Groupoids and relations among Reidemeister and among Nielsen numbers

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Abstract

In this work we simplify, generalize and extend results and methods concerning the relationships between various Reidemeister numbers, and applications thereof to four different Nielsen theories (fixed point, ordinary coincidence, semi-index coincidence and root theory). We do this in two distinct contexts. The first context deals with the relationship between the Nielsen numbers of the maps involved, and those of representative lifts to regular covering spaces. We have a special interest in homogeneous spaces. The second context of our applications, namely to Nielsen theories of fibre-preserving maps, is rather curiously dual to the first. In both contexts, our results improve on those previously given.

Our main tool is a collection of 8 term exact sequences (of groups and sets) whose inspiration and proof comes from the theory of fibrations of groupoids. We give a complete analysis of our sequences which yields previously unknown upper and lower bounds on the Reidemeister and Nielsen numbers we are wanting to compute (in one case it sharpens a previously known lower bound). When the upper and lower bounds coincide, generalizations of familiar formulas are forthcoming. The process also gives a uniform approach to proofs in both the underlying algebra (Reidemeister considerations) and to the two distinct contexts of our applications to the four Nielsen theories.

New results include a new formula generalization of the averaging formula in both the algebra and the geometry. We also generalize the original coincidence averaging formula for oriented infra-nilmanifolds to the smooth non-orientable category, and also to a pair of self maps of smooth infra-solvmanifolds of type R. Other generalizations concern the finiteness of Reidemeister numbers. Finally we fill in proofs and details missing from previous work.

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1 Introduction

We start with the algebra. Consider the following diagram of groups and homomorphisms

$$\begin{array}{ccccccc} 1\to & H_1 & \xrightarrow{i_1} & G_1 & \xrightarrow{p_1} & \bar{G}_1 & \to 1 \\ & f'\downarrow\downarrow g' & & f\downarrow\downarrow g & & \bar{f}\downarrow\downarrow \bar{g} & \\ 1\to & H_2 & \xrightarrow{i_2} & G_2 & \xrightarrow{p_2} & \bar{G}_2 & \to 1, \end{array} \tag{1}$$

in which the top and bottom rows are exact. Thus f' and g' are the restrictions of f and g respectively and $\bar{f}$ and $\bar{g}$ are the corresponding induced homomorphisms.

In [26] it was stated (without proof) that Diagram (1) gives rise to the 8 term exact sequence

$$1 \to \mathrm{Coin}(f',g') \to \mathrm{Coin}(f,g) \xrightarrow{\hat{p}_1} \mathrm{Coin}(\bar{f},\bar{g}) \xrightarrow{\delta} \mathcal{R}(f',g') \xrightarrow{\hat{i}_{2*}} \mathcal{R}(f,g) \xrightarrow{\hat{p}_{2*}} \mathcal{R}(\bar{f},\bar{g}) \to 1 \tag{2}$$

The first 4 terms are groups and homomorphisms, and the rest sets and functions. Here for example $\mathrm{Coin}(f,g)=\{x\in G_1 | f(x)=g(x)\}$ (we use the notation $Fix\ f$ when $g=1$), and $\mathcal{R}(f,g)$

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($\mathcal{R}(f)$ when $g = 1$) is the set of Reidemeister classes of f and g. The statement and sketch proof of a non-technical fixed point version was given earlier in [25].

For our purposes here, we need (and give) a more subtle version of these sequences. We say sequences (plural) because when G_2 in Diagram (1) is not Abelian, it gives rise to a whole collection of exact sequences that arise out of conjugation of the f', f and $\bar{f}$ (see Theorem 2.4 and Example 2.7). The more subtle version allows for a deeper analysis which gives rise to new results including upper and lower bounds which we outline first in the algebra. These bounds then carry over to both contexts of our applications.

In fact there are essentially two formulations of the upper and lower bounds even in the algebra, depending on whether $\bar{G}_1$ and $\bar{G}_2$ are finite or not (equations (3) and (4) below). In the case they are not finite, we can use the weaker condition that the $[Coin(\tau_{\bar{\alpha}}\bar{f}, \bar{g}); p_{1*}(Coin(\tau_\alpha f, g))]$ (denoted $[C_{\bar{\alpha}} : \hat{p}_1^\alpha(C_\alpha)]$ below) are bounded. Under these conditions we have the inequalities

$$\frac{\sum_{\bar{\alpha}\in\mathcal{R}(\bar{f},\bar{g})} R(\tau_\alpha f', g')}{Min_\alpha([C_{\bar{\alpha}} : \hat{p}_1^\alpha(C_\alpha)])} \geq R(f, g) \geq \frac{\sum_{\bar{\alpha}\in\mathcal{R}(\bar{f},\bar{g})} R(\tau_\alpha f', g')}{Max_\alpha([C_{\bar{\alpha}} : \hat{p}_1^\alpha(C_\alpha)])} \tag{3}$$

(the τ_α represent conjugation by $\alpha \in G_2$). Actually we will also need a second, slightly different version of (3) (see Theorem 2.12).

When $\bar{G}_1$ and $\bar{G}_2$ are finite the second formulation of our upper and lower bounds generalize the so called averaging formula. In this formulation we have:–

$$\frac{Max_{\beta\in G_1}(|\hat{p}_1(C_\beta)|)}{|\bar{G}_1|} \sum_{\alpha\in\Xi} R(\tau_\alpha f', g') \geq R(f, g) \geq \frac{Min_{\beta\in G_1}(|\hat{p}_1(C_\beta)|)}{|\bar{G}_1|} \sum_{\alpha\in\Xi} R(\tau_\alpha f', g'), \tag{4}$$

where $\hat{p}_1^\alpha(C_\alpha) := p_{1*}(Coin(\tau_\alpha f, g))$. Since $Min_{\beta\in G_1}(|\hat{p}_1(C_\beta)|) \geq 1$, the right hand bound gives a sharper lower bound than that given in the Reidemeister averaging formula (see Remark 3.18), which is recovered when $Max_{\beta\in G_1}(|\hat{p}_1(C_\beta)|) = Min_{\beta\in G_1}(|\hat{p}_1(C_\beta)|) = 1$ (Theorem 2.16(b)).

In fact when, in our various formulations, the upper and lower bounds coincide, the inequalities give rise to a number of new and familiar Reidemeister formulas (see Corollary 2.14). All this translates into analogous results in both (i.e in the dual) contexts of our applications in each of four Nielsen theories (fixed point, ordinary coincidence, semi-index coincidence and root theory - see Theorem 3.11, Corollary 3.12 and Theorems 3.15, 3.30 and 3.32).

We describe first the, perhaps more familiar, fibre space context of our applications, in which we extend results given in [26]. In this context the corresponding horizontal terms in Diagram (1), are truncated homotopy ladders of fibre preserving maps. That is fibre preserving maps f and g of fibrations p_1 and p_2 (left hand diagram below) give rise to a diagram of the form of (1) (right hand diagram)

$$\begin{array}{ccccccccc} E_1 & \xrightarrow{p_1} & B_1 & & Ker\ p_{1*} & \longrightarrow & \pi_1(E_1) & \xrightarrow{p_{1*}} & \pi_1(B_1) & \to 1 \\ f \downarrow\downarrow g & & \bar{f} \downarrow\downarrow \bar{g} & & \downarrow\downarrow & & f_* \downarrow\downarrow g_* & & \bar{f}* \downarrow\downarrow \bar{g}_* & \\ E_2 & \xrightarrow{p_2} & B_2, & & Ker\ p_{2*} & \longrightarrow & \pi_1(E_2) & \xrightarrow{p_{2*}} & \pi_1(B_2) & \to 1. \end{array} \tag{5}$$

Specifying basepoints allows us to identify the Kernels in Diagram (5). Let $b \in \Phi(\bar{f}, \bar{g})$, we use the symbols f_b, g_b to denote the restriction of f and g to the fibre of p_1 over b. Thus $Ker\ p_{1*} = \pi_1(F_b, x)/K_1$ where $K_1 = Ker\ i_{1*} : \pi_1(F_b, x) \to \pi_1(E, x)$, for $x \in \Phi(f, g)$ with $p_1(x) = b$. Similarly $Ker\ p_{2*} = \pi_1(F_{\bar{f}(b)}, f(x))/K_2$. The right hand side of (5) now gives rise to a sequence of the form of (2). Under appropriate conditions, we can then form the subsequence below (Lemma 3.26) by using the the usual embeddings of Nielsen into Reidemeister classes

$$1 \to \mathrm{Coin}_K(f_{b*}, g_{b*}) \to \mathrm{Coin}(f_*, g_*) \overset{\hat{p}_{1*}}{\to} \mathrm{Coin}(\bar{f}, \bar{g}) \overset{\delta}{\to} \mathcal{E}_K(f_b, g_b) \overset{j_*}{\to} \mathcal{E}(f, g) \overset{p_{\mathcal{E}}}{\to} \mathcal{E}(\bar{f}, \bar{g}). \tag{6}$$

Here $\mathcal{E}(f, g)$ is the set of essential Nielsen classes of f and g, and $\mathcal{E}_K(f_b, g_b)$ the essential mod K (for kernel) Nielsen classes (see section 3.3 for full details). Under certain other conditions these subsequences are exact. When this happens the new algebraic results and proofs can be mimicked to give new Nielsen theory results. In particular under suitable conditions we have

$$\frac{\sum_{b\in\mathcal{E}_\chi} N_K^{\mathcal{I}}(f_b, g_b)}{Min_{x\in\mathcal{E}\Theta_\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])} \geq N^{\mathcal{I}}(f, g) \geq \frac{\sum_{b\in\mathcal{E}_\chi} N_K^{\mathcal{I}}(f_b, g_b)}{Max_{x\in\mathcal{E}\Theta_\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])},$$

where the superscript $\mathcal{I}$ refers to one of four Nielsen theories, and K refers to the mod K version of these numbers (see Theorem 3.32 for more details). Here too it is convenient to have two versions of the inequalities in order to generalize known formulas for fibre preserving maps which then fall out when the upper and lower bounds coincide (Theorems 3.30 and 3.32).

