotopy group $\pi_n(X, A, *)$ for $n \geq 2$.

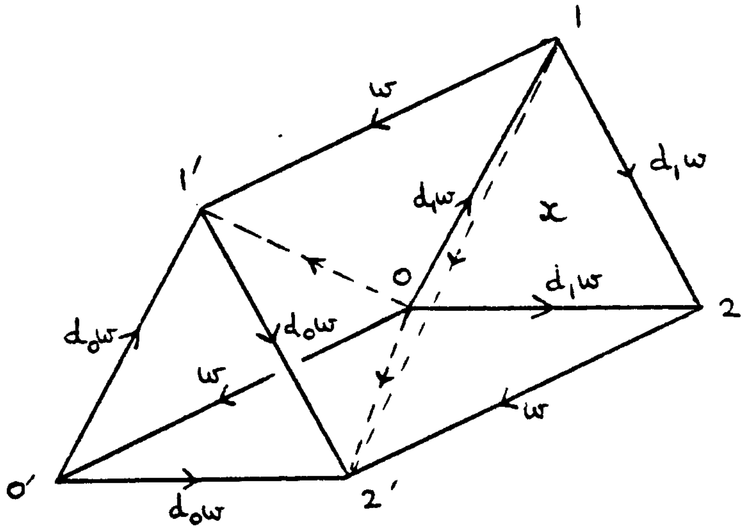
First we describe the action of $\pi_1(X, *)$ on $\pi_n(X, *)$. In order to do this we need to describe a more general construction following Lamotke. Suppose that $w \in X_1$. There exists a map

$$o(w) : \pi_n(X, d_1 w) \longrightarrow \pi_n(X, d_o w)$$

where $o(w)a$ for a a member of $\pi_n(X, d_1 w)$ is constructed as follows : let $a = [x]$ and regard x as a map $(\Delta^n, \dot{\Delta}^n) \longrightarrow (X, d_1 w)$ where by $d_1 w$ we really mean, by abuse of notation, the subcomplex generated by the 0-simplex $d_1 w$. Further regard w as a map $w : I \rightarrow X$ where $I = \Delta^1$. Define a map

$$f : \dot{\Delta}^n \times I \cup \Delta^n \times 0 \longrightarrow X$$

by $f(u,t) = w(t)$ for all $u \in \dot{\Delta}^n$ and $t \in I$ and $f(u, 0) = x(u)$ for all $u \in \Delta^n$.



By the Homotopy Extension Property (HEP) (see [4]), f extends to $f' : \Delta^n \times I \to X$, and $f'(\Delta^n \times 1)$ is the n-simplex we require. Define $o(w)a$ to be the class of this simplex in $\pi_n(X, d_0w)$. Lamotke proves that $o(w)a$ is well defined and satisfies the following proposition.

PROPOSITION 2.1 [4] <u>The map</u>

$$o(w) = \pi_n(X, d_1w) \longrightarrow \pi_n(X, d_0w)$$

<u>is a homomorphism satisfying</u>

$$o(d_0r) \circ (d_2r) = o(d_1r)$$

<u>for any member r of X_2. Further $o(w)$ depends only on the homotopy class of w.</u>

For the proof see [4].

It follows from the above proposition that if $b \in \pi_1(X, *)$ then we have a well-defined homomorphism $o(b) : \pi_n(X,*) \longrightarrow \pi_n(X,*)$

PROPOSITION 2.2 [4] <u>The map</u>

$$\pi_n(X, *) \times \pi_1(X, *) \longrightarrow \pi_n(X, *)$$

$$(a, b) \longmapsto o(b)a$$

<u>constitutes a group operation of $\pi_1(X, *)$ on $\pi_n(X, *)$ for each $n \geq 1$. If $a, b \in \pi_1(X, *)$ then</u>

$$o(b)a = b^{-1}a\ b$$

For the proof, see [4].

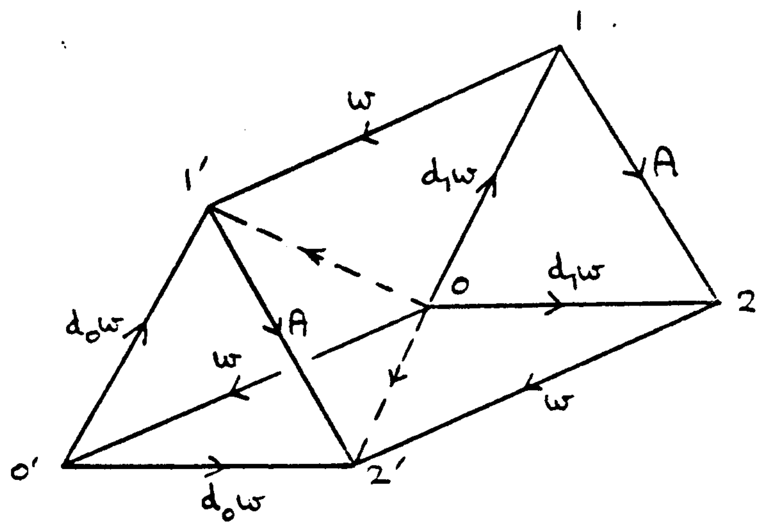
In a similar way to the above, Lamotke also describes the case of the relative homotopy groups $\pi_n(X, A, *)$. Suppose that w is a 1-simplex of A. We may define a map

$$o(w) : \pi_n(X, A, d_1w) \rightarrow \pi_n(X, A, d_0w)$$

as follows : let $a \in \pi_n(X, A, d_1w)$ and suppose that $a = [x]$. Regard a and w as maps $(\Delta^n, \dot{\Delta}^n, \Lambda_n^0) \longrightarrow (X, A, d_1w)$ and $I \rightarrow A$ respectively. Define the map

$$f : \Lambda_n^0 \times I \cup \dot{\Delta}^n \times 0 \longrightarrow A$$

by $f(u, t) = w(t)$ for all $u \in \Lambda_n^0$ and $t \in I$ and $f(u, 0) = x(u)$ for all $u \in \dot{\Delta}^n$.



By the HEP, f extends to a map $f' : \dot{\Delta}^n \times I \to A$, and we further extend f' to give

$$g : \dot{\Delta}^n \times I \cup \Delta^n \times 0 \longrightarrow X$$

by defining $g(u, 0) = x(u)$ for all $u \in \Delta^n$. Then using the HEP a second time, g extends to $g' : \Delta^n \times I \longrightarrow X$ and $g'(\Delta^n \times 1)$ is the n-simplex we require. Define $o(w)a$ to be the class of this n-simplex in $\pi_n(X, A, d_o w)$. Lamotke shows that $o(w)$ is well-defined and satisfies the following proposition.

PROPOSITION 2.3 [4] <u>The map</u>

$$o(w) : \pi_n(X,A,d_1 w) \longrightarrow \pi_n(X,A, d_o w)$$

<u>is a homomorphism satisfying</u>

a) <u>$o(d_o r)\, o(d_2 r) = o(d_1 r)$ for all $r \in A_2$</u>

b) <u>$o(w)\, da = d o(w) a$ where d is the boundary homomorphism $\pi_n(X, A, d_1 w) \longrightarrow \pi_{n-1}(A, d_1 w)$.</u>

<u>Further $o(w)$ depends only on the homotopy class of w.</u>

Similar to the absolute case, it follows from the above proposition that if $b \in \pi_1(A,*)$, then we have a well-defined homomorphism o(b)
$o(b) : \pi_n(X,A,*) \longrightarrow \pi_n(X,A,*)$.

PROPOSITION 2.4 [4] *The map*

$$\pi_n(X, A, *) \times \pi_1(A, *) \longrightarrow \pi_n(X, A, *)$$

$$(a, b) \longmapsto o(b)a$$

constitutes a group operation of $\pi_1(A, *)$ *on* $\pi_n(X, A, *)$ *for each* $n \geqslant 2$. *If* $a, b \in \pi_2(X, A, *)$, *then*

$$o(db)a = b^{-1} a b$$

where d *is the boundary homomorphism* $\pi_2(X, A, *) \rightarrow \pi_1(A, *)$.

This completes the results on the homotopy groups of a Kan-complex which we shall need to make use of.

DEFINITION 2.5 [9] A *Crossed Module* (A, B, d) consists of a morphism of groups $d : A \longrightarrow B$ and a group operation of B on the right of A, written $(a, b) \longmapsto a^b$ for $a \in A$ and $b \in B$ satisfying

(i) $d(a^b) = b^{-1}d(a)b$

(ii) $a_1^{da} = a^{-1}a_1 a$ for $a, a_1 \in A$.

In a moment we shall show that the action of $\pi_1(A, *)$ on $\pi_2(X, A, *)$ gives $d : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$ the structure of a crossed module. First we note that Brown and Spencer in [3] have defined the notion of a morphism of crossed modules as follows:

DEFINITION 2.6 [3] A <u>morphism</u> $(f, g) : (A, B, d) \to (A^1, B^1, d^1)$ of crossed modules consists of morphisms of groups $f : A \to A^1$ and $g : B \to B^1$ satisfying

(i) $gd = d^1 f$

(ii) $f(a^b) = f(a)^{g(b)}$

for all $a \in A$ and $b \in B$.

We thus have a category of crossed modules which we shall call $\underset{\sim}{C}$.

PROPOSITION 2.7 <u>Let X be a Kan complex, let A be a sub Kan complex of X and let * be a base-point belonging to A. The action of $\pi_1(A, *)$ on $\pi_2(X, A, *)$ together with the boundary homomorphism $d : \pi_2(X, A, *) \to \pi_1(A, *)$ constitutes a crossed module.</u>

PROOF. We have to check conditions (i) and (ii) of Definition 2.5. Condition (ii) is given in Proposition 2.4 and condition (i) follows from Propositions 2.2 and 2.3, for by Proposition 2.3 part (b) we have $d\, o(b)a = o(b)da$ where $a \in \pi_2(X, A, *)$ and $b \in \pi_1(A, *)$ and by Proposition 2.2 we have $o(b)da = b^{-1}d(a)b$.

For the purposes of our next chapter, we are interested in a particular crossed module, where we use the T-filtration $X = \{\overline{X}^n\}_{n \geq 1}$ or rather the bottom two members of it, to construct the crossed module

$$\pi_2(X^2, X^1, *) \xrightarrow{d} \pi_1(X^1, *)$$

where * is a base point for the T-complex X. We now show that the homotopy groups used in the above crossed modules are simply vertex groups of the groupoid structures existing on X_2 and X_1. Let $X_2\{*\}$ denote the vertex group of X_2 together with its groupoid structure o_2 consisting of elements x satisfying $d_1 x = d_2 x = *$. Let $X_1\{*\}$ denote the vertex group of X_1 together with its groupoid structure consisting of elements x based at *, that is satisfying $d_o x = d_1 x = *$.

