Addendum
to
"In search of new "Homology" functors having a close relationship to K-theory"
by
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I. For G any group, following Loday (Les matrices monomiales et le groupe de Whitehead Wh_2, Lecture Notes in Math. 551, pp. 155-163), define $M_n(G)$ ($\approx G^n \rtimes S_n = G \wr S_n$) to be the $n \times n$ monomial matrices with entries in G and $M(G) = \varinjlim M_n(G)$. The subgroup consisting of elements whose associated permutation is alternating and the product of whose elements lies in the commutator subgroup of G, $[G,G]$, is the commutator subgroup of $M(G)$. This subgroup is perfect and will be denoted $E(G)$. Similarly $[M_n(G), M_n(G)] = E_n(G)$ can be so described and is perfect for $n \geq 5$. There then exist spaces $B_{M(G)}$ and $B^+_{M(G)}$ (+ with respect to $E(G)$). For any group G there are canonical maps to and from the trivial group. Hence $\pi_i(B^+_{M(G)})$ is canonically the direct sum of two groups $\pi_i(B^+_{M(1)})$ (= the stable homotopy groups of spheres) and a second group which we denote $\bar{H}_i(G)$. It is clear that $\bar{H}_1(G) = H_1(M(G)) = H_1(G)$ and that $\bar{H}_2(G) = H_2(E(G))$. If $S(G)$ denotes the universal central extension of $E(G)$, then also $\bar{H}_3(G) = H_3(S(G))$. A computation analogous to that showing that K_2 of a field is generated by symbols (in fact, only that part dealing with the "Weyl group" W;

See J. Milnor, Introduction to Algebraic K-theory, Annals of Math. Studies No. 72, pp. 76-78) shows that

$$\overline{H}_2(G) \approx \widetilde{H}_2(G) \qquad (\text{the } \widetilde{H}_2 \text{ defined in the paper}).$$

In fact, $H_2 E_n(G) \approx H_2 E(G)$ for all $n \geq 5$.

Further as in §3, $S(G) = (E(G), E(G))$, $(M(G), M(G)) \approx (E(G), E(G)) \times (M(G)^{ab}, M(G)^{ab})$ (note that $M(G)^{ab} \approx G^{ab}$) and hence there is a canonical split exact sequence

$$1 \longrightarrow \widetilde{H}_2 E(G) \longrightarrow \widetilde{H}_2 M(G) \longrightarrow \widetilde{H}_2 M(G)^{ab} \longrightarrow 1 .$$

Further one can show that the functors $\overline{H}_i(G)$ have the properties:

1) There is a homomorphism $\overline{H}_i(G) \longrightarrow H_i(G)$ (only known to be surjective for $i \leq 2$);
2) Given a ring R there are maps $\overline{H}_i(R^*) \to K_i(R)$;
3) For G abelian, $\overline{H}_*(G)$ has a multiplicative structure and the map $\overline{H}_*(R^*) \longrightarrow K_*(R)$ is a ring homomorphism for R a commutative ring.

These 3 remarks are immediate from results of Loday appearing in the paper mentioned above and in "K-théorie algébrique et représentations de groupes", Ann. Sci. École Norm. Sup. (4) 9 (1976), 309-377.

It is not yet clear that this is the "correct" sequence of functors sought in the original note.

II. The following computations are a variation on those of Loday which he showed me on Feb. 18, 1977. We will prove that relation (3) is a consequence of relations (1) and (2). In fact, we will prove a somewhat stronger statement. Recall the relations

(1) $\langle x, y\rangle \langle y, x\rangle$

(2) $\langle {}^x y, {}^x z\rangle \langle x, z\rangle \langle xy, z\rangle^{-1}$ (2′) $\langle x, yz\rangle^{-1} \langle x, y\rangle \langle {}^y x, {}^y z\rangle$

(3) $\langle {}^x y, {}^x z\rangle \langle z, y\rangle \langle x, [y,z]\rangle^{-1}$.

Let (3*) denote the "more natural" relation

(3*) $\langle {}^x y, {}^x z\rangle \langle y, z\rangle^{-1} \langle x, [y,z]\rangle^{-1}$.

We will now show that (3*) is a consequence of (2) and (2′). In particular, it follows that (3) is a consequence of (1) and (2).

First $\langle y \cdot y^{-1}xy, z\rangle \equiv {}^y\langle y^{-1}xy, z\rangle \langle y, z\rangle$ by (2)

(*) or $\langle xy, z\rangle \equiv \langle x, {}^y z\rangle \langle y, z\rangle$.

Thus $\langle z^{-1}z, x\rangle \equiv \langle z^{-1}, {}^z x\rangle \langle z, x\rangle$ by (*)

$1 = \langle 1, x\rangle \equiv \langle z^{-1}, {}^z x\rangle \langle z, x\rangle$ by (2)

(**) $1 \equiv \langle z^{-1}, {}^z x\rangle \langle z, x\rangle$ as above by (2)

and $1 \equiv \langle x, z\rangle \langle {}^z x, z^{-1}\rangle$ similarly by (2′).