We call the other context of our applications the covering space context. As we see below it is dual to that of the fibre space context. In this context (which comes first in our exposition) we have Nielsen analogues of both of the inequalities in (3) and (4).

The reader may be finding the need for two contexts a little puzzling. After all covering spaces are themselves fibrations. The point though, is that taking p_1 and p_2 in Diagram (1) to be induced by covering projections does not work. For starters the Kernels in (5) would be trivial since the fibres of covering projections are discrete. In addition neither p_1 nor p_2 would be surjective, so the corresponding horizontal sequences in Diagram (1) would not be exact. If, on the other hand, we let i_1 and i_2 be the homomorphisms induced by our covering projections $q_1 : \tilde{X}_1 \to X_1$ and $q_2 : \tilde{X}_2 \to X_2$ respectively, then by taking Cokernels we have a diagram of the form of (1) that is dual to the diagram in (5)

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{q_1} & X_1 \\ \alpha\tilde{f} \downarrow\downarrow \tilde{g} & & f \downarrow\downarrow g \\ \tilde{X}_2 & \xrightarrow{q_2} & X_2 \end{array} \qquad \begin{array}{ccccccc} 1 \to & \pi_1(\tilde{X}_1) & \xrightarrow{q_{1*}} & \pi_1(X_1) & \longrightarrow & Coker\ q_{1*} \\ & \alpha\tilde{f}_* \downarrow\downarrow \tilde{g}_* & & f_* \downarrow\downarrow g_* & & \bar{f}_* \downarrow\downarrow \bar{g}_* \\ 1 \to & \pi_1(\tilde{X}_2) & \xrightarrow{q_{2*}} & \pi_1(X_2). & \longrightarrow & Coker\ q_{2*}. \end{array} \tag{7}$$

Here $\tilde{f}$ and $\tilde{g}$ are chosen lifts of maps f and g respectively, and $\alpha \in \pi_1(X_2)$. Under appropriate conditions (see Proposition 3.4), we then have the following not quite complete dual

$$1 \to \mathrm{Coin}(\alpha\tilde{f}_*, \tilde{g}_*) \to \mathrm{Coin}(f_*, g_*) \stackrel{\hat{p}_{1*}}{\to} \mathrm{Coin}(\bar{f}_*, \bar{g}_*) \stackrel{\delta}{\to} \mathcal{E}(\alpha\tilde{f}, \tilde{g}) \stackrel{\mathcal{E}_{q_1}}{\to} \mathcal{E}(f, g). \tag{8}$$

of subsequences (6) (sequence (6) has 7 terms, sequence (8) has only 6). Under conditions that imply exactness, we can again mimic the algebra to produce generalizations and extensions of Nielsen theory results in this covering space context. As we said, this time we have results that are analogous to both types of the inequalities in (3) and (4). In the first case, in the context of Diagram (7), we have (also new) the analogue of a slightly different (the position of the $\sum$), but usefull version of (3) (see Theorem 3.11 for comparisons, notation and details).

$$\sum_{\alpha_k \in \tilde{\mathcal{X}}} \frac{N^{\mathcal{I}}(\alpha_k\tilde{f}, \tilde{g})}{Min_{\theta \in \mathcal{E}\Theta\tilde{\alpha}_k}([C_{\tilde{\alpha}_k} : \hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])} \geq N^{\mathcal{I}}(f, g) \geq \sum_{\alpha_k \in \tilde{\mathcal{X}}} \frac{N^{\mathcal{I}}(\alpha_k\tilde{f}, \tilde{g})}{Max_{\theta \in \mathcal{E}\Theta\tilde{\alpha}_k}([C_{\tilde{\alpha}_k} : \hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])} \tag{9}$$

Before we discuss the analogue of (4) we indicate how (9) allows us to extend and generalize results in the literature. At the time [26] was being prepared, Jezierski was working on his paper "Nielsen number of a covering map" ([33]). The basic goal was to give conditions under which the Nielsen number $N(f)$ of a self map f can be written as a linear combination of the Nielsen numbers of various lifts $\alpha\tilde{f}$, to a finite index covering space ($q_1 = q_2$ and $g = 1$ in the left hand diagram of (7) above). This was later generalized to coincidences in a non-trivial way by Moh'D ([43, 44]). Though Jezierski and Moh'D used neither the sequences nor our formulation of their results, they did in fact prove them under conditions that imply the upper and lower bounds in (9) coincide (Theorem 3.11). Our result generalizes those just mentioned, freeing them from the constraint that $\bar{G}_1$ and $\bar{G}_2$ are finite (Corollary 3.12). This generalization then allows us to give an application which compares ordinary and Mod H Nielsen numbers. Of particular interest is the specific application to Mod K Nielsen numbers of the fibres of fibre-preserving maps, which in fact conncts the two applications of the sequences that we give here (Proposition 3.22).

In this same covering space context, in addition to (9) and its variant, we also have a Nielsen analogue of the inequalities in (4). We come at this through the historical development. Though the full sequences (2) are unique to our work, portions of them have appeared in the literature. In particular after the fixed point version of our sequences appeared in [25], but both before and after [26], the following part

$$\mathcal{R}(f', g') \to \mathcal{R}(f, g) \to \mathcal{R}(\bar{f}, \bar{g}) \to 1. \tag{10}$$

of sequence (2) appeared on its own, first in the fixed point case, then later in coincidence versions (see [13, 15, 16, 17, 48] and [8, 12, 19]). In all cases (10) was used to study the relationship between $R(f,g)$, $R(\bar{f},\bar{g})$ and the $R(\tau_\alpha f',g')$. At times, usually under very strong conditions (which we weaken), it was used to produce formulas, or to investigate when $\mathcal{R}(f,g)$ is finite (we also generalize this, see Corollary 2.15). Some of the applications of (10) given in the works cited above fit well into fibre space context, others do not. This was noted in [26], and came with the cryptic comment “One could, I suppose, debate whether these are fibre techniques or not” ([26, p.544]). Behind this comment was the thought that all Nielsen formula could somehow or other be deduced from the collection of conjugates of (2). As we shall see these are in fact the exact sequences associated with a fibration of groupoids (section 4). What I did not see clearly at the time, was the need for the separate (dual) covering space context of our applications. This came firstly with Jezierski's work in [33] and then later when the series [38, 39, 22] caught my attention with the presentation of [22] at the Nielsen theory conference in China in 2011.

But sequence (10) was not the only part of sequence (2) that appeared in the literature. About the same time that [26] appeared, the fixed point version of the following conjugates of the first part of sequence (2) also appeared ([38])

$$1 \to \mathrm{Coin}(\tau_\alpha f',g') \xrightarrow{\hat{i}_1} \mathrm{Coin}(\tau_\alpha f,g) \xrightarrow{\hat{p}_1} \mathrm{Coin}(\tau_{\bar{\alpha}}\bar{f},\bar{g}), \tag{11}$$

under the condition that $\bar{G}_1$ and $\bar{G}_2$ are finite. Coincidence versions followed later in [39, 22] under the same finiteness condition. However, the crucial Bockstein type boundary function connecting (10) and (11) was absent (see Remark 4.7). The main goal in [38] and [39, 22] was to give fixed and coincidence versions respectively, of the averaging formula. The Nielsen coincidence version of which is represented by equality in the equation

$$N(f,g) \geq \frac{1}{|\bar{G}_1|}\sum_{[\bar{\alpha}]\in\bar{G}_2} N(\alpha\tilde{f},\tilde{g}). \tag{12}$$

A Reidemeister version of (12) also appeared in [22], and this Reidemeister version is generalized here in equation (4) by giving an upper bound and sharpening the given lower one. With many details, the authors of [38] (fixed points) and [39] (coincidences) used the two partial sequences (10) and (11) to exhibit the inequality (12). In addition, equality was shown to hold if and only if $\hat{p}_1(Fix(\tau_\alpha f)) \subseteq H_1$ respectively $\hat{p}_1(Coin(\tau_\alpha f,g)) \subseteq H_1$ for all α. This condition was seen to be automatic for maps f and g of infra-nilmanifolds. It was then used to deduce that the fixed point averaging formula always holds for self maps of infra-nilmanifolds, and that the coincidence version holds for oriented infra-nilmanifolds X_1 and X_2 of the same dimension.