PROPOSITION 2.8 <u>The morphism of groups</u>

$$d : \pi_2(\bar{X}^2, \bar{X}^1, *) \longrightarrow \pi_1(\bar{X}^1, *)$$

<u>is precisely the face map</u>

$$d_o : X_2\{*\} \longrightarrow X_1\{*\}$$

PROOF. By definition we have

$$\pi_2(\bar{X}^2, \bar{X}^1, *) = \bar{X}^2_2(\bar{X}^1, *)/\sim \text{ rel } \bar{X}^1$$

and

$$\bar{X}^2_2(\bar{X}^1, *) = \{ x \in \bar{X}^2_2 : d_o x \in X^1, \ d_i x = *, \ i \geq 1\}$$

$$= \{ x \in X_2 : d_o x \in X, \ d_i x = *, \ i \geq 1\}$$

$$= X_2\{*\}$$

Further, if $x \sim y$ rel $\bar{X}^1$ in $\bar{X}^2$, then there exists a homotopy w from x to y, that is a simplex w of $\bar{X}^2_3$ such that $d_o w$ is a homotopy u from $d_o x$ to $d_o y$ in $\bar{X}^1$, $d_1 w = *$, $d_2 w = x$, $d_3 w = y$. Since u is a 2-simplex of $\bar{X}^1$, it must be thin and hence, since $d_o u = *$, $u = s_1 d_o x = s_1 d_o y$. Further, since w is a 3-simplex of $\bar{X}^2$, it is thin and so $w = T(\Lambda^3 w) = T(s_1 d_o x, *, x, -) = s_2 x$. It follows that $x = d_3 w = y$.

Also, by definition,

$$\pi_1(\bar{X}^1, *) = \bar{X}^1_1(*)/\sim$$

and

$$\bar{X}_1^1(*) = \{x \in \bar{X}_1^1 : d_i x = *, i > 0\}$$
$$= \{x \in X_1 : d_i x = *, i \geq 0\}$$
$$= X_1\{*\}$$

If $x \sim y$ where x and y belong to $\bar{X}_1^1$ then there exists a homotopy u from x to y in $\bar{X}^1$, that is a simplex u of $\bar{X}_2^1$ such that $d_o u = *$, $d_1 u = x$, $d_2 u = y$. Since u is a 2-simplex of $\bar{X}^1$ it must be thin and hence, since $d_o u = *$, degenerate. It follows that $x = y$.

Finally, by definition the morphism d is, in this case, the face map d_o.

In Chapter 5 we shall give an explicit description of the action of $X_1\{*\}$ on $X_2\{*\}$ involved in the above crossed module. To complete this chapter, we define the higher dimensional extension of a crossed module, namely a crossed chain complex and show how a crossed chain complex $\pi_*(\underset{\sim}{X})$ is obtained from the T-filtration X. The following definition extends the notion of homotopy system as defined by Whitehead[9]. The name is due to R. Brown.

DEFINITION 4.9 A Crossed Chain Complex consists of a family $C = \{C_n\}_{n \geq 1}$ of groups, abelian for $n \geq 3$, together with morphisms $d : C_n \longrightarrow C_{n-1}$ for $n \geq 2$ such that $d^2 = 0$ and such that the following conditions hold :

a) $d : C_2 \longrightarrow C_1$ admits the structure of a crossed module

b) each C_n, for $n > 2$, is a C_1/dC_2-module

c) for each $n \geq 3$, $d : C_n \longrightarrow C_{n-1}$ is an operator homomorphism,

that is, if $a \in C_1$ and $\bar{a}$ denotes the class of a in C_1/dC_2, then, regarding a and $\bar{a}$ as operators,
$d\bar{a} = \bar{a}d$ for $n \geq 4$ and $d\bar{a} = ad$ for $n = 3$.

We define the crossed chain complex $\pi_*(\underset{\sim}{X}, *)$ associated with the T-filtration $\underset{\sim}{X} = \{X^n\}_{n \geq 1}$ and a fixed base-point $*$ of the T-complex X as follows : let $\pi_*(\underset{\sim}{X}, *)$ be the collection of homotopy groups $\{\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)\}_{n \geq 1}$, where, in the case $n = 1$, we understand $\pi_1(\bar{X}^1, \bar{X}^o, *)$ to mean the absolute homotopy group $\pi_1(\bar{X}^1, *)$, together with morphisms $d : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \to \pi_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}, *)$ for each n defined in the following manner : for each $n \geq 2$ we have a boundary homomorphism $d' : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \longrightarrow \pi_{n-1}(\bar{X}^{n-1}, *)$ and for each $n \geq 3$ there is an induced map $j_* : \pi_{n-1}(\bar{X}^{n-1}, *) \longrightarrow \pi_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}, *)$ where j is the inclusion of pairs $(\bar{X}^{n-1}, \bar{X}^{n-2})$. When $n \geq 3$, we define d to be the composite $j_*d' : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \to \pi_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}, *)$ and when $n = 2$ we define d to be the map $d' : \pi_2(\bar{X}^2, \bar{X}^1, *) \longrightarrow \pi_1(\bar{X}^1, *)$.

We already know that $d : \pi_2(\bar{X}^2, \bar{X}^1, *) \to \pi_1(\bar{X}^1, *)$ admits the structure of a crossed module, we must show further that there is an operation of $\pi_1(\bar{X}^1, *)/d\pi_2(\bar{X}^2, \bar{X}^1, *)$ on $\pi_n(\bar{X}^n, \bar{X}^{n-1})$ for each $n \geq 3$ such that each d is an operator homomorphism with respect to this operation.

Suppose $n \geq 3$, and let $i : \bar{X}^1 \to \bar{X}^{n-1}$ be the inclusion so that we have an induced morphism $i_* : \pi_1(\bar{X}^1, *) \to \pi_1(\bar{X}^{n-1}, *)$. By propositions 2.3 and 2.4, we know that there is a group action of $\pi_1(\bar{X}^{n-1}, *)$ on $\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$. written $o(b) : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \to \pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ for b a member of $\pi_1(\bar{X}^{n-1}, *)$. If $w \in \bar{X}_1^{n-1}$ represents b then the morphism $o(b)$ is induced by the morphism $o(w) : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \to \pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ of Proposition 2.3 Define an operation of $\pi_1(\bar{X}^1, *)$ on $\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$, written x^b for $x \in \pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ and $b \in \pi_1(\bar{X}^1, *)$, by

$$x^b = o(i_* b)x$$

Since i_* is a group homomorphism, it follows that this is a group operation.

Next, in order to obtain an operation of the factor group $\pi_1(\bar{X}^1, *)/d\pi_2(\bar{X}^2, \bar{X}^1\ *)$ on $\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$, we must check that, under the above operation, operating by an element of $d\pi_2(X^2, X^1, *)$ gives no change. Suppose, then, that $a \in \pi_2(\bar{X}^2, \bar{X}^1, *)$ and that $a = [v]$ for some $v \in \bar{X}_2^2(\bar{X}^1, *)$. Then $d_o v$ is an element of $\bar{X}_1^1(*)$ representing da in $\pi_1(\bar{X}^1, *)$ and $d_1 v = d_2 v = *$. But $d_o v$ also represents $i_* da$ in $\pi_1(\bar{X}^{n-1}, *)$ and so, if $x \in \pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$, we have

$$x^{da} = o(i_* da)x = o(d_o v)x$$

Now by Proposition 2.3, $o(d_o v)o(d_2 v) = o(d_1 v)$ and since $d_1 v = d_2 v$ $(= *)$, it follows that $o(d_o v)$ must be the identity. Hence $x^{da} = x$.

It follows from the above that $\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ has the structure of a $\pi_1(\bar{X}^1, *)/\pi_2(\bar{X}^2, \bar{X}^1, *)$- module for each $n \geq 3$ and we have

PROPOSITION 2.10 <u>$\Pi_*(X, *)$ is a crossed chain complex.</u>

PROOF. We have already shown that $d : \pi_2(\bar{X}^2, \bar{X}^1, *) \to \pi_1(\bar{X}^1, *)$ admits the structure of a crossed module and that, for $n \geq 3$, $\pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ is a $\pi_1(\bar{X}^1, *)/d\pi_2(\bar{X}^2, \bar{X}^1, *)$-module. It remains to check that $d^2 = 0$ and that, for each $n \geq 3$, $d : \pi_n(\bar{X}^n, \bar{X}^{n-1}, *) \to \pi_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}, *)$ is an operator homomorphism.

The fact that $d^2 = 0$ follows from the following diagram of exact sequences of pairs.

$$\begin{array}{ccccc}
\pi_n(\bar{X}^n, \bar{X}^{n-1}, *) & \xrightarrow{d'} & \pi_{n-1}(\bar{X}^{n-1}, *) & & \\
& \searrow^{d} & \downarrow j_* & & \\
& & \pi_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}, *) & \xrightarrow{d'} & \pi_{n-2}(\bar{X}^{n-2}, *) \\
& & & \searrow^{d} & \downarrow j_* \\
& & & & \pi_{n-2}(\bar{X}^{n-2}, \bar{X}^{n-3}, *)
\end{array}$$

When $n > 3$, we have $d^2 = j_* d' j_* d'$ and, by exactness, $d' j_* = 0$. When $n = 2$, $d^2 = d' j_* d'$ and so, similarly, $d^2 = 0$.

In order to check that d is an operator homomorphism, suppose that $x \in \pi_n(\bar{X}^n, \bar{X}^{n-1}, *)$ for some $n \geq 3$ and that $b \in \pi_1(\bar{X}^1, *)$. Let $\bar{b}$ denote the class of b in the factor group $\pi_1(\bar{X}^1, *)/d\pi_2(\bar{X}^2, \bar{X}^1 *)$ and suppose that $b = [u]$ for some $u \in \bar{X}_1^1(*)$ then, writing the operation of $\bar{b}$ on x as $x^{\bar{b}}$, we have

$$\begin{aligned} d(x^{\bar{b}}) &= d(x^b) \\ &= d(o(i_* b)x) \\ &= d(o(u)x) \end{aligned}$$

Now by Proposition 2.3 we know that $d(o(u)x) = o(u)dx$ and so we have

$$\begin{aligned} d(x^{\bar{b}}) &= o(u)dx \\ &= o(i_* b)dx \\ &= (dx)^b \end{aligned}$$

In the case $n = 3$, this is exactly what we require and, when $n > 3$, $(dx)^b$ is by definition equal to $(dx)^{\bar{b}}$ as required. This completes the proof.