We compute

$$\begin{aligned}
\langle {}^x y, {}^x z\rangle &\equiv \langle xy, z\rangle \langle x, z\rangle^{-1} && \text{by (2)}\\
&\equiv \langle x, {}^y z\rangle \langle y, z\rangle \langle x, z\rangle^{-1} && \text{by (*)}\\
&\equiv \langle x, {}^y z\rangle \langle y, z\rangle \langle {}^z x, z^{-1}\rangle && \text{by (**)}\\
&\equiv \langle x, {}^y z\rangle \langle {}^{[y,z]z} x, {}^{[y,z]} z^{-1}\rangle \langle y, z\rangle && \text{by (6) which follows from (2) and (2′)}\\
\star \quad &= \langle x, {}^y z\rangle \langle {}^{yz} x, {}^{yz} z^{-1}\rangle \langle y, z\rangle
\end{aligned}$$

Next applying (2′) as in the derivation of (*) yields

$\langle x, yz\rangle \equiv \langle x, z\rangle \langle {}^z x, y\rangle$ by (2′)

which implies

$$\langle {}^\beta \gamma, \alpha\rangle \equiv \langle \gamma, \beta\rangle^{-1} \langle \gamma, \alpha\beta\rangle .$$

Taking $\alpha = {}^{yz} z^{-1}$, $\beta = {}^y z$, $\gamma = x$ yields

$$\begin{aligned}
\langle {}^{yz} x, {}^{yz} z^{-1}\rangle &\equiv \langle x, {}^y z\rangle^{-1} \langle x, {}^{yz} z^{-1} \cdot {}^y z\rangle \\
&\equiv \langle x, {}^y z\rangle^{-1} \langle x, [y,z]\rangle .
\end{aligned}$$

Thus continuing our computation from point ★ above

$$\langle {}^{x}y, {}^{x}z\rangle \equiv \langle x, {}^{y}z\rangle \langle x, {}^{y}z\rangle^{-1} \langle x,[y,z]\rangle \langle y,z\rangle$$

or $\langle {}^{x}y, {}^{x}z\rangle \equiv \langle x,[y,z]\rangle \langle y,z\rangle$ which is (3^*).

III. Let G be a <u>perfect</u> group. Then we know $\tilde{H}_2 G = H_2 G$ and hence relation (4) $\langle x,x\rangle$ is a consequence of (1)–(3) [and hence of (1)–(2) in view of remark II above]. In fact, the relations (2) and (2') suffice to define the universal cover $(\tilde{G}, \tilde{\omega})$ of G. This is easy to see from universal properties as the extension of G defined using (2) and (2') is already known to be central [see equation (6), p. 7, of the original note].

More precisely, define for <u>any</u> group G, $G\otimes G$ to be the free group on the pairs $G\times G$ modulo the relations (2) and (2'). If G is abelian, this really is $G\otimes G$! In general, $G\otimes G$ is not abelian, of course. Denote the images of the pairs $\langle x,y\rangle$ in $G\otimes G$ by $x\otimes y$. For $x,y\in G$, let $\{x,y\} = x\otimes y \cdot y\otimes x \in G\otimes G$. This is a central element of $G\otimes G$ by equation (6).

Thus $\{x,y\} = \{y,x\}$ (conjugate $\{x,y\}$ by $x\otimes y$).

Next
$$\begin{aligned}
\{x,yz\} &= x\otimes yz \cdot yz\otimes x \\
&= x\otimes y \cdot {}^{y}(x\otimes z) \cdot {}^{y}(z\otimes x) \cdot y\otimes x \\
&= x\otimes y \ {}^{y}\{x,z\} \ y\otimes x \\
&= \{x,y\} \ {}^{y}\{x,z\} \qquad \text{(by centrality).}
\end{aligned}$$

Similarly $\{xy,z\} = {}^{x}\{y,z\}\{x,z\}$.

Also
$$\begin{aligned}
\{x,[y,z]\} &= x\otimes [y,z] \cdot [y,z]\otimes x \\
&= {}^{x}(y\otimes z)\cdot (y\otimes z)^{-1} \cdot (z\otimes y)^{-1} \cdot {}^{x}(z\otimes y) \\
&= {}^{x}(y\otimes z) \ \{z,y\}^{-1} \ {}^{x}(z\otimes y)
\end{aligned}$$

= [illegible]

If $x^2 = 1$, then $\{x,x\} = x\otimes x \cdot x\otimes x$

$$= {}^{x}(x\otimes x)\cdot x\otimes x$$
$$= x^2\otimes x$$
$$= 1\otimes x = 1.$$

Via (6) we have ${}^{z}\{x,y\} = \{x,y\}$ if $z\in[G,G]$.
Thus these elements $\{x,y\}$ satisfy

M0. They commute.
M1. $\{x,y\} = \{y,x\}$
M2. $\{xy,z\} = {}^{x}\{y,z\}\{x,z\}$ M2'. $\{x,yz\} = \{x,y\}{}^{y}\{x,z\}$
M3. ${}^{z}\{x,y\} = \{x,y\}$ if $z\in[G,G]$
M4. $\{x,x\} = 1$ if $x^2 = 1$
M5. $\{x,[y,z]\} = {}^{x}\{y,z\}\{y,z\}^{-1}$ (follows from M1 + M2)

Let $L(G)$ denote the group with this presentation. There is then an exact sequence

$$L(G)\longrightarrow G\otimes G\longrightarrow (G,G)\longrightarrow 1.$$

Thus if G is perfect, by M3, M2, M2', $\{\ ,\ \}$ is bilinear in each variable. But $L(G)$ is abelian and G is perfect, hence $L(G)=1$.
If $[G,G]$ is perfect, then arguments similar to those in R.C. Alperin and R.K. Dennis, K_2 of quaternion algebras, preprint, Corollary 1.3, p.3, show that $\{\ ,\ \}$ is bilinear.
If G is abelian, then (modulo errors) the sequence

$$1\to L(G)\to G\otimes G\to (G,G)\to 1$$

is exact. Is this true for arbitrary G?