We generalize and extend all of this in the covering space context of our applications in several ways. Firstly we give conditions under which the inequalities

$$\frac{Max_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{|\bar{G}_1|}\sum_{[\bar{\alpha}]\in\bar{G}_2} N^{\mathcal{I}}(\alpha\tilde{f},\tilde{g}) \geq N^{\mathcal{I}}(f,g) \geq \frac{Min_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{|\bar{G}_1|}\sum_{\alpha\in[\bar{\alpha}]\in\bar{G}_2} N^{\mathcal{I}}(\alpha\tilde{f},\tilde{g})$$

hold for our four distinct Nielsen numbers $N^{\mathcal{I}}(f,g)$. Secondly, this not only adds an upper bound and sharpens the lower bound given in (12), but also, under conditions under which the upper and lower bound coincide gives rise to a new formula generalizing the averaging formula itself (Corollary 3.16). Thirdly, using the work of Vendrúscolo ([45]), we generalize the averaging formulas of [41, Theorem 4.2] and [39, Theorem 4.9] to smooth non-orientable infra-nilmanifolds, as well as generalizing the fixed point version in [41], to a coincidence version for pairs of smooth self maps of a smooth infra-solvmanifold of type R (Corollary 3.17 includes both of these results).

There seems to me to be beauty and value in a number of things in our presentation that are worth pointing out. Firstly in addition to giving many new results, the sequences allow for an expression of the them that make both results and proofs almost obvious. For example in light of the sequences, it is clear where the coefficients $[Coin(\bar{\alpha}_k\bar{f}_*,\bar{g}_*) : \hat{j}_1^{\theta\alpha_k}(Coin(\theta\alpha_k\tilde{f}_*,\tilde{g}_*)]$ come from in our simultaneous formulation (Corollary 3.12) of the work of Jezierski and Moh'D in [33, 43, 44]. With our formulation the truth of the statements then seem almost obvious (but see the caution below). In addition the sequences point to a way to compute these coefficients in a different way. Secondly once the Reidemeister theory has been set up, the multitude of details of the analagous topological results are often reduced to two or three lines (see for example 2.13, 3.11, 3.12, 3.15, 3.16, 3.32). Thirdly it seems to me that there is value in bringing a large number of results and proofs together in a way that unites both results and methodology. In particular in both the algebra and the geometry all results are deduced from appropriate sequences, the analogous bounds are shown to exist, and formulas deduced when the upper and lower bounds coincide. In this way the algebraic and geometric proofs are essentially identical. Fourthly we see value in reproducing important results by a different method, especially when such results fall out easily from the methodology (see Corollary 3.35 for example). Finally the details we give here, though elementary in the sense that they are mostly set theoretic, are at times subtle and so worth including. It would be easy for example, to think that the equation below follows from the exactness of sequences (2) alone

$$\#(Im\ \delta) = [\mathrm{Coin}(\bar{f}, \bar{g}) : \hat{p}_1(\mathrm{Coin}(f, g)]. \tag{13}$$

This is not the case as the following example shows.

Example 1.1. Consider the exact sequence of groups and homomorphisms (first three terms)

$$0 \to 3\mathbb{Z} \hookrightarrow \mathbb{Z} \xrightarrow{\delta} \{a, b\} \to 1,$$

and base point preserving functions (remaining terms), where δ takes $3\mathbb{Z}$ to the point a (the base point of $\{a, b\}$), and everything else to b. Note that $\#(Im\ \delta) = 2$, while $[\mathbb{Z} : 3\mathbb{Z}] = 3$.

As indicated above, equation (13) does indeed hold true. In order to give insight into what is happening here, we reveal the intuition that lies behind this paper (and in fact behind [25] and [26]). In particular with the right interpretation (see section 4), our sequences are the exact sequences associated with a fibration of groupoids. Such sequences are entirely analogous to the bottom part of the long exact homotopy sequence of a topological fibration $F \to E \xrightarrow{p} B$. In this context, the analogous equality $\#(Im\ \delta) = [\pi_1(B) : p_*(\pi_1(E))]$ holds, where $\delta : \pi_1(B) \to \pi_0(F)$ is the usual boundary. As here, this is not deduced from exactness alone, but rather from the properties of lifting functions associated with the fibration (see [10]). Equation (13) holds true for similar reasons, and this, together with the Bockstein type boundary, turn out to be key in terms of what allows for the deeper and simpler analysis and proofs we give here.

Although our primary focus is not to produce a survey of results, this comes as a byproduct of our considerations in the covering space context of our applications. This is not true of the applications in the fibre space context, and we refer the reader to [26] for those looking for a more complete set of applications of this context. In the algebra and both contexts of our applications, new results include the existence of upper and lower bounds on the Nielsen or Reidemeister numbers we are seeking to compute. In the fibre space context we also include a number of proofs of results stated, but not proved, in [26] (i.e. Theorem 3.34). The work also gives a simultaneous proof of a number of results in the literature (Corollary 3.33 includes all the naïeve Nielsen addition (product) formulas of [47, 31, 32, 30] and [26, Corollary 11.4]). We also give a fibre space proof (Corollary 3.35) of a Theorem of Dobreńko and Jezierski [9, Theorem 2.5]. The proof in [9] used a prototype of the averaging formula. The technical versions of our sequences in both the fixed point ([25]) and coincidence version ([26]) are also new. The more technical versions are necessary for the deeper analysis given here. We include a complete and rigorous proof of our technical version Theorem 2.4 of our sequences. The original fixed point case contained only a sketch of the non-technical version ([25]), and the non-technical coincidence version of Theorem 2.4 was stated in [26] without proof.

The paper is divided as follows. Section two, following this introduction, is devoted to the algebraic side of our considerations. The proof of the main Theorem (2.4) is however, delayed until section 4. Section three is divided into three subsections, starting with a brief review of index and semi index considerations. The next two subsections give firstly the covering space context, then secondly the fibre space context of our applications of the algebraic section. The fourth section gives the delayed proof of Theorem 2.4.

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2 The Reidemeister sequences and the analysis thereof.

In this section, which is divided up into three subsections, we deal exclusively with the algebraic side of our story. In the first subsection we briefly review Reidemeister theory, introduce our notation and give other relevant material. In the second subsection we state the main Theorem giving our 8 term sequences together with the crucial extra information. We also look at the special case when G_2 is Abelian, where we are able to exhibit a formula relating $R(f,g)$, $R(f',g')$ and $R(\bar{f},\bar{g})$. This gives us a kind of model or pattern to aim for when G_2 is not Abelian. In such cases there is often a lack of uniformity in the cardinality of pre-images of various functions, and we need multiple "conjugate" sequences in order to study the appropriate pre-images individually. The third subsection then gives our complete analysis of the general case including the promised upper and lower bounds.

2.1 Notation and preliminaries (algebraic side)

Let $f,g : A \to B$ be group homomorphisms, the Reidemeister set $\mathcal{R}(f,g)$, is the quotient set of B defined by the relation $\alpha \sim \beta$ if and only if there is $\gamma \in A$ such that $\alpha = g(\gamma^{-1})\beta f(\gamma)$. The Reidemeister number $R(f,g)$ of f and g, is cardinality $\#(\mathcal{R}(f,g))$ of $\mathcal{R}(f,g)$. The algebraic coincidence set $Coin(f,g)$ is the subgroup $Coin(f,g) =: \{\alpha \in A | f(\alpha) = g(\alpha)\} \subseteq A$. When $g = 1$ the identity we will sometimes write $\mathcal{R}(f,g)$ as $\mathcal{R}(f)$, and $Coin(f,g)$ as $Fix\ f$. When $g = *$ the zero homomorphism, we may write $RR(f)$ in place of $\mathcal{R}(f,*)$, and of course $Ker\ f$ for $Coin(f,*)$.

Proposition 2.1. *(c.f. [25, 21]) There is an exact sequence of sets and functions*

$$1 \to Coin(f,g) \xrightarrow{i} A \xrightarrow{f\cdot^{-1}g} B \xrightarrow{j} \mathcal{R}(f,g) \to 1,$$

where i is the inclusion, $f \cdot^{-1} g$ the function that takes α to $g(\alpha^{-1})f(\alpha)$, and j places an element β into its Reidemeister class $[\beta] \in \mathcal{R}(f,g)$. If $(B,+)$ is Abelian, there is a canonical Abelian group structure $+$ on $\mathcal{R}(f,g)$ with $[\alpha]+[\beta] = [\alpha+\beta]$, and $(f \cdot^{-1} g)(\alpha) = f(\alpha) - g(\alpha)$.

Furthermore a commutative diagram of groups and homomorphisms of the form

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ f_1 \downarrow\downarrow g_1 & & f_2 \downarrow\downarrow g_2 \\ B & \xrightarrow{q} & D, \end{array}$$

induces a morphism of exact sequences (of groups if C and D are Abelian) in the obvious way. $\square$

Remark 2.2. Our results incorporate coincidence, fixed point and root theory. If $A = B$ and $g = 1$ is the identity then we have classical Reidemeister fixed point theory with $(f \cdot^{-1} g)(\alpha) = \alpha^{-1}f(\alpha)$, $Fix\ f := \{\alpha | f(\alpha) = \alpha\} = Coin(f,1)$ and $R(f) := R(f,1)$. If $g = *$ is the trivial homomorphism, we have classical Reidemeister root theory with $(f \cdot^{-1} g)(\alpha) = f(\alpha)$, $Coin(f,*) = Ker\ f$ and $RR(f) := R(f,*) = [B : f(A)]$, where $RR(f)$ is the Reidemeister root number. Furthermore $Coin(f,g) = Ker(f-g)$, and if B is Abelian $\mathcal{R}(f,g) \cong Coker(f-g)$.