We state as a conjecture at this point that there is a reverse procedure for obtaining a T-complex from a crossed chain complex. In our next chapter we show in particular how the nerve functor N of Segal [9] extends to a functor from crossed modules to T-complexes of rank 2 and we assume that this can be further generalised. It is a consequence of the work of the next chapter that the category of T-complexes of rank 2 with only one vertex is equivalent to the category of crossed modules and, to generalise, we conjecture that the category of all T-complexes possessing only one vertex is equivalent to the category of crossed chain complexes.

CHAPTER 5

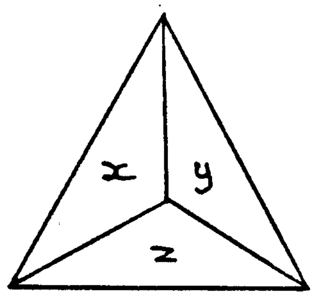
T-Complexes of Rank 2

In Chapter 4 we showed how a T-complex gives rise to a crossed chain complex and in particular to a crossed module. It is the aim of this chapter to show how the axioms for a T-complex enable us to set up a method of subdivision of the 2-simplices and how, using this, we can give a more explicit description of the crossed module. Ultimately, we wish to prove an equivalence of categories and the difficulty here is that in order to obtain a crossed module from a T-complex we have to first select a base-point . In order to eliminate this difficulty we shall use the notion of a crossed module over a groupoid, due to R. Brown and P.J. Higgins,[2] where we work with a groupoid, rather than a group, and so avoid the need to select one particular base-point. The main result of the chapter will be that the category of T-complexes of rank 2 is equivalent to the category of crossed modules over groupoids.

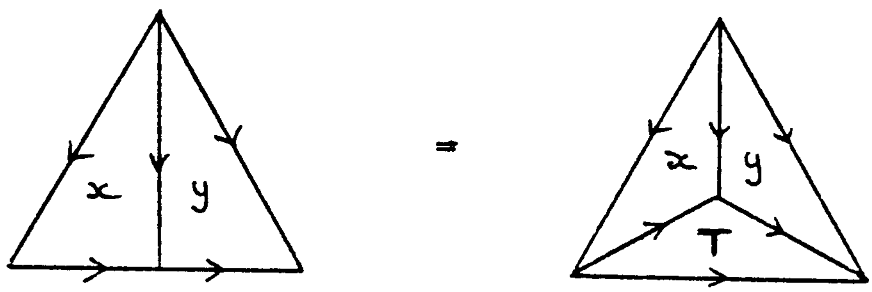
Many of the proofs in this chapter will be done largely by the use of diagrams rather than formulae. We make no apology for this since, in the present state of the work, the diagrams are less cumbersome and more explanatory than the corresponding formulae would be.

§1. Subdivision of 2-Simplicies

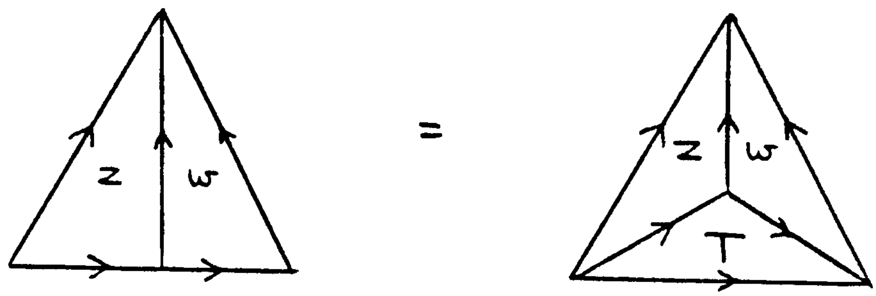
Let X be a T-complex. In Chapter 3 we showed how, for each $n \geq 1$, the set X_n of n-simplices of X possesses n groupoid structures. We now look at the 2-simplices of X and show how these may be combined in a more general fashion than by using the groupoid structures. The basic idea is as follows : suppose we are given three 2-simplices x, y and z of X fitting together under a scheme of the form



that is x, y and z satisfy certain conditions of equality between their faces so that they form a horn in X. By filling in the horn formed by x, y and z with the unique thin filler, we may obtain a new 2-simplex w, namely the new face of the thin filler. We shall use diagrams of the above form to denote the simplex w subdivided into x, y and z. In the case where one of the simplices x, y or z is thin then, provided there is no ambiguity in the ordering of the vertices, we shall condense these diagrams : for example we shall let



Recalling that, according to Chapter 3, there are two groupoid structures on the set X_2 of 2-simplices of X, the above diagram in fact denotes $x \circ_2 y$. Similarly, the diagram



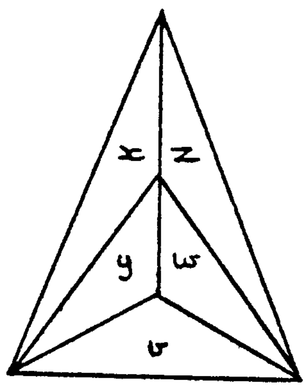
denotes $z \circ_1 w$.

If X_2 together with its two groupoid structures were to form a double groupoid in the sense of Brown and Spencer [3] or a groupoid version of the double categories of Wyler [12] , then we would need an "interchange law", that is a law stating that, provided both sides exist, the identity

$$(x \circ_1 y)\circ_2(z \circ_1 w) = (x \circ_2 z)\circ_1(y \circ_2 w)$$

holds. If one pictures the elements of a double groupoid as squares, as do Brown and Spencer [3] , with a horizontal law of composition and a vertical law of composition, then the interchange law is easy to picture, but here, where we are dealing with triangles, it is difficult to deal with. However we do have the following result of which we shall make extensive use throughout this chapter.

Consider a diagram of 2-simplices of the form

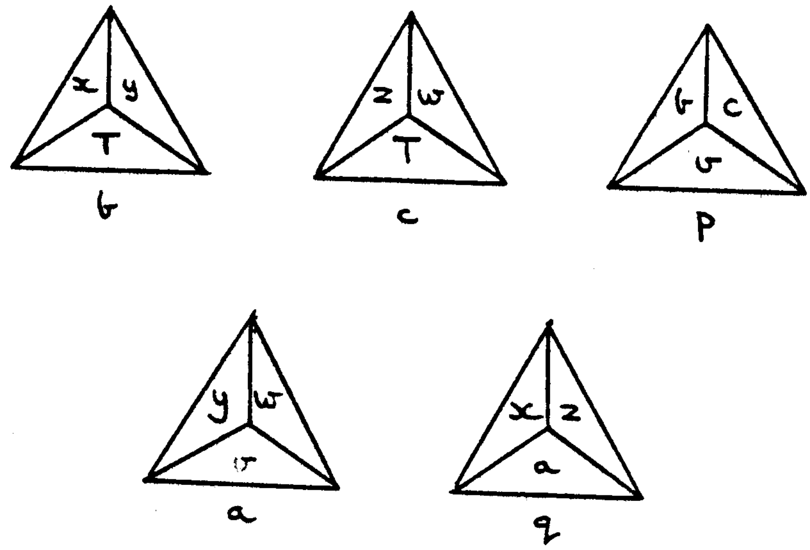


(*)

with some particular orientation of the simplices. There are two ways of composing such a diagram : one either first composes x with y and z with w and then fills in the resulting horn with a thin filler and takes the new face, or alternatively one first fills the horn formed by y, w and v and takes the new face, which we will denote by a, and then one fills in the horn formed by x, z and a and, again, takes the new face.

LEMMA 1.1 Either method of composing a diagram of the form (*) yields the same resulting 2-simplex.

PROOF. In composing the diagram (*) in both ways, the following five three dimensional thin fillers arise. In each case the letter beneath the diagram denotes the fourth face of the filler.



The four thin 3-simplices above which determine b, c, p and a form a horn which we may fill with a thin 4-simplex. By axiom A3 of the definition of a T-complex, it follows that the new face of this 4-simplex is itself thin. But its faces are x, z, a and p and so, since a thin 3-simplex is uniquely determined by any three of its faces it follows immediately that p = q which is the required result.

§2. The crossed module over a groupoid associated with a T-complex

We now define the notion of a crossed module over a groupoid, due to R. Brown and P.J. Higgins, and give an explicit description, in terms of diagrams of the type described in §1, of the way in which we can obtain a local crossed module from a T-complex.

DEFINITION 2.1 A <u>crossed module C over a groupoid</u> consists of a groupoid C_1 with objects C_o and, for each $p \in C_o$, a group $C_2(p)$ and a morphism $d : C_2(p) \rightarrow C_1(p) = C_1(p, p)$. For each $a \in C_1(p,q)$, there is an induced morphism $a_* : C_2(p) \rightarrow C_2(q)$ with 1_* being the identity and $(aa^1)_* = a^1_* \, a_*$. Further C satisfies the axioms

$$\text{(C1)} \quad da_*(x) = a^{-1}(dx)a \qquad x \in C_2$$

$$\text{(C2)} \quad (dy)_*(x) = y^{-1} \, x \, y \qquad x, y \in C_2$$

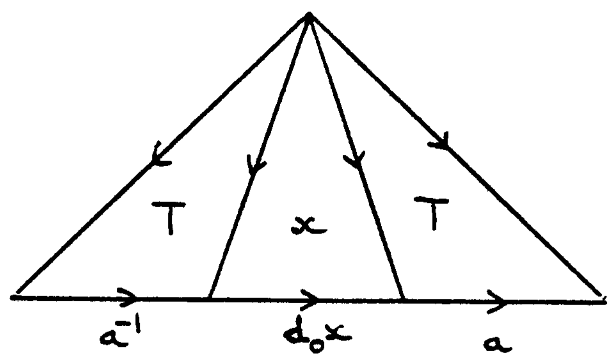
We may obtain a category C of crossed modules over groupoids by defining a morphism $f : C \rightarrow D$ to be a triple $(f_2, f_1, f_o) : (C_2, C_1, C_o) \rightarrow (D_2, D_1, D_o)$ where (f_1, f_o) is a morphism of groupoids and f_2 is a collection of morphisms of groups $f_2 : C_2(p) \rightarrow D_2(f_op)$ for each $p \in C_o$ satisfying $df_2 = f_1d$ and $f_2a_* = (f_1a)_* \, f_2$ for each $a \in C_1$.

Now let X be a T-complex and regard the set X_1 as a groupoid with the composition described in Chapter 2 and the set X_2 as agroupoid with the law of composition o_2. For brevity, in this chapter we shall write o for o_2. Note that, if we wished, we could equally well use the composition o_1.

First we define a partial operation of the groupoid X_1 on the groupoid X_2 as follows : if $x \in X_2$ and $a \in X_1$ are such that $d_1a = d_od_ox$, then define

$$x^b = T(a^{-1}, d_2x, -) \circ x \circ T(a, -, d_1x)$$

that is diagramatically



The following proposition is immediately obvious.

PROPOSITION 2.2 <u>If x and y belonging to X_2 and a and b belonging to X_1 are composable as necessary, then</u>

(i) $x^1 = x$ where 1 is a suitable identity in X_1

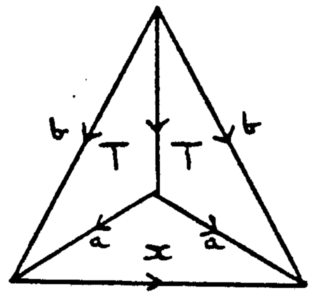
(ii) $x^{ab} = (x^a)^b$

(iii) $(x \circ y)^a = x^a \circ y^a$

(iv) $d_o(x^a) = a^{-1}(d_o x)a$

PROOF. The only non-trivial part is (ii) and this follows immediately from axiom A3 of the definition of a T-complex.

Next, given a pair a and b of 1-simplic of the T-complex X such that $d_1 a = d_1 b$, we can define a map $\phi_{a,b} : X_2(a) \longrightarrow X_2(b)$, where $X_2(a)$ and $X_2(b)$ are the vertex groups of the groupoid X_2 at the points a and b, by

$\phi_{a,b}(x) =$ 

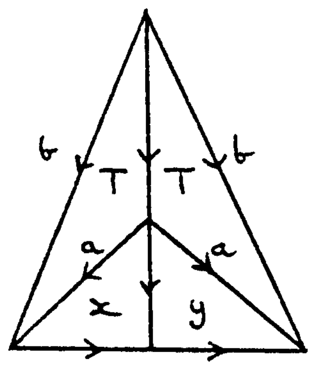
PROPOSITION 2.3 <u>The maps $\phi_{a,b}$ are isomorphisms of groups and satisfy</u>

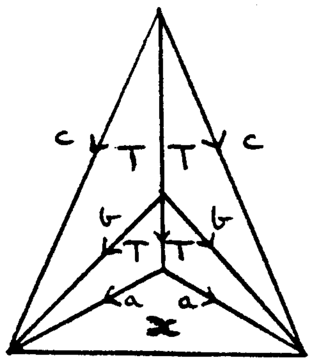
(i) $\phi_{b,c} \circ \phi_{a,b} = \phi_{a,c}$

(ii) $\phi_{a,b}^{-1} = \phi_{b,a}$

(iii) $\left[\phi_{b,c}(x)\right]^{a} = \phi_{ba,ca}(x^{a})$

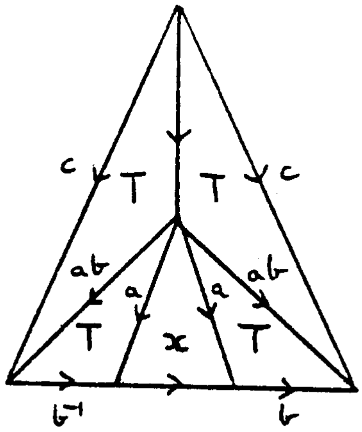
PROOF. Firstly, applying Lemma 1.1 to the diagrams





shows respectively that $\phi_{a,b}$ is a homomorphism and that (i) is satisfied. For (ii), note that, for any $a \in X_1$, $\phi_{a,a}$ is the identity for $X_2(a)$ as the thin filler used to construct $\phi_{a,a}$ is degenerate, and apply (i). The proof of (iii) is by successive applications of Lemma 1.1 as follows :

$\phi_{ab,c}(x^b) =$



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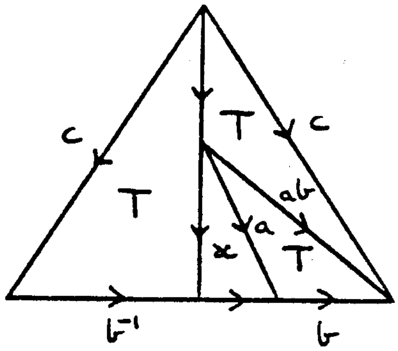

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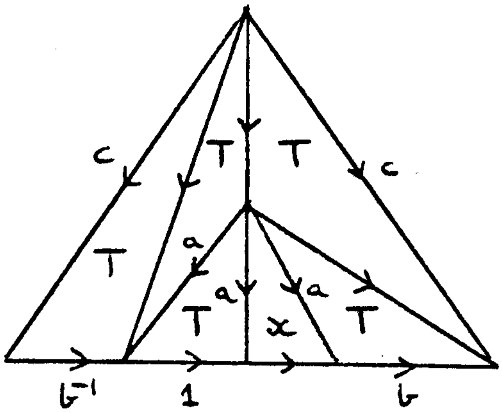

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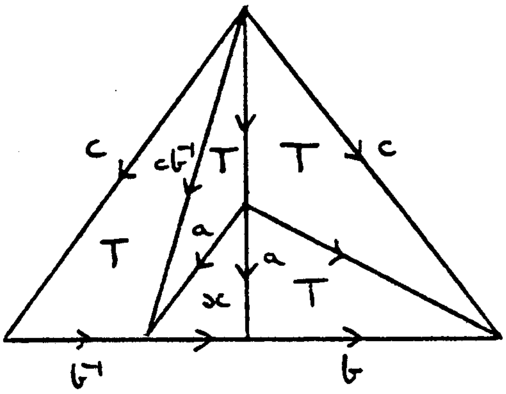

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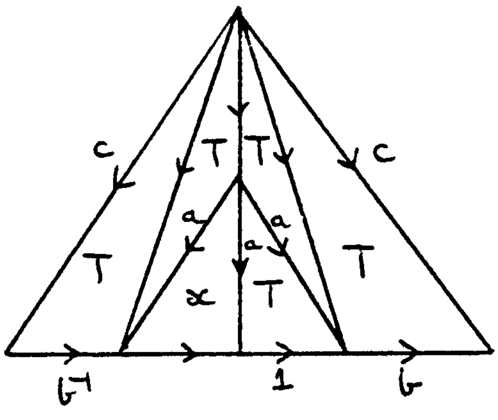

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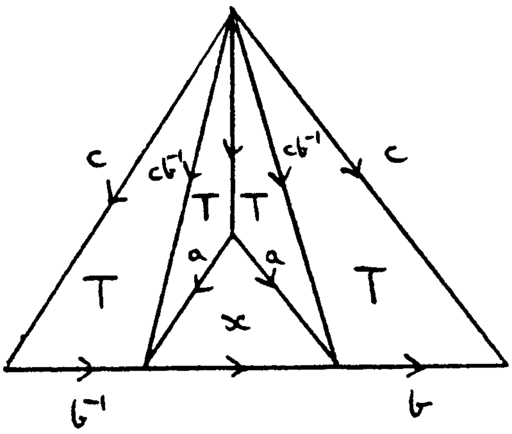


$= \left[\phi_{a,cb^{-1}}(x)\right]^{b}$

Given the T-complex X, we can now construct the crossed module C(X) over a groupoid associated with X. We define the groupoid $C(X)_1$ to be the set X_1 of 1-simplices of X together with the induced groupoid structure defined in Chapter 2. Thus $C(X)_o$ is the set of 0-simplicies X_o of X. For each p of $C(X)_o$, we define the group $C(X)_2(p)$ to be the vertex group $X_2(s_op)$ of the groupoid X_2 where the law of composition on X_2 is $o = o_2$. Then for each p of $C(X)_o$ we have a morphism $d: C(X)_2(p) \longrightarrow C(X)_1(p)$ which is just the face map d_o.

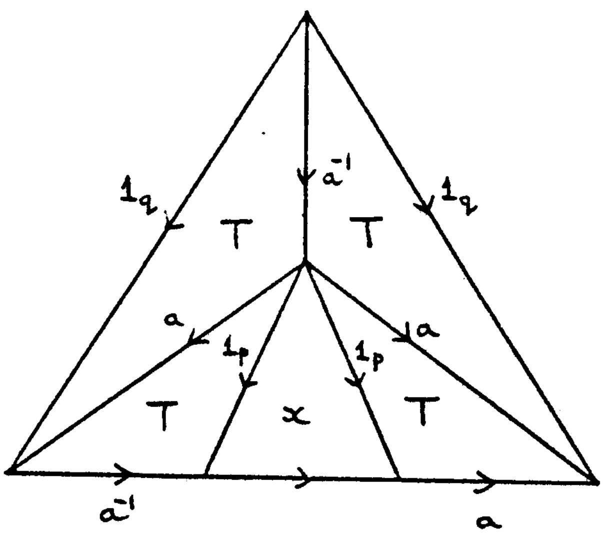
Next if p and q belong to $C(X)_o$ and $a \in C_1(X)(p,q)$ then we define a map

$$a_* : C_2(X)(p) \longrightarrow C_2(X)(q)$$

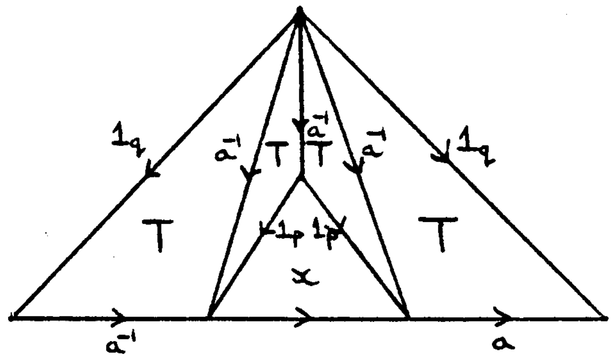
by

$$a_*(x) = \phi_{a,1_q}(x^a)$$

where 1_q is the identity s_oq of the groupoid X_1. Pictorially this is



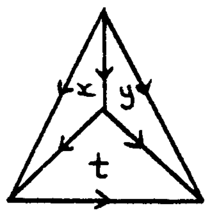
or equivalently, by Proposition 2.3 (iii),



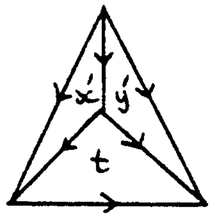
PROPOSITION 2.4 <u>C(X) is a crossed module over a groupoid.</u>

In order to prove this proposition we shall need the following lemma.

LEMMA 2.5 If the 2-simplices



and



where t is a thin 2-simplex, are equal, then, on replacing t by any 2-simplex z with $d_1 z = d_1 t$ and $d_2 z = d_2 t$, the equality remains true.

We leave the proof of this lemma until later.