These formulations easily give the following Reidemeister analogy/generalization to coincidences and roots of the well known Theorem from [2].

Corollary 2.3. *Let $f,g :\ \mathbb{Z}^n \to \mathbb{Z}^n$ be homomorphisms with linearizations[1] F and G respectively. If $det(F-G) = 0$, then $R(f,g) = \infty$. If $det(F-G) \neq 0$ then $R(f,g) = |det(F-G)|$, $Coin(f,g)$ $= 1$, $R(f) = |det(F-I)|$ (where I is the identity matrix), and $RR(f) = |det(F)|$.*

[1] that is a matrix representation of f and g with respect to a fixed basis for $\mathbb{Z}^n$ (see also ([37]).

Proof. By Proposition 2.1, $R(f,g) = \#(Coker(F-G))$, and this is ∞ if $det(F-G) = 0$. If $det(F-G) \neq 0$, then $F-G$ is injective and $Ker(f-g) = Coin(f,g) = 1$. For $n = 1$ it is easy to see that $R(f,g) = |det(F-G)|$. For $n > 1$ (as in the proof of the fixed point case in [2]) there are unimodular matrices A and B such that $D := A(F-G)B$ is diagonal. So then the order of $Coker(F-G)$ is the same as the product of 1 dimensional cases, each of order the absolute value of one of the diagonal entries of D. $\square$

2.2 The sequences and product formulas

When the groups in Diagram (1) are not Abelian, we need the following well known "conjugate" version of Diagram (1):-

$$\begin{array}{ccccccc} 1 \to & H_1 & \xrightarrow{i_1^\alpha} & G_1 & \xrightarrow{p_1^\alpha} & \bar{G}_1 & \to 1 \\ & \tau_\alpha f' \downarrow\downarrow g' & & \tau_\alpha f \downarrow\downarrow g & & \tau_{\bar{\alpha}} \bar{f} \downarrow\downarrow \bar{g} & \\ 1 \to & H_2 & \xrightarrow{i_2^\alpha} & G_2 & \xrightarrow{p_2^\alpha} & \bar{G}_2 & \to 1, \end{array} \tag{14}$$

where for example $\tau_h(u) = huh^{-1}$ for any $u, h \in G$.

The normality of H_2 allows the composite $\tau_\alpha f i_1$ to factor through H_2 (denoted by $\tau_\alpha f'$). We also use $\bar{\alpha}$ for $p_2(\alpha)$. It is useful to incorporate redundancies into our notation. In particular the *homomorphisms* i_t^α and p_t^α are simply i_t and p_t respectively for $t = 1, 2$.

A simplified version of the fixed point case of our main Theorem, which we now give, was sketched in [25, Theorem 1.8]. A simplified version of the coincidence case was stated but not proved in the survey article [26, Theorem 9.18].

Theorem 2.4. *(c.f. [25, Theorem 1.8], [26, Theorem 9.18]) For each $\alpha \in G_2$, the following sequence of groups and homomorphisms (first four terms) and based sets and base point preserving functions (next 4 terms) is exact*

$$1 \to Coin(\tau_\alpha f', g') \stackrel{\hat{i}_1^\alpha}{\to} Coin(\tau_\alpha f, g) \stackrel{\hat{p}_1^\alpha}{\to} Coin(\tau_{\bar{\alpha}} \bar{f}, \bar{g}) \stackrel{\delta^\alpha}{\to} \mathcal{R}(\tau_\alpha f', g') \stackrel{\hat{i}_2^\alpha}{\to} \mathcal{R}(\tau_\alpha f, g) \stackrel{\hat{p}_2^\alpha}{\to} \mathcal{R}(\tau_{\bar{\alpha}} \bar{f}, \bar{g}) \to 1,$$

where $\delta^\alpha(\bar{\beta}) = [g(\beta^{-1})\tau_\alpha f(\beta)]$, for any $\beta \in G_1$ with $p_1(\beta) = \bar{\beta}$. Furthermore if $\bar{\beta}, \bar{\theta} \in Coin(\tau_{\bar{\alpha}} \bar{f}, \bar{g})$ then $\delta^\alpha(\bar{\beta}) = \delta^\alpha(\bar{\theta})$ iff there is a $\gamma \in Coin(\tau_\alpha f, g)$ with $\hat{p}_1(\gamma) = \bar{\beta}^{-1}\bar{\theta}$. In particular $\#(Im\ \delta^\alpha) = [Coin(\tau_{\bar{\alpha}} \bar{f}, \bar{g}); p_{1*}^\alpha(Coin(\tau_\alpha f, g))]$. The cardinality $R(\tau_\alpha f', g')$ of $\mathcal{R}(\tau_\alpha f', g')$ is independent of $\alpha \in (p_2^\alpha)^{-1}([\bar{\alpha}]) \subset G_2$[2].

Finally, if G_2 is Abelian, then there are canonical group structures on the three Reidemeister sets, and the whole sequence becomes an exact sequence of groups and homomorphisms

An immediate Corollary which is useful in examples is the following:-

Corollary 2.5. *If both $Coin(\tau_\alpha f', g')$ and $Coin(\tau_{\bar{\alpha}} \bar{f}, \bar{g})$ are trivial, so also is $Coin(\tau_\alpha f, g)$. Moreover the formula*

$$[Coin(\bar{f}, \bar{g}); \hat{p}_1(Coin(f,g))] R(f,g) = R(f', g') R(\bar{f}, \bar{g})$$

holds if (a) G_2 Abelian, or (b) $G_1 = G_2$, $g = 1$ and $f^n(G_1)$ is commutative for some $n \in \mathbb{N}$. Under these same conditions $R(f,g) = R(f', g') R(\bar{f}, \bar{g})$ iff $[Coin(\bar{f}, \bar{g}); \hat{p}_1(Coin(f,g))] = 1$. $\square$

Condition (b) is dubbed "eventually commutative" by Jiang in [34]. We will show later that the Root version of this product formula also holds (Corollary 2.14).

2.3 The analysis, bounds and formulas.

In this subsection we give a complete analysis of the situation when G_2 is not abelian. Equation (10) is well known and follows from the surjectivity of $\hat{p}_2$ (i.e. [19] and [22]).

$$\mathcal{R}(f,g) = \bigsqcup_{[\bar{\alpha}] \in \mathcal{R}(\bar{f}, \bar{g})} \hat{p}_2^{-1}[\bar{\alpha}]. \tag{15}$$

[2]It is known that $R(\tau_\alpha f', g')$ is independent of $\alpha \in (p_2)^{-1}(\bar{\alpha})$ (as opposed to $\alpha \in (p_2^\alpha)^{-1}([\bar{\alpha}])$ (see [22] and 2.10), but this is not enough (see proof of Corollary 2.15). A first principles proof is complex (see Remark 4.5).

Our 8 term sequence has allowed us to replace $Ker\ \hat{p}_2^\alpha$ with $Im\ \hat{i}_2^\alpha$ in the next Lemma which otherwise is well known.

Lemma 2.6. *(c.f. [17, 38, 39]) For each $\alpha \in G_2$, the designation $[\theta] \rightsquigarrow [\theta\alpha^{-1}]$ determines a well defined bijection $(\alpha^{-1})^* : \mathcal{R}(f,g) \to \mathcal{R}(\tau_\alpha f, g)$. Thus $R(f,g) = R(\tau_\alpha f, g)$. Furthermore $(\alpha^{-1})^*$ restricts to a bijection $\widehat{\alpha}^{-1} : \hat{p}_2^{-1}([\bar{\alpha}]) \to Ker\ \hat{p}_2^\alpha$, and if $\bar{\alpha} = p_2(\alpha)$, then the Diagram*

$$\begin{array}{ccccc} \hat{p}_2^{-1}([\bar{\alpha}]) & \to & \mathcal{R}(f,g) & \stackrel{\hat{p}_2}{\to} & \mathcal{R}(\bar{f},\bar{g}) \\ (\widehat{\alpha}^{-1})^* \downarrow & & \downarrow (\alpha^{-1})^* & & \downarrow\ (\bar{\alpha}^{-1})^* \\ Im\ \hat{i}_2^\alpha & \to & \mathcal{R}(\tau_\alpha f, g) & \stackrel{\hat{p}_2^\alpha}{\to} & \mathcal{R}(\tau_{\bar{\alpha}}\bar{f},\bar{g}) \end{array}$$

is commutative. Thus $\#(\hat{p}_2^{-1}([\bar{\alpha}])) = \#(Ker\ \hat{p}_2^\alpha) = \#(Im\ \hat{i}_2^\alpha)$ for all $\alpha \in G_1$.

Proof. The function $\alpha^* : \mathcal{R}(\tau_\alpha f, g) \to \mathcal{R}(f,g)$ given by $\alpha^*([\theta]) = [\theta\alpha]$ induces an inverse to each vertical function, which in turn gives rise to an "inverse" commutative diagram. $\square$

The point, as in the various references, is that we can transfer the study of the cardinality of $\hat{p}_2^{-1}([\bar{\alpha}])$ to the study of $\#(Im\ \hat{i}_2^\alpha)$ in the "α" sequence. As we now see, they can be very different.