PROOF OF 2.4 Firstly a_* is a homomorphism since each $\phi_{a,b}$ is a homomorphism and we have $(x \circ y)^a = x^a \circ y^a$. The proof that a_* is an identity whenever a is an identity is trivial and given a, b $\in$ C $(X)_1$ such that ab is defined we have

$$b_*a_*(x) = \phi_{b,1}(\phi_{a,1}(x^a)^b)$$

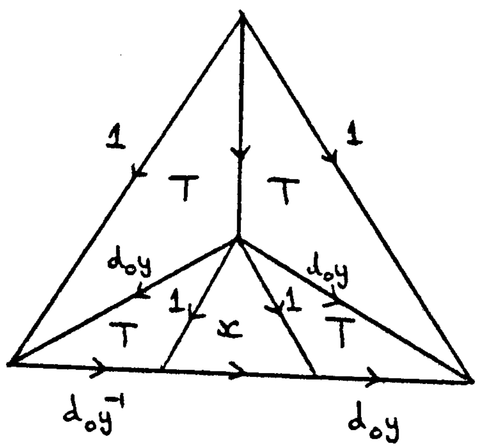
$$= \phi_{b,1}\ \phi_{ab,b}(x^{ab})$$

$$= \phi_{ab,1}(x^{ab})$$

$$= (ab)_*(x)$$

where we have used Propositions 2.2 and 2.3. Secondly, by definition of $a_*(x)$, axiom C1 of the definition of a crossed module over a groupoid is trivially true, and axiom C2 is proved using Lemma 2.5. We have

$(dy)_*(x) =$



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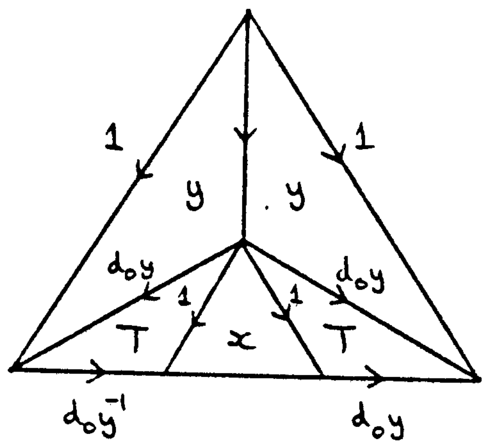
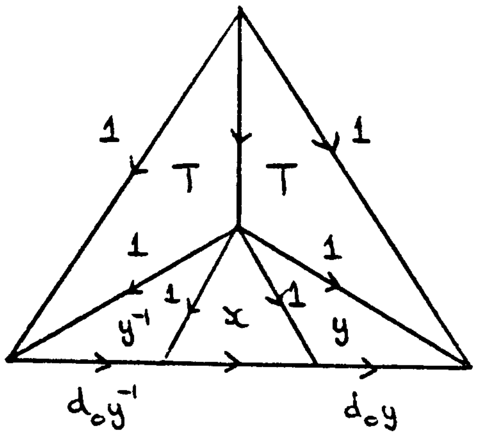


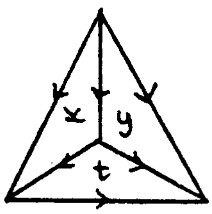
by Lemma 2.5

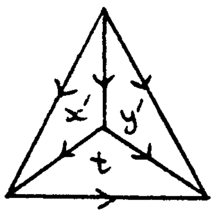
= 

using Lemma 1.1 successively

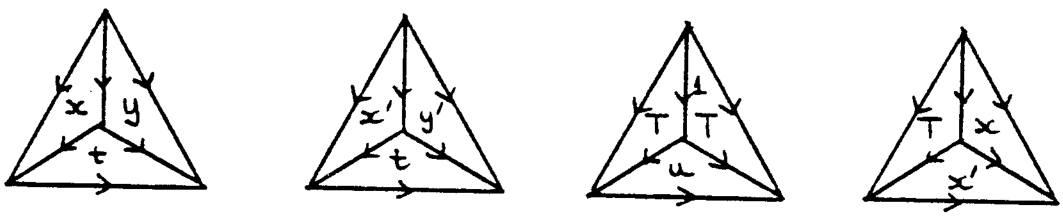
$= y^{-1} \; x \; y$

PROOF OF 2.5 In the following diagrams the letter beneath denotes the 2-simplex. Let

u = 

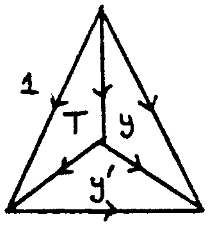
= 

The four thin 3-simplices



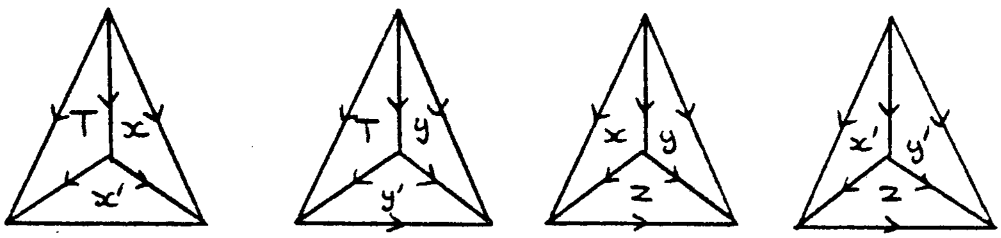
u u u v

form a horn in X_3 and on taking the thin filler of this horn we obtain the new thin 3-simplex



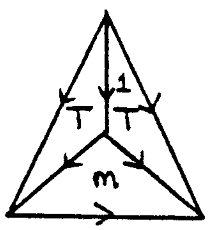
v

Now consider the four thin 3-simplices



v v 1 m

These also form a horn in X_3 and on taking the thin filler of this horn we obtain the new thin 3-simplex



1

The thin faces of this 3-simplex are degenerate and so it follows that the 3-simplex itself must be degenerate. Hence $l = m$ which is the required result.

We have thus associated to a T-complex X the crossed module C(X). This construction is functorial for, given a morphism $f : X \to Y$ of T-complexes, the restriction of f to $C_2(X)$ gives us a morphism of crossed modules over groupoids. We shall denote the functor by $C : \underset{\sim}{T} \to \underset{\sim}{C}$

§3. The nerve of a crossed module over a groupoid

We now give the reverse construction for obtaining a T-complex from a crossed module. In order to do this construction we extend Segal's nerve functor [9] as used in Chapter 2 to crossed modules.

Let C be a crossed module over a groupoid. The nerve NC of C is the simplicial set defined as follows: let $NC_o = C_o$, $NC_1 = C_1$ and

$$NC_2 = \{ w = (x; a_o, a_1, a_2) : x \in C_2, a_i \in C_1, dx = a_2 a_o a_1^{-1} \}$$

Recalling that C_1 is a groupoid over C_o we take the obvious face maps and degenerate elements for NC_1. The face maps for NC_2 are given by $d_i w = a_i$ and the degenerate 2-simplices are $s_o a = (1; a, a, 1)$ and $s_1 a = (1; 1, a, a)$ where the identities are the obvious ones. Next, writing $w_i = (x_i; a_o^i, a_1^i, a_2^i)$, the 3-simplices of NC are defined to be quadruples

(w_0, w_1, w_2, w_3) of 2-simplices satisfying the relations $a^i_j = a^{j+1}_i$ for $0 \leqslant i \leqslant j \leqslant 2$, which simply ensure that the faces w_i fit together as required, and also the relation

$$x_0 = (a^2_2)_*(x_3 \circ x_1 \circ x_2^{-1}) \qquad (*)$$

or alternatively

$$(a_2^{2-1})_* (x_0) \circ x_2 \circ x_1^{-1} \circ x_3^{-1} = 0$$

which, in this latter form, is similar to the formula given by the homotopy addition lemma for the boundary $d(t)$ in $\pi_2(\Delta^{3,2}, \Delta^{3,1}, *)$ of the single generating element t of $\pi_3(\Delta^3, \Delta^{3,2}, *)$. Face maps for NC_3 are given by $d_i(w_0, w_1, w_2, w_3) = w_i$ and degenerate elements by

$$s_0 w = (w, w, s_0 d_1 w, s_0 d_2 w)$$

$$s_1 w = (s_0 d_0 w, w, w, s_1 d_2 w)$$

$$s_2 w = (s_1 d_0 w, s_1 d_1 w, w, w)$$

In higher dimensions NC is defined inductively by

$$NC_{n+1} = \{ (x_0, \ldots, x_{n+1}) : x_i \in NC_n, \; d_j x_i = d_i x_{j+1}, \; i \leqslant j \}$$

that is, an $(n+1)$-simplex simply consists of $n+2$ n-simplices fitting together as required. Face and degeneracy maps are the obvious ones.

It is clear that this is an extension of the idea of a nerve of a category for, if we let the group $C_2(p)$ of the crossed module C be trivial for each p, then NC becomes simply the usual nerve of the groupoid C_1.

PROPOSITION 3.1 <u>The construction NC gives a functor N from the category of crossed modules over groupoids to the category of T-complexes of rank 2.</u>

PROOF. Define sets T_n of thin elements by

$$T_1 = \{\text{identity elements of } C_1\}$$

$$T_2 = \{(1;\ a_o,\ a_1,\ a_2) \in NC_2\}$$

$$T_n = NC_n \qquad n \geqslant 3$$

We must show that these satisfy the axioms for a T-complex. Firstly, by definition all degenerate elements are thin. Secondly, horns in NC_o and NC_1 certainly have unique thin fillers and in higher dimensions any n faces of a n-simplex uniquely determine the simplex and so any horn automatically has a unique thin filler. Thirdly, suppose x is a thin element of NC having all faces but one themselves thin. If $x = (1;\ a_o,\ a_1,\ a_2) \in NC_2$ then two of the a_i are an identity and so, since $a_2 \circ a_o \circ a_1^{-1} = 1$, the third must be also. If $x = (w_o,\ w_1,\ w_2,\ w_3)\quad T_3$ where $w_i = (x_i;\ a_o^i,\ a_1^i,\ a_2^i)$ then three out of the four x_i must be identities and it follows by the formula (*) that the fourth x_i must be also. In higher dimensions, since then all elements are thin, there is nothing to prove. Thus NC is a T-complex of rank 2.

Now suppose that $f : C \longrightarrow D$ is a morphism of crossed modules over groupoids then in an obvious fashion f determines a simplicial map $Nf : NC \longrightarrow ND$. The only point we need to check is that Nf is well-defined in dimension 3. Suppose that $w = (w_o,\ w_1,\ w_2,\ w_3)$ is a 3-simplex of

NC and that $w_i = (x_i;\, a_o^i,\, a_1^i,\, a_2^i)$. Then we must check that condition (*) holds for $Nf(w)$. We have $Nf(w) = (v_o, v_1, v_2, v_3)$ where $v_i = (f_2x_i;\, f_1a_o^i,\, f_1a_1^i,\, f_1a_2^i)$ and

$$(f_1a_2^2)_*(f_2x_3 o f_2x_1 o f_2x_2^{-1}) = f_2(a_2^2)_*(x_3 o x_1 o x_2^{-1})$$
$$= f_2x_o$$

which is the required condition that $Nf(w)$ be a well-defined 3-simplex of ND. Nf certainly preserves thin elements and so it is a morphism of T-complexes. Finally the functoriality of N is obvious.

We end this section with a result on the homotopy groups of NC. From the way in which we were able to construct a crossed module from a T-complex in Chapter 4 using homotopy groups, we would expect NC to have first and second homotopy groups isomorphic to the cokernel and kernel of the maps $d : C_2(p) \longrightarrow C_1(p)$. We now prove this.

PROPOSITION 3.2 Let C be a crossed module over a groupoid, then

$$\pi_1(NC,p) = \text{coker } d : C_2(p) \longrightarrow C_1(p)$$

$$\pi_2(NC,p) = \ker d : C_2(p) \longrightarrow C_1(p)$$

$$\pi_i(NC,p) = 0 \text{ for } i > 2$$

PROOF. Firstly $\pi_1(NC,p) = NC_1(p)/\sim \;= C_1(p)/\sim$ where $a \sim b$ if there exists a 2-simplex $(x; 1, a, b)$, that is if $ba^{-1} \in \mathrm{Im}\ d: C_2(p) \to C_1(p)$. Hence $\pi_1(NC,p)$ is as required. Secondly

$$\pi_2(NC,p) = NC_2(p)/\sim \quad \cong (\ker d : C_2(p) \longrightarrow C_1(p))/\sim$$

where $(x; 1_p, 1_p, 1_p) \sim (y; 1_p, 1_p, 1_p)$ if there exists a 3-simplex w with $d_2 w = (1; 1_p, 1_p, 1_p)$. But then, by the formula (*), $x = y$ and so $\pi_2(NC,p)$ is as required. Finally, since NC has rank 2, $\pi_i(NC,p) = 0$ for $i \quad 2$.

§4. The equivalence of categories

We now prove the following

THEOREM The category $\underset{\sim}{T}^2$ of T-complexes of rank 2 is equivalent to the category $\underset{\sim}{C}$ of crossed modules over groupoids.

We prove the theorem by means of the two Propositions 4.1 and 4.2.

PROPOSITION 4.1 Let D be a crossed module over a groupoid. There exists a natural isomorphism

$$\psi : CN(D) \longrightarrow D$$

PROOF. Let d be the homomorphism $D_2(p) \to D_1(p)$ for each p of the crossed module D and let the corresponding homomorphism for CN(D) be d'

Now we have by definition

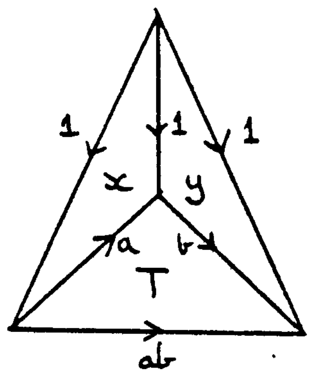
$$CN(D)_o = N(D)_o = D_o$$

$$CN(D)_1 = N(D)_1 = D_1$$

$$CN(D)_2(p) = N(D)_2(1_p) \qquad \text{for } p \in D_o$$

$$= \{ w = (x;\ a_o,\ 1_p,\ 1_p) : x \in D_2(p),\ dx = a_o \}$$

We define the isomorphism $\Psi = (\Psi_2, \Psi_1, \Psi_o)$ be letting Ψ_1 and Ψ_o be the identities and letting $\Psi_2(x;\ a_o,\ 1_p,\ 1_p,) = x$ so that Ψ_2 is certainly a bijection. Firstly, we check that Ψ_2 is a homomorphism $CN(D)_2(p) \longrightarrow D_2(p)$ for each $p \in D_o$. Let $v, w \in CN(D)_2(p)$ and let $v = (x;\ a,\ 1,\ 1)$ and $w = (y;\ b,\ 1,\ 1)$ where we have written 1 instead of 1_p for brevity. Then $v \circ w = (z;\ ab,\ 1,\ 1)$ where z is given by $d_2 t$ and t is the thin 3-simplex pictured by

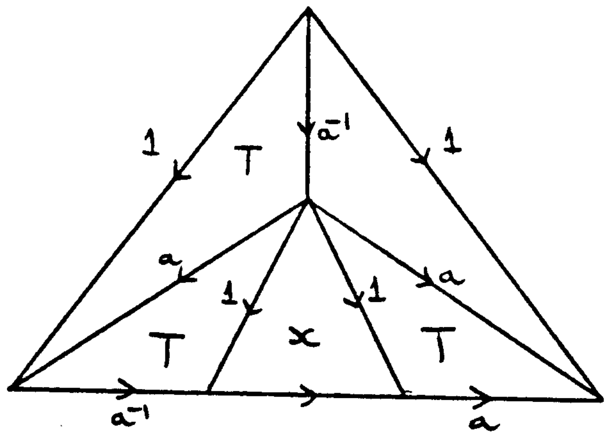


By definition of the 3-simplices of N(D) the identity $1 = 1_*(x \circ y \circ z^{-1})$ must hold and so z $= x \circ y$. Thus $\Psi_2(v \circ w) = \Psi_2(v) \circ \Psi_2(w)$ as required.

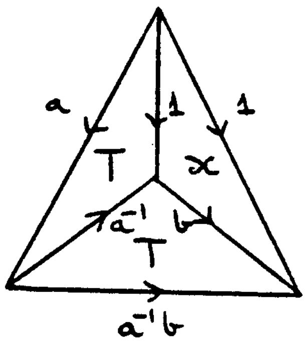
Secondly, Ψ is required to satisfy $d'\Psi_2 = \Psi_1 d$, which is trivially true, and also to satisfy $\Psi_2 a_* = (\Psi_1 a)_* \Psi_2$ for any $a \in CN(D)_1$.

In other words, if $w = (x;\ b,\ 1, 1)$ then we must have $a_*(w) = (a_* x;\ a^{-1}ba,\ 1,\ 1)$

The diagram for $a_*(w)$ is



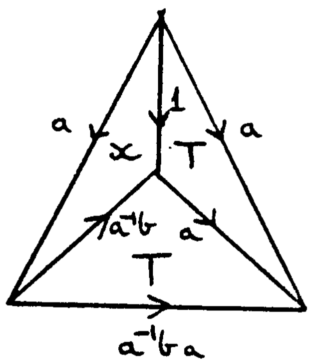
and this is constructed in three stages as follows where each diagram represents a subdivided 2-simplex :



$= (u;\ a^{-1}b,\ 1,\ 1)$

where $1 = a_*(1 \circ x \circ u^{-1})$

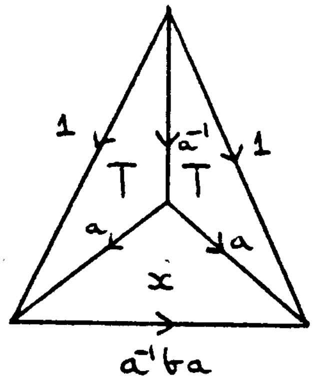
so that $u = x$



$= (v;\ a^{-1}ba,\ 1,\ 1)$

where $1 = a_*(x \circ 1 \circ v^{-1})$

so that $v = x$



$= (z;\ a^{-1}ba,\ 1,\ 1)$

where $x = (a^{-1})_*(1 \circ z \circ 1)$

so that $z = a_*(x)$

Thus we have $a_*(w) = (a_*x;\ a^{-1}ba,\ 1,\ 1)$ as required.

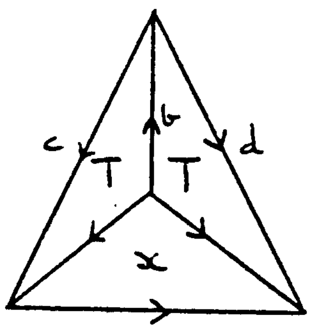
Finally, the naturality of Ψ with respect to morphisms is trivially satisfied and this completes the proof.

PROPOSITION 4.2 <u>Let X be a T-complex of rank 2. There exists a natural isomorphism</u>

$$\phi : NC(X) \longrightarrow X$$

In order to facilitate the proof we first state some simple lemmas.

LEMMA 4.3 <u>For any 2-simplex x and 1-simplices b, c and d, the following equality of 2-simplices holds :</u>



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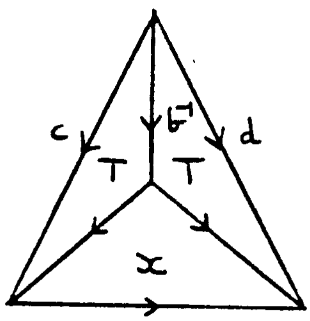
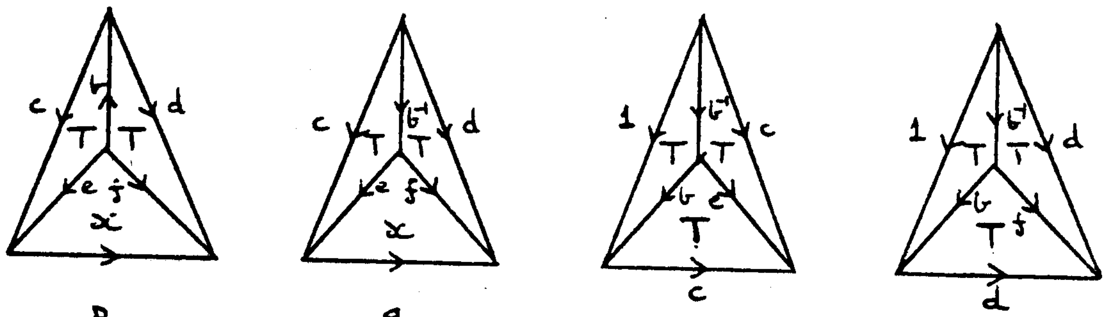
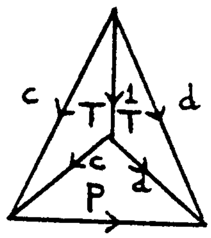


PROOF Consider the four thin 3-simplices



p q

where p and q are the 2-simplices determined by the thin fillers. These form a horn which we may fill with a thin 4-simplex. The new face of this 4-simplex has faces as in the diagram

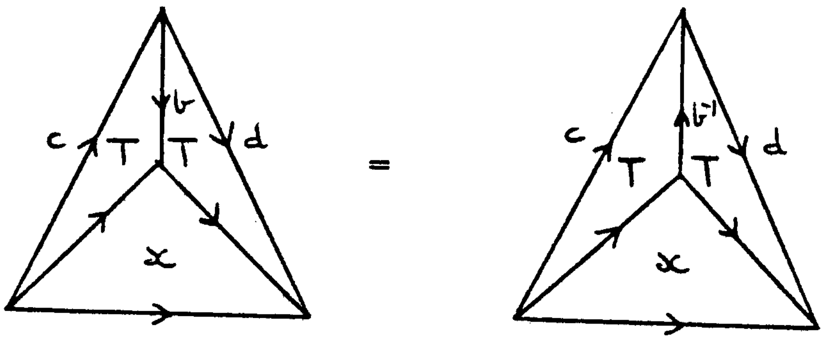


q

But, by axiom A3 of the definition of a T-complex, this must be a thin 3-simplex and hence it must in this case be degenerate. Thus p = q which is the required result.