Example 2.7. Let K^2 denote the Klein bottle thought of as the quotient space of $\mathbf{R}^2$, under the equivalence relation defined by $(s,t) \sim ((-1)^k s, t+k)$ and $(s,t) \sim (s+k, t)$ for any $k \in \mathbf{Z}$. Note that K^2 fibres as $S^1 \hookrightarrow K^2 \stackrel{p}{\to} S^1$ where p is induced by projection on the second factor. The bottom end of the exact homotopy sequence is $1 \to \mathbb{Z} \to \pi_1(K^2) \to \mathbb{Z} \to 1$. The correspondence $(s,t) \to (-s,-t)$ induces a well defined fibre preserving map f on K^2, which in turn induces a self morphism of the above short exact sequence. If we choose different base points $x = \overline{(0,0)}$ and $y = \overline{(0,\frac{1}{2})}$ in K^2 (with corresponding base points in base and fibre), the two sequences in Theorem 2.4 end with $1 \to \mathbb{Z}_2 \to \mathcal{R}(f_*^x) \stackrel{\hat{p}_{2*}}{\to} \mathbb{Z}_2 \to 1,$ and $1 \to \mathbb{Z} \to \mathcal{R}(f_*^y) \stackrel{\hat{p}_{2*}^\alpha}{\to} \mathbb{Z}_2 \to 1$ respectively.

The 8 term sequences allows us to continue our analysis by examining the relationship between the $\mathcal{R}(\tau_\alpha f', g')$ and the $Im(\hat{i}_2^\alpha) = Ker(\hat{p}_2^\alpha)$ (and hence between $\hat{p}_2^{-1}([\bar{\alpha}])$). For each $\alpha \in G_1$, define an equivalence relation on $\mathcal{R}(\tau_\alpha f', g')$ as follows: $[\theta] \sim [\mu]$ if and only if $i_{2*}^\alpha([\theta]) = i_{2*}^\alpha([\mu])$. Denote the set of equivalence classes by $\overline{\mathcal{R}}(\tau_\alpha f', g')$ and its cardinality by $\overline{R}(\tau_\alpha f', g')$.

Definition 2.8. A set $\tilde{\chi} \subseteq G_2$ is said to be a *set of Reidemeister lifts* for Diagram (1), if for each $[\bar{\alpha}] \in \mathcal{R}(\bar{f},\ \bar{g})$, there is exactly one $\alpha \in \tilde{\chi}$ with $[p_2(\alpha)] = [\bar{\alpha}]$.

The first equation in the Lemma below is the Reidemeister analogy of an equation in Nielsen fixed point fibre space theory ([30, Theorem 3.3]).

Lemma 2.9. *Let $\tilde{\chi}$ be a set of Reidemeister lifts for Diagram (1), then*

$$R(f,g) = \sum_{\alpha \in \tilde{\chi}} \overline{R}(\tau_\alpha f', g') \ \textit{and} \ \sum_{\alpha \in \tilde{\chi}} R(\tau_\alpha f', g') \geq R(f,g).$$

Proof. From Theorem 2.4 we have, for purely set theoretic reasons, that $\overline{R}(\tau_\alpha f', g') = \#(Ker\ \hat{p}_2^\alpha)$ $= \#(Im(\hat{i}_2^\alpha))$. The first equation now follows directly from Lemma 2.6, the definitions and equation (15). The inequality also follows, since (clearly) $R(\tau_\alpha f', g') \geq \overline{R}(\tau_\alpha f', g')$ for all α. $\square$

The continuation of sequence (2) to the left by $\delta^\alpha : \mathrm{Coin}(\tau_{\bar{\alpha}}\bar{f}, \bar{g}) \to \mathcal{R}(\tau_\alpha f', g')$ allows us to perform the same sort of analysis on the $\mathcal{R}(\tau_\alpha f', g')$ given earlier in Lemmas 2.6 and 2.9 on $\mathcal{R}(f,g)$. We start, by analogy with equation (15), with the disjoint union

$$\mathcal{R}(\tau_\alpha f', g') = \bigsqcup_{[\gamma] \in Ker(\hat{p}_2^\alpha) = Im(\hat{i}_2^\alpha)} (\hat{i}_2^\alpha)^{-1}([\gamma]). \tag{16}$$

As with the $\hat{p}_2^{-1}[\bar{\alpha}]$, there need be no uniformity among the the cardinalities of the $(\hat{i}_2^\alpha)^{-1}([\gamma])$. A topologically inspired example of this is given in [30, Example 1.3]. Our sequences allow us (in the Lemma below) to express each $(\hat{i}_2^\alpha)^{-1}([\gamma])$ in equation (16) in terms of the image of some δ^β. For $\bar{G}_1$ and $\bar{G}_2$ finite, much of the essence of the Lemma is given in [22], in a multitude of technical details, but of course phrased there in terms of $ker\ \hat{i}_2^\beta$ rather than $Im\ \delta^\beta$.

Lemma 2.10. *Let $\alpha,\ \beta \in G_2$, then $(\alpha\beta^{-1})^* : \mathcal{R}(\tau_\alpha f, g) \to \mathcal{R}(\tau_\beta f, g)$. If $\alpha\beta^{-1} \in H_2$, then of course $(\alpha\beta^{-1})^* : \mathcal{R}(\tau_\alpha f', g') \to \mathcal{R}(\tau_\beta f', g')$ is a bijection which in turn restricts to a bijection $(\hat{i}_2^\alpha)^{-1}([\beta\alpha^{-1}]) \to Im\ \delta^\beta$ shown in the commutative diagram*

$$\begin{array}{ccccc}
(\hat{i}_2^\alpha)^{-1}([\beta\alpha^{-1}]) & \hookrightarrow & \mathcal{R}(\tau_\alpha f', g') & \stackrel{\hat{i}_2^\alpha}{\to} & \mathcal{R}(\tau_\alpha f, g) \\
\downarrow & & \downarrow (\alpha\beta^{-1})^* & & \downarrow (\alpha\beta^{-1})^* \\
Im\ \delta^\beta & \hookrightarrow & \mathcal{R}(\tau_\beta f', g') & \stackrel{\hat{i}_2^\beta}{\to} & \mathcal{R}(\tau_\beta f, g),
\end{array}$$

so that $\#((\hat{i}_2^\alpha)^{-1}([\beta\alpha^{-1}])) = \#(Im\ \delta^\beta) = [Coin(\tau_{\bar\alpha}\bar f, \bar g) : \hat{p}_1^\beta(Coin(\tau_\beta f, g)]).$

Notation: We often use the abbreviations $C_{\bar\alpha}$ for $Coin(\tau_{\bar\alpha}\bar f, \bar g)$ and $\hat{p}_1^\beta(C_\beta)$ for $\hat{p}_1^\beta(Coin(\tau_\beta f, g))$.

Proof. The element $\beta\alpha^{-1}$ induces an inverse to each vertical function. That $ker\ \hat{i}_2^\beta = Im\ \delta^\beta$ and that $\#(Im\ \delta^\beta) = [C_{\bar\alpha} : \hat{p}_1^\beta(C_\beta)]$ comes from Theorem 2.4. □

Let $\tilde\chi$ be a set of Reidemeister lifts and $\alpha \in \tilde\chi$. We choose $\Theta\alpha \subseteq \{\theta \in H_2 | p_2(\theta\alpha) = p_2(\alpha)\}$, one θ for each Reidemeister class in $Ker\ \hat{p}_2^\alpha$. Clearly $\#(\Theta\alpha) = \#(Ker\ \hat{p}_2^\alpha) = \overline{\mathcal{R}}(\tau_\alpha f', g')$. Equation (17) below appears in [22] under the hypothesis that $\bar G_1$ and $\bar G_2$ are finite.

Corollary 2.11. *Let $\tilde\chi$, $\alpha \in \tilde\chi$ and $\Theta\alpha$ be as above, then*

$$R(\tau_\alpha f', g') = \sum_{\theta\in\Theta\alpha} [C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})]. \tag{17}$$

Moreover, if the set of numbers $[C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})]$ is bounded over $\theta \in \Theta\alpha$ then we have that

$$\frac{R(\tau_\alpha f', g')}{Min_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})])} \geq \overline{R}(\tau_\alpha f', g') \geq \frac{R(\tau_\alpha f', g')}{Max_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})])}.$$

Proof. It is clear from the definition of $\Theta\alpha$, that each $[\gamma] \in Ker(\hat{p}_2^\alpha))$ can be written as $[\gamma] = [\theta\alpha]$ for some $\theta \in \Theta\alpha$. Since $\#(\Theta\alpha) = \#(Ker\ \hat{p}_2^\alpha)$, we have from equation (16) that $R(\tau_\alpha f', g') = \sum_{\theta\in\Theta\alpha} \#((\hat{i}_2^\alpha)^{-1}([\theta\alpha])) = \sum_{\theta\in\Theta\alpha} \#(Im\ \delta^{\theta\alpha}) = \sum_{\theta\in\Theta\alpha} [C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})]$, giving the first part. Next since $\#(\Theta\alpha) = \overline{\mathcal{R}}(\tau_\alpha f', g')$, then $R(\tau_\alpha f', g') \geq Min_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})]) \cdot \overline{R}(\tau_\alpha f', g')$, and $R(\tau_\alpha f', g') \leq Max_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})]) \cdot \overline{R}(\tau_\alpha f', g')$. The second part follows. □

Let $\Theta\tilde\chi$ denote the union over $\alpha \in \tilde\chi$ of the sets $\Theta\alpha$ (defined just prior to Corollary 2.11). We call $\Theta\tilde\chi$ a *complete set of representative lifts for Diagram (1).*

Theorem 2.12. *If the $[C_{\bar\beta} : \hat{p}_1^\beta(C_\beta)]$ are bounded over $\beta \in \Theta\tilde\chi$, for some complete set $\Theta\tilde\chi$ of representative lifts for Diagram (1), then*

$$\sum_{\alpha\in\tilde\chi} \frac{R(\tau_\alpha f', g')}{Min_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})])} \geq R(\tau_\alpha f, g) \geq \sum_{\alpha\in\tilde\chi} \frac{R(\tau_\alpha f', g')}{Max_{\theta\in\Theta\alpha}([C_{\bar\alpha} : \hat{p}_1^{\theta\alpha}(C_{\theta\alpha})])}$$

and

$$\frac{\sum_{\alpha\in\tilde\chi} R(\tau_\alpha f', g')}{Min_{\beta\in\Theta\tilde\chi}([C_{\bar\beta} : \hat{p}_1^\beta(C_\beta)])} \geq R(f, g) \geq \frac{\sum_{\alpha\in\tilde\chi} R(\tau_\alpha f', g')}{Max_{\beta\in\Theta\tilde\chi}([C_{\bar\beta} : \hat{p}_1^\beta(C_\beta)])}.$$

Proof. The inequalities of Corollary 2.11 remain true if we replace $Max_{\beta\in\Theta\alpha}$ by $Max_{\beta\in\Theta\tilde\chi}$, and $Min_{\beta\in\Theta\alpha}$ by $Min_{\beta\in\Theta\tilde\chi}$. Taking the sum over $\alpha \in \tilde\chi$ gives both results by 2.9 and 2.11 . □

In our first application of Theorem 2.12 we discuss the finiteness of $R(f, g)$. The Corollary below generalizes a number of published results. These include that $R(\bar f, \bar g) = \infty$ implies that $R(f, g) = \infty$ ([12, 19]). In [48, Theorem 1] Wong proved that if $R(\bar f, \bar g)$ and all the $R(\tau_\alpha f', g')$ are finite, then so also is $R(f, g)$. Alternatively in both fixed point and coincidence cases when $R(\bar f, \bar g)$ is finite and if either $Coin(\tau_{\bar\alpha}\bar f, \bar g) = 1$ (or $Fix\ \tau_{\bar\alpha}\bar f = 1$) for all $\bar\alpha$, or if $Coin(\tau_{\bar\alpha}\bar f, \bar g)$ or $\bar G_1$ are finite, and if $R(\tau_\alpha f', g') = \infty$ for some α, then $R(f, g) = \infty$ ([18, 19, 12]). Other results include that when $\bar G_1$ and $\bar G_2$ are finite, then $R(f, g) < \infty$ if and only if $R(\tau_\alpha f', g') < \infty$ for all α ([8]) (see also [19, 12, 15]). Our result most closely resembles the last formulation in that for equivalence to hold we require only that $R(\bar f, \bar g)$ and the $[Coin(\tau_{\bar\alpha}\bar f, \bar g); p_{1*}^\alpha(Coin(\tau_\alpha f, g))]$ are bounded. The easy proof that it generalizes all the above is left to the reader.

Corollary 2.13. *If $R(\bar{f},\bar{g})$ is infinite, so also is $R(f,g)$. If the $[C_{\bar{\alpha}} : \hat{p}_1^{\alpha}(C_\alpha)]$ are bounded and $R(\bar{f},\bar{g})$ is finite, then $R(f,g)$ is finite if and only if $R(\tau_\alpha f',g')$ is finite for every $\alpha \in \tilde{\chi}$. Of course the $[C_{\bar{\alpha}} : \hat{p}_1^{\alpha}(C_\alpha)]$ are bounded and $R(\bar{f},\bar{g})$ is finite if both $\bar{G}_1$ and $\bar{G}_2$ are finite.*

Proof. The first part is obvious since $\hat{p}_{2*}$ is surjective. The second part is forced by the inequalities in Theorem 2.12. Finally $[C_{\bar{\alpha}} : \hat{p}_1^{\alpha}(C_\alpha)] \leq \#(C_{\bar{\alpha}}) \leq |\bar{G}_1|$, and $R(\bar{f},\bar{g}) \leq |\bar{G}_2|$. □

We investigate next conditions under which we have formulas. Part (a) of the Corollary below gives Reidemeister analogues of Nielsen theory formulas given by Jezierski in [33, Theorem 4.2] and by Moh'D in [43, Theorem 4.9] (see Corollary 3.12). The Nielsen results in these references are for $\bar{G}_1$ and $\bar{G}_2$ finite (where the boundedness conditions given below are automatic). It is also the Reidemeister analogue of a fixed point Nielsen fibre space formula proved in [30, Theorem 4.1], and of a coincidence version stated but not proved in [26, Theorem 9.9] (see Theorem 3.32).

Corollary 2.14. *Suppose that the $[C_{\bar{\alpha}} : \hat{p}_1^{\theta\alpha}(C_\beta)]$ are bounded over $\Theta\tilde{\chi}$ and are independent (a) of $\theta \in \Theta\alpha$, or (b) of $\theta \in \Theta\tilde{\chi}$ and in (b) that $R(\tau_\alpha f',g')$ are also independent of $\alpha \in G_2$. Then*

$$\text{(a)}\quad R(f,g) = \sum_{\alpha\in\tilde{\chi}} \frac{R(\tau_\alpha f',g')}{[C_{\bar{\alpha}} : \hat{p}_1(C_\alpha)]}, \quad \text{respectively (b)}\quad [C_{\bar{\alpha}} : \hat{p}_1(C_\alpha)]R(f,g) = R(f',g')R(\bar{f},\bar{g}).$$

In particular, if $g = *$ (root theory) then $[Ker\ \bar{f} : \hat{p}_1(Ker\ f)]RR(f) = RR(f')RR(\bar{f})$.

Proof. Under hypothesis (a), the $\overline{R}(\tau_\alpha f',g')$ of Lemma 2.9 have the required form by Corollary 2.11. Part (b) follows as in the proof of Corollary 2.5 when $[C_{\bar{\alpha}} : \hat{p}_1(C_\alpha)]$ is finite. When it is not, then both sides are infinite.

In the root theory case note that $Coin(\tau_\alpha f, *) = Ker\ \tau_\alpha f$. Let $\alpha \in G_2$, since H_2 is normal then $\tau_\alpha : G_2 \to G_2$ restricts to $\tau_\alpha : H_2 \to H_2$ (by abuse of notation). This gives a morphism

$$\begin{array}{ccccccccc}
1 \to & Ker\ f' & \xrightarrow{i} & H_1 & \xrightarrow{f'} & H_2 & \to & \mathcal{R}R(f') \to 1 \\
& \downarrow 1 & & \downarrow 1 & & \downarrow \tau_\alpha & & \downarrow [\tau_\alpha] \\
1 \to & Ker\ \tau_\alpha f' & \xrightarrow{i} & H_1 & \xrightarrow{\tau_\alpha f'} & H_2 & \to & \mathcal{R}R(\tau_\alpha f') \to 1,
\end{array}$$

of the sequences of Proposition 2.1, where $[\tau_\alpha]$ is induced by τ_α. Clearly $\tau_{\alpha^{-1}}$ induces the inverse morphism. In particular $Ker\ f' = Ker\ \tau_\alpha f'$ and $RR(f') = RR(\tau_\alpha f')$. Similarly we have that $[Ker\ \bar{f} : \hat{p}_1(Ker\ f)] = [Ker\ \tau_{\bar{\alpha}}\bar{f} : \hat{p}_1(Ker\ \tau_\alpha f)]$ for all α. The result follows. □

The first part of the next Corollary generalizes several results in the literature given under the hypothesis that the $Coin(\tau_{\bar{\alpha}}\bar{f},\bar{g})$ (or the $Fix(\tau_{\bar{\alpha}}\bar{f})$) =1 ([13, 17, 22, 48]). The last part is a Reidemeister and coincidence analogue of a result by Fadell on natural fibre splittings ([11]).

Corollary 2.15. *If $R(\bar{f},\bar{g})$ and the $R(\tau_\alpha f',g')$ are finite, then*

$$R(f,g) = \sum_{\alpha\in\tilde{\chi}} R(\tau_\alpha f',g') \iff [Coin(\tau_{\bar{\alpha}}\bar{f},\bar{g}); \hat{p}_1(Coin(\tau_\alpha f,g))] = 1 \text{ for all } \alpha.$$

In particular, if both p_1 and p_2 have sections σ_1 and σ_2 respectively, with $\sigma_2(\bar{G}_2)$ normal in G_2, and if $f(\sigma_1(\bar{G}_1)) \subseteq \sigma_2(\bar{G}_2)$ and $g(\sigma_1(\bar{G}_1)) \subseteq \sigma_2(\bar{G}_2)$, then the equations hold true.