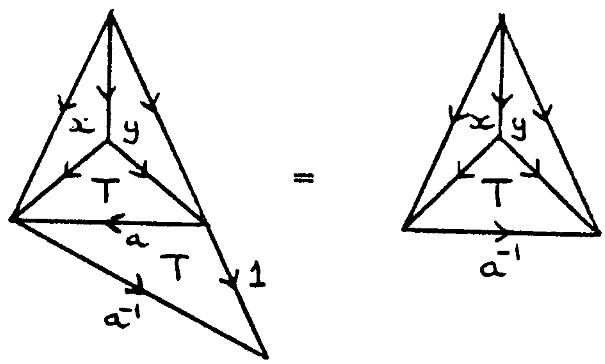
In an exactly similar way we may also prove the following lemmas.

LEMMA 4.4 <u>For any 2-simplex x and 1-simplices b, c and d, the following equality of 2-simplices holds:</u>



Note that this is the same result as Lemma 4.3 except that the orientation of c has been reversed.

LEMMA 4.5 <u>For any 2-simplices x and y and 1-simplex a suitably fitting together, the following equality of 2-simplices holds :</u>



PROOF OF 4.2 By Theorem 3.1 of Chapter 1, it is sufficient for us to define ϕ up to and including dimension 3. We have

$$NC(X)_0 = C(X)_0 = X_0$$

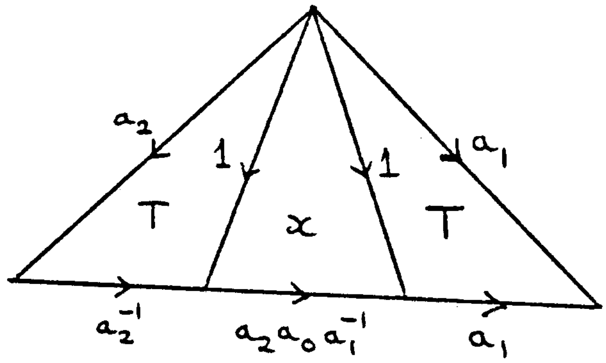
$$NC(X)_1 = C(X)_1 = X_1$$

So we define ϕ_0 and ϕ_1 to be the identity maps. Further

$$NC(X)_2 = \{ (x; a_0, a_1, a_2) : x \in X_2(1_p) \text{ where } p \in X_0,\ a_i \in X_1,\ d_0 w = a_2 a_0 a_1^{-1} \}$$

and we define ϕ_2 by

$\phi_2(x;\ a_0,\ a_1,\ a_2) =$



The 3-simplices of NC(X) are quadruples of 2-simplices $w_i = (x_i;\ a_0^i,\ a_1^i,\ a_2^i)$ satisfying $a_j^i = a_i^{j+1}$ for $i \leq j$ and $x_0 = (a_2^2)_*(x_3 \circ x_1 \circ x_2^{-1})$. We define

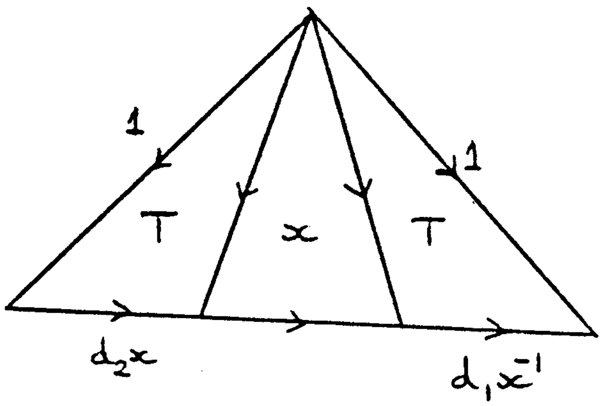
$$\phi_3(w_0, w_1, w_2, w_3) = T(-, \phi_2 w_1, \phi_2 w_2, \phi_2 w_3)$$

At this point we assert that

$$d_0 T(-, \phi_2 w_1, \phi_2 w_2, \phi_2 w_3) = \phi_2 w_0 \qquad (*)$$

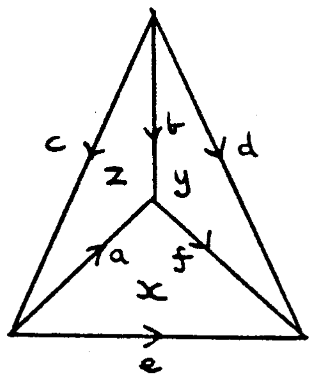
so that we have $d_i \phi_3 = \phi_2 d_i$ for all i as required. We shall prove (*) later. We can define an inverse for ϕ_2 by

$\phi_2^{-1}(x) =$



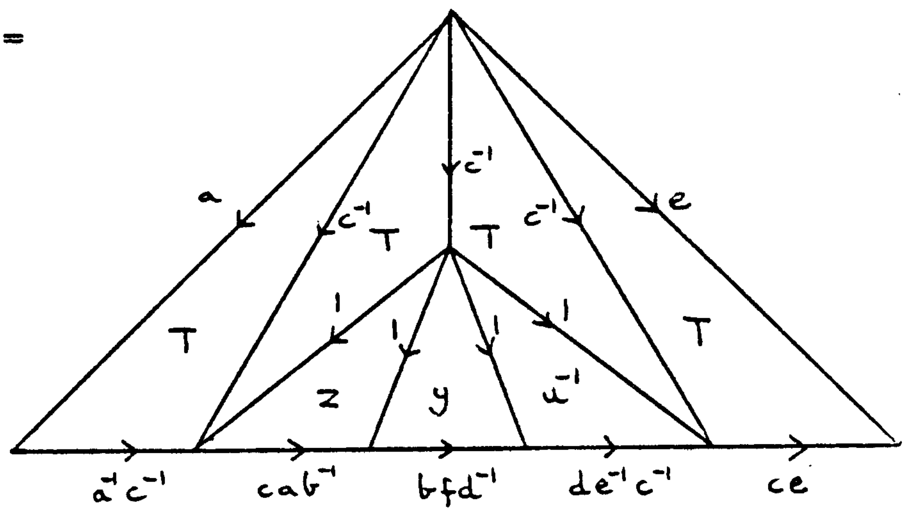
and it follows that ϕ_2 and ϕ_3 are bijective. The map ϕ_2 certainly preserves thin elements and it now follows from Theorem 3.1 of Chapter 1 that we have an isomorphism ϕ of T-complexes. The naturality of ϕ in morphisms of T-complexes is immediate.

It now remains for us to check (*). Suppose that a 3-simplex (w_0, w_1, w_2, w_3) of NC(X) is represented by the diagram



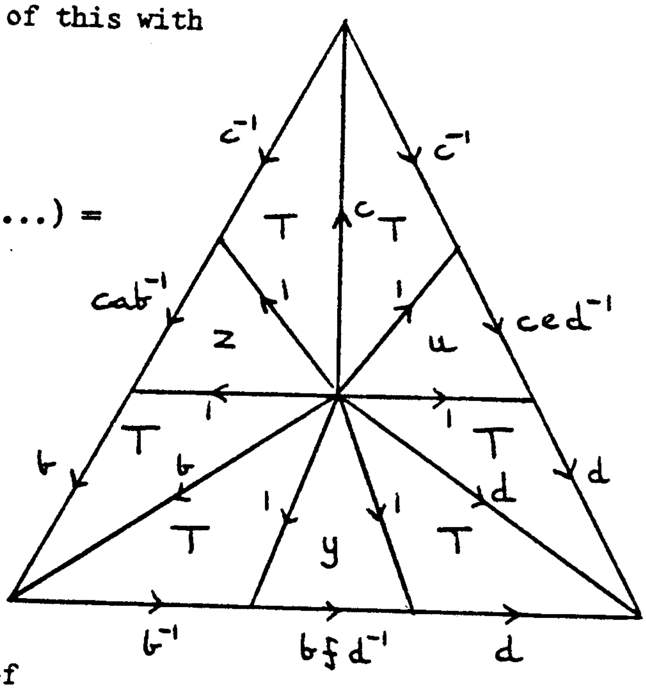
so that $w_0 = (x;\ f,\ e,\ a)$, $w_1 = (y;\ f,\ d,\ b)$, $w_2 = (u;\ e,\ d,\ c)$ and $w_3 = (z;\ a,\ b,\ c)$. Then $x = c_*(z \circ y \circ u^{-1})$ and so we have