Proof. *Sufficiency:* Clearly if $[C_{\bar{\beta}} : \hat{p}_1^{\alpha}(C_\beta)] = 1$ for all $\beta \in G_1$, equality holds by Corollary 2.14. *Necessity:* Suppose when $R(\bar{f},\bar{g})$ and the $R(\tau_\alpha f',g')$ are finite, that $[C_{\bar{\beta}} : \hat{p}_1^{\alpha}(C_\beta)] > 1$ for some $\beta \in G_1$. Then $R(\tau_\beta f',g') > \overline{R}(\tau_\beta f',g')$ by Corollary 2.11. Since the $R(\tau_\alpha f',g')$ are independent[3] of $\alpha \in (p_2^{\alpha})^{-1}([\bar{\alpha}]) \subset G_2$ (Theorem 2.4) we can, without loss assume that $\beta \in \tilde{\chi}$, so $\sum_{\alpha\in\tilde{\chi}} R(\tau_\alpha f',g') > \sum_{\alpha\in\tilde{\chi}} \overline{R}(\tau_\alpha f',g') = R(f,g)$ (by Lemma 2.9), a contradiction.

The given conditions in the last part give rise to a section to $\mathrm{Coin}(\tau_\alpha f,g) \to \mathrm{Coin}(\tau_{\bar{\alpha}}\bar{f},\bar{g})$. □

We end our analysis by generalizing and extending the recent Reidemeister averaging inequality ([22]). We do this by adding an upper bound, sharpening the given lower bound and by giving a new formula in situations where $Coin(\tau_\alpha f,g)$ might not be contained in H_1.

Theorem 2.16. *Suppose that $\bar{G}_1$ and $\bar{G}_2$ are finite, and that $\Xi \subset G_2$ contains exactly one representative $\alpha \in p_2^{-1}(\bar{\alpha})$ for each $\bar{\alpha} \in \bar{G}_2$ (we call Ξ a set of lifts for p_2). Then*

$$\frac{Max_{\beta\in G_1}(|\hat{p}_1(C_\beta)|)}{|\bar{G}_1|} \sum_{\alpha\in\Xi} R(\tau_\alpha f',g') \geq R(f,g) \geq \frac{Min_{\beta\in G_1}(|\hat{p}_1(C_\beta)|)}{|\bar{G}_1|} \sum_{\alpha\in\Xi} R(\tau_\alpha f',g').$$

[3] See footnote 2 and Remark 4.5

Moreover if $|\hat{p}_1(C_\beta)|$ is independent of $\beta \in G_1$ then

$$\text{(a)} \quad R(f,g) = \frac{|\hat{p}_1(C_\beta)|}{|\bar{G}_1|} \sum_{\alpha \in \Xi} R(\tau_\alpha f', g'), \quad \text{and in particular (b)} \quad R(f,g) = \frac{1}{|\bar{G}_1|} \sum_{\alpha \in \Xi} R(\tau_\alpha f', g')$$

if $Coin(\tau_\alpha f, g) \subset H_1$ for all α. If all the $R(\tau_\alpha f', g')$ are finite, then this last condition is both necessary and sufficient for (b) to hold. □

Proof. Now $\sum_{\alpha\in\Xi} R(\tau_\alpha f', g') = |\bar{G}_1| \sum_{\alpha\in\Xi} \frac{1}{\#([\bar{\alpha}])} \sum_{\theta\in\Theta\alpha} \frac{1}{\hat{p}_1^{\theta\alpha}(|\mathrm{Coin}(\tau_{\theta\alpha} f, g)|)}$ from [22]. Clearly $\frac{1}{\#([\bar{\alpha}])} \cdot \frac{\sum_{\theta\in\Theta\alpha} 1}{Min_{\beta\in G_1}(|\hat{p}_1^\beta(C_\beta)|)} \geq \frac{1}{\#([\bar{\alpha}])} \sum_{\theta\in\Theta\alpha} \frac{1}{\hat{p}_1^{\theta\alpha}(|\mathrm{Coin}(\tau_{\theta\alpha} f, g)|)} \geq \frac{1}{\#([\bar{\alpha}])} \cdot \frac{\sum_{\theta\in\Theta\alpha} 1}{Max_{\beta\in G_1}(|\hat{p}_1^\beta(C_\beta)|)}$ for each $\alpha \in \Xi$. Now $\sum_{\theta\in\Theta\alpha} 1 = \overline{R}(\tau_\alpha f', g')$, so $\sum_{\alpha\in\Xi} \frac{1}{\#([\bar{\alpha}])} \sum_{\theta\in\Theta\alpha} 1 = \sum_{\alpha\in\tilde{\Xi}} \overline{R}(\tau_\alpha f', g') = R(f,g)$. Using this, summing the inequality over Ξ and multiplying by $|\bar{G}_1|$ we have from above that $\frac{|\bar{G}_1| \cdot R(f,g)}{Min_{\beta\in G_1}(|\hat{p}_1^\beta(C_\beta)|)} \geq \sum_{\alpha\in\Xi} R(\tau_\alpha f', g') \geq \frac{|\bar{G}_1| \cdot R(f,g)}{Max_{\beta\in G_1}(|\hat{p}_1^\beta(C_\beta)|)}$. Rearranging this gives the result. □

3 Applications to Nielsen theory

The aim of this section is to give applications of section 2 to four different Nielsen theories in two distinct contexts. We deal first with the covering space context in which the i_1 and i_2 in Diagram (1) are induced by covering projections. The second and dual context is the fibre space context, where p_1 and p_2 are induced by fibrations. The four Nielsen theories are fixed point, classical coincidence theory on oriented closed manifolds, semi-index coincidence theory, and finally root theory.

Our technique in both contexts, is to outline precise conditions under which the usual inclusions of Nielsen classes into the corresponding Reidemeister classes, give rise to exact subsequences of the corresponding Reidemeister versions. In this way the algebraic results and proofs of section 3 can be mimicked almost word for word.

3.1 Notation and preliminaries

For maps $f, g : X_1 \to X_2$, we use the symbols $\Phi(f,g) := \{x \in X_1 | f(x) = g(x)\}$, $\tilde{\Phi}(f,g)$ and $\tilde{\Phi}_H(f,g)$ ($\Phi(f)$ and $\tilde{\Phi}_H(f)$ in the fixed point case) to denote the set of coincidences and their Nielsen classes respectively their mod (H) Nielsen classes where H_1 and H_2 are normal subgroups of $\pi_1(X_1)$ and $\pi_1(X_2)$ respectively. Without loss of generality we assume all the $\Phi(f,g)$ are finite.

When $x \in \Phi(f,g)$ we use the symbols $f_*^x, g_*^x : \pi_1(X_1, x) \to \pi_1(X_2, f(x))$ to denote the induced homomorphisms. With the usual notation we have well known injections

$$\rho : \tilde{\Phi}(f,g) \to \mathcal{R}(f_*^x, g_*^x) \;\text{ and }\; \rho_H : \tilde{\Phi}_H(f,g) \to \mathcal{R}_H(f_*^x, g_*^x) \tag{18}$$

defined, (for example) on a class $[yH_1] \in \tilde{\Phi}_H(f,g)$ by $\rho_H([aH_1]) = [g(\lambda) f(\lambda^{-1}) H_2]$, where $\lambda : x \to y$ is any path, and y is any representative of $[yH]$, and where $\mathcal{R}(f_*^x, g_*^x)$ and $\mathcal{R}_H(f_*^x, g_*^x)$ are respectively the ordinary and mod H Reidemeister numbers, and $x \in X_1$ represents the basepoint.

We assume the reader is familiar with the following notions of index and semi index.

Definition 3.1. The phrase *a (semi) index $\mathcal{I}$* (sometimes an essentiality $\mathcal{I}$) refers to one (or more) of the following scenarios:-

(i) $\mathcal{I}_1$ is a fixed point index defined for self maps f (i.e. $g = 1$) on either compact connected ANRs, or compact connected manifolds ([34, 5])

(ii) $\mathcal{I}_2$ is the usual coincidence index defined for maps $f, g : X_1 \to X_2$, of compact oriented closed manifolds X_1 and X_2 of the same dimension (i.e. [46])

(iii) $\mathcal{I}_3$ is the coincidence semi - index defined for smooth maps $f, g : X_1 \to X_2$ defined on closed smooth manifolds X_1 and X_2 of the same dimension (see [9]).

(iv) $\mathcal{I}_4$ which takes values in $\mathbb{Z} \cup \mathbb{Z}_2$, is multiplicity for root classes as defined in [4, page 9]. In particular g is the constant map.

Convention 3.2. We adopt the convention that each (semi) index indicated in Definition 3.1 is attached to specific category of (fibre) spaces, maps and homotopies. So when using $\mathcal{I}_3$ for example, the base, total space and fibres of any fibration, covering space, map, lifts and all homotopies are smooth. Of course we exclude homotopies of $g(= 1)$ when using $\mathcal{I}_1$.