$\phi_2 w_o =$



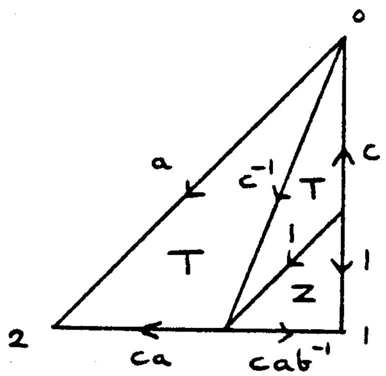
We must prove the equality of this with

$d_oT(-, \phi_2 w_1, \ldots) =$

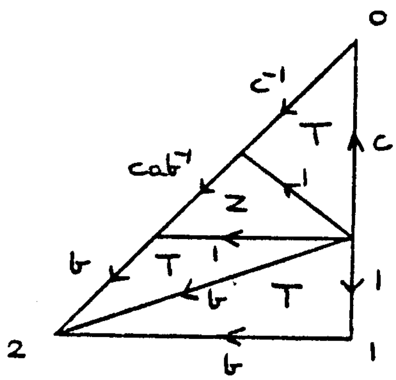


First we show the equality of

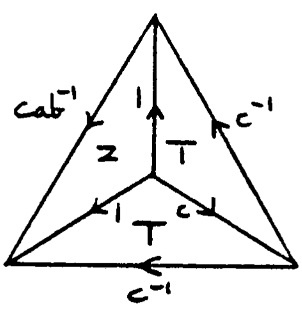
p =



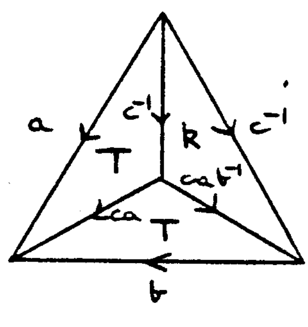
and q =



p is given in two stages by

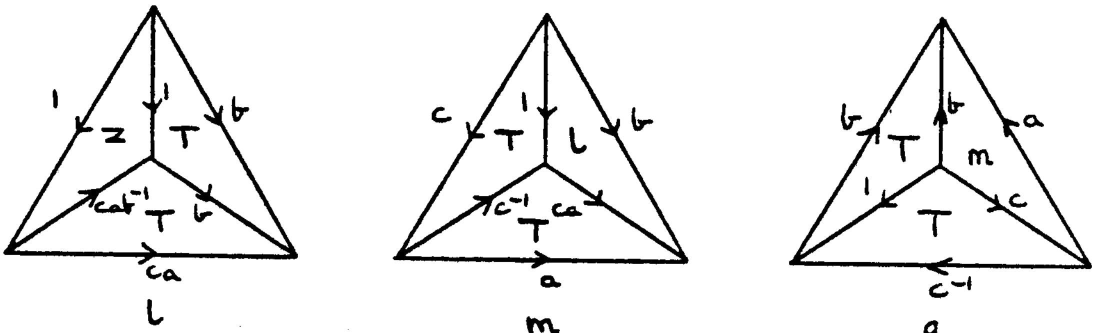


k



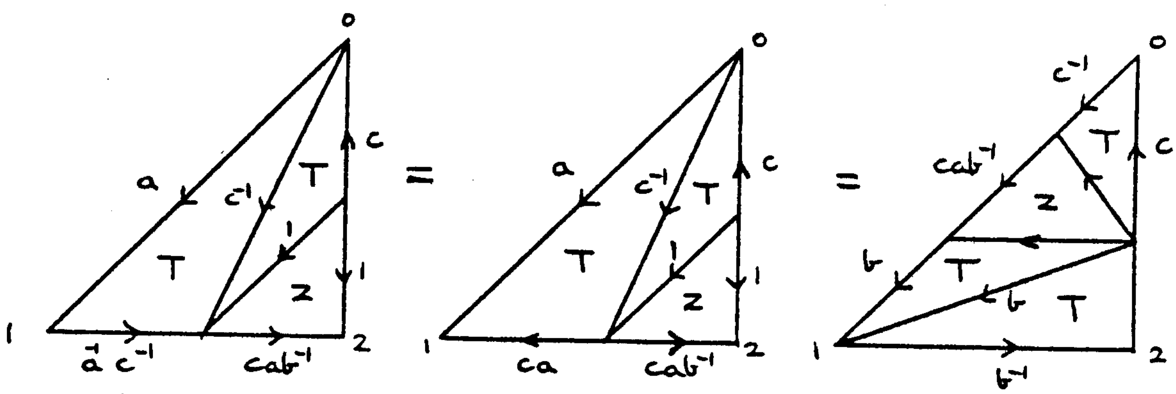
P

and q is given in three stages by

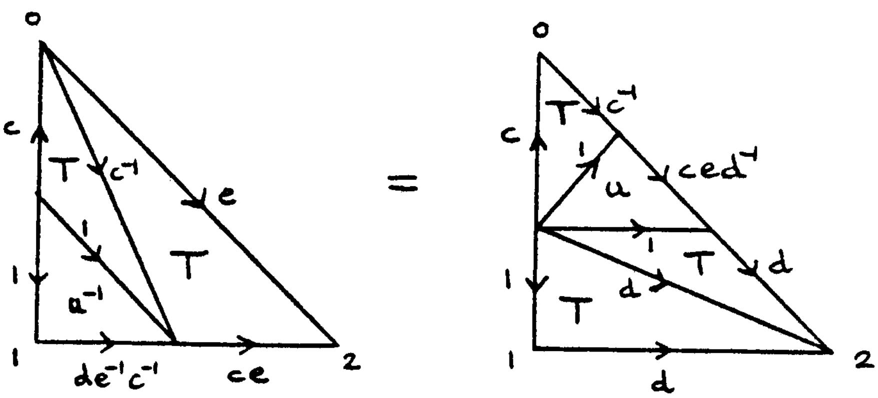


The first four of these thin 3-simplices form a horn in X_3 and on taking the thin filler we obtain a new thin 3-simplex which is precisely the thin simplex which determines q. Hence p = q.

Using Lemmas 4.4 and 4.5 we now have

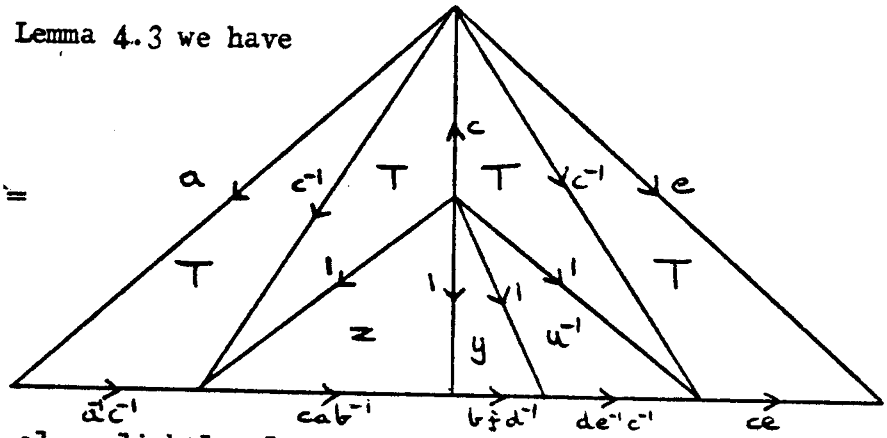


In an exactly similar way we may show that



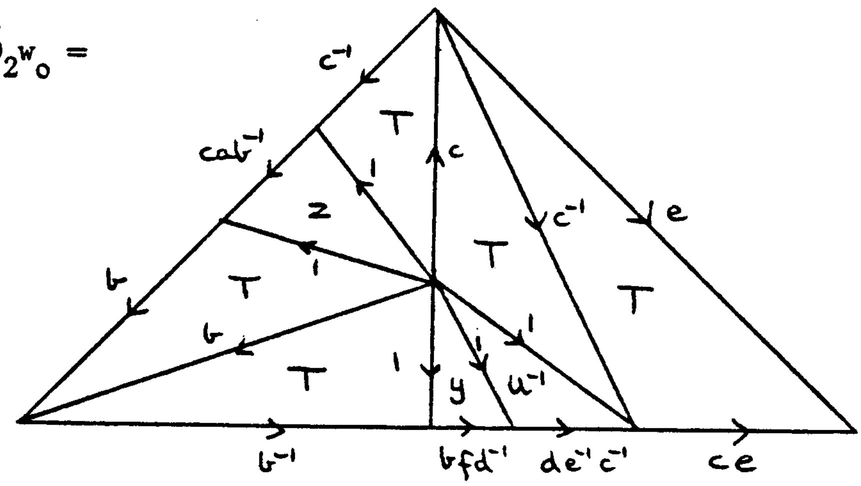
and putting these two equalities together, the proof is concluded as follows. By Lemma 4.3 we have

$\phi_2 w_o$ =



where we have also slightly altered the shape of the diagram in order to make the next step clear. Now using Lemma 1.1 successively, together with the two equalities we have proved above, we have

$\phi_2 w_o$ =



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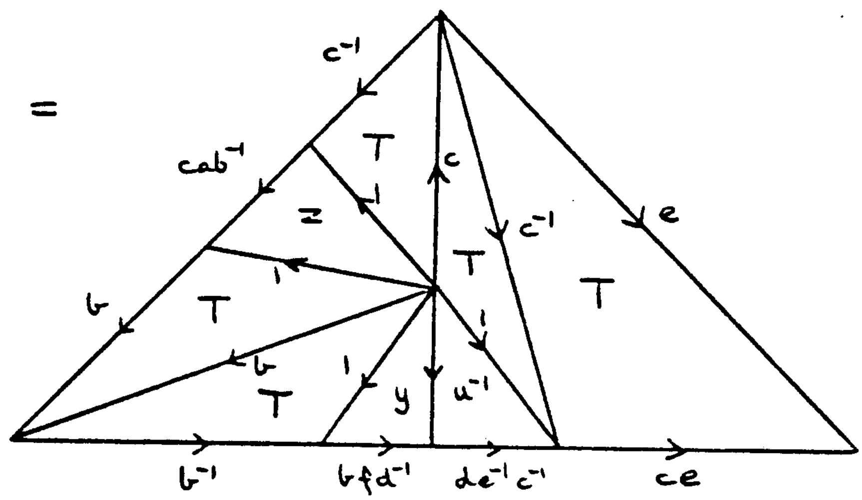

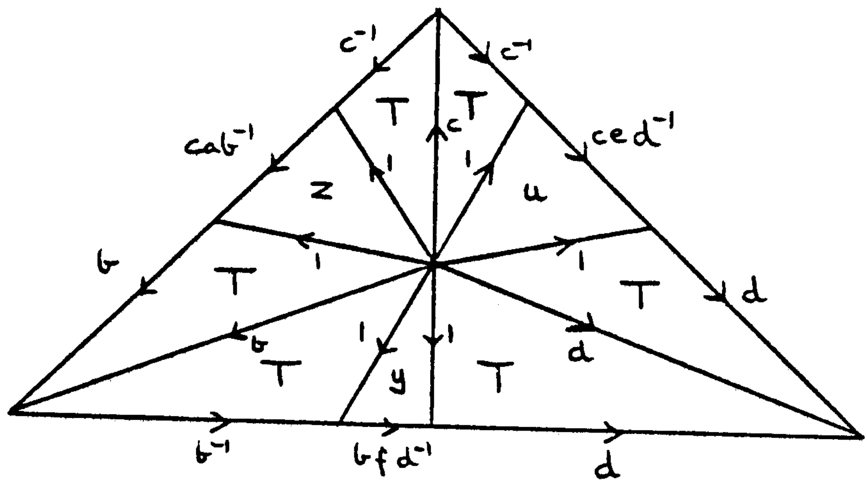

$$= d_0T(-, \phi_2w_1, \phi_2w_2, \phi_3w_3)$$

as required.

This completes the proof of Proposition 4.2 and combining Propositions 4.1 and 4.2 gives the equivalence theorem stated at the beginning of the section.

A particular case is where we restrict ourselves to T-complexes having only one vertex. Then we obtain the result that the category of T-complexes of rank 2 possessing only one vertex is equivalent to the category of crossed modules.

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