Each (semi) index $\mathcal{I}$ in Definition 3.1 gives rise to a notion of essentiality. A class $[x]$ (sometimes A) is said to be *essential* if its (semi) index $\mathcal{I}([x]) \neq 0$ ($\neq 0$ or $[0]$ for $\mathcal{I}_4$). Similarly a mod H class is essential if it's index or semi index is non-zero. For each $\mathcal{I}$ in Definition 3.1 the *($\mathcal{I}$) Nielsen number* $N^{\mathcal{I}}(f,g)$ is defined to be the number of essential (with respect to $\mathcal{I}$) classes of $\tilde{\Phi}(f,g)$. We use a subscript for mod H versions (i.e $N^{\mathcal{I}}_H(f,g)$). In more usual notation

$$N^{\mathcal{I}_1}(f,g) = N(f),\ N^{\mathcal{I}_2}(f,g) = N(f,g),\ N^{\mathcal{I}_3}(f,g) = N(f,g) \text{ and } N^{\mathcal{I}_4}(f,g) = NR(f), \quad (19)$$

where $N(f)$ is the classical Nielsen number and $NR(f)$ is the Nielsen root number (i.e. [4]).

We extend each (semi) index to Reidemeister classes (this is the essence of the modified fundamental group approach - see [24, 20]). Let $[\alpha] \in \mathcal{R}(f^x_*, g^x_*)$, for each $t = 1, \cdots 4$

$$\mathcal{I}_t([\alpha]) := \begin{cases} \mathcal{I}_t([a]) & \text{if } [\alpha] = \rho_H([a]),\ \text{for some } [a] \in \tilde{\Phi}(f,g), \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

With this definition the functions ρ and ρ_H are index preserving.

Recall that the linearization of a self map $f : T \to T$ of a torus T is defined, up to conjugation, to be the matrix associated with the linear transformation $f_* : \pi_1(T) \to \pi_1(T)$. The linearization of a self map $f : N \to N$ of a nilmanifold N is the block diagonal matrix whose blocks are formed from the fibred toral decomposition (i.e. [27, Theorem 3.1]).

Proposition 3.3. *(c.f [2, 31]). Let N be a nilmanifold (in particular a torus) and $f,g : N \to N$ maps with linearizations F respectively G (chosen with respect to the same basis). Then for each $\mathcal{I}$ of Definition 3.1 we have that $N^{\mathcal{I}}(f,g) = |det(F-G)|$ (note $G = I$ for $\mathcal{I}_1$, and $G = 0$ for $\mathcal{I}_4$).*

Proof. Since for orientable manifolds we have that $N^{\mathcal{I}_1}(f,g) = N^{\mathcal{I}_2}(f,1) = N(f)$, $N^{\mathcal{I}_2}(f,g) = N^{\mathcal{I}_3}(f,g)$ and $N^{\mathcal{I}_4}(f,g) = N^{\mathcal{I}_2}(f,*) = NR(f)$, then we need only consider $\mathcal{I}_2$. Now nilmanifolds (and hence tori) are Jiang type spaces for coincidences ([16]), and $L(f,g) = det(F-G)$. If this is zero so is $N^{\mathcal{I}_2}(f,g)$. When $det(F-G) \neq 0$ then $N^{\mathcal{I}_2}(f,g) = R(f_*, g_*) = |det(F-G)|$ for tori by Corollary 2.3. For nilmanifolds$N^{\mathcal{I}_2}(f,g) = |det(F-G)|$ by the product theorem ([31, Lemma 7.3]) on the fibred toral decomposition (i.e. [27, Theorem 3.1]). □

3.2 Nielsen numbers, covering maps, homogeneous spaces etc.

In this subsection we give our applications of section 2 to four Nielsen theories in the covering space contex. We generalize the work of Jezierski and Moh'D from finite index covering spaces ([33, 43, 44]) as well as providing upper and lower bounds on their numbers. This work relates the Nielsen numbers of maps to a linear combination of the Nielsen numbers of representative lifts of the maps involved. In a similar way our results also generalize and extend those of a number of authors who have exhibited averaging formulas that compute both Nielsen fixed point and coincidence point numbers on infra-nilmanifolds or infra-solvmanifolds of type R (fixed point cases) and oriented infra-nilmanifolds (coincidence case). In particular existing results are all extended to give upper and sharpen lower, bounds on the $N^{\mathcal{I}}(f,g)$ in all four theories. Finally this subsection also considers the relationship between ordinary and "Mod H" Nielsen numbers.

Let $H_1 = \pi_1(\tilde{X}_1)$, $H_2 = \pi_1(\tilde{X}_2)$, $\tilde{x} \in \Phi(\tilde{f}, \tilde{g})$ and $q_1(\tilde{x}) = x$. Using the notation of subsection 3.1 we identify the right hand side of Diagram (7) below obtaining, for each such $\tilde{x}$, the diagram

$$\begin{array}{ccccccc} 1 \to & H_1 & \xrightarrow{\hat{q}_{1*}} & \pi_1(X_1, x) & \xrightarrow{j_1} & \pi_1(X_1,x)/H_1 & \to 1 \\ & \tilde{f}^{\tilde{x}}_* \downarrow\downarrow \tilde{g}^{\tilde{x}}_* & & f^x_* \downarrow\downarrow g^x_* & & \bar{f}^x_* \downarrow\downarrow \bar{g}^x_* & \\ 1 \to & H_2 & \xrightarrow{\hat{q}_{2*}} & \pi_1(X_2, f(x)) & \xrightarrow{j_2} & \pi_1(X_2, f(x))/H_2 & \to 1. \end{array} \quad (21)$$

Theorem 2.4 gives exactness of the top sequences of the various diagrams presented in Proposition 3.4 below. Our aim is to give necessary and sufficient conditions for exactness at $\hat{j}^{x_k}_{1*}$, $\tilde{\delta}^{x_k}$ and $\tilde{q}^{\alpha_k}_1$ in all of the various subsequences that diagram represents. With these subsequences we can mimic the proofs and results of section 2.

Let $\tilde{\chi} = \{\alpha = 1, \alpha_2 \cdots, \alpha_{R(\bar{f}_*, \bar{g}_*)}\} \subset \pi_1(X_2)$ be a chosen set of Reidemeister lifts for Diagram (21). Then $\Phi(f,g) = \bigsqcup_{\alpha_k \in \tilde{\chi}} q_1(\Phi(\alpha_k \tilde{f}, \tilde{g}))$, the union is disjoint, and $q_1(\Phi(\alpha\tilde{f}, \tilde{g}))$ is a union of coincidence classes of f and g. For each $\alpha_k \in \tilde{\chi}$ let $\mathcal{E}_k(f,g)$ denote the subset of essential classes contained in $p_1(\tilde{\Phi}(\alpha_k\tilde{f},\tilde{g}))$ (some of the $\mathcal{E}_k(f,g)$ may be empty). Then for each of the essentialities $\mathcal{I}$ of Definition 3.1 we have that

$$\mathcal{E}(f,g) = \bigsqcup_{\alpha_k\in\tilde{\chi}} \mathcal{E}_k(f,g), \quad \text{so} \quad N^{\mathcal{I}}(f,g) = \sum_{\alpha_k\in\tilde{\chi}} \#(\mathcal{E}_k(f,g)), \tag{22}$$

Proposition 3.4. *For $\mathcal{I} \in \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$, if $Im\ \delta^{x_k} \cap Im\ \tilde{\rho} \neq \emptyset$ then $\delta^{x_k} : Coin(\bar{f}_*^{x_k},\bar{g}_*^{x_k}) \to \mathcal{R}(\tilde{f}_*^{\tilde{x}_k},\tilde{g}_*^{\tilde{x}_k})$ factors through $\mathcal{E}(\alpha_k\tilde{f},\tilde{g})$ as shown, $\tilde{q}_1 : \tilde{\Phi}(\alpha_k\tilde{f},\tilde{g}) \to \tilde{\Phi}(f,g)$ restricts to $\tilde{q}_1^{\alpha_k} : \mathcal{E}(\alpha_k\tilde{f},\tilde{g}) \to \mathcal{E}(f,g)$ which in turn factors through $\mathcal{E}_k(f,g)$ as shown. This gives rise to the following commutative diagram which exhibits subsequences of the form given in (8)*

$$\begin{tikzcd}
Coin(f_*^{x_k},g_*^{x_k}) \arrow[r, "\hat{j}_{1*}^{x_k}"] & Coin(\bar{f}_*^{x_k},\bar{g}_*^{x_k}) \arrow[r, "\delta^{x_k}"] \arrow[dr, dotted, "\tilde{\delta}^{x_k}"'] & \mathcal{R}(\alpha_k\tilde{f}_*^{\tilde{x}_k},\tilde{g}_*^{\tilde{x}_k}) \arrow[r, "q_{2*}^{x_k}"] & \mathcal{R}(f_*^{x_k},g_*^{x_k}) \\
& & \mathcal{E}(\alpha_k\tilde{f},\tilde{g}) \arrow[u, "\tilde{\rho}"] \arrow[r, dotted, "\tilde{q}_1^{\alpha_k}"] \arrow[dr, dotted, "\tilde{q}_1^{\alpha_k}"'] & \mathcal{E}(f,g). \arrow[u, "\rho"'] \\
& & & \mathcal{E}_k(f,g) \arrow[u, dotted]
\end{tikzcd}$$

If $\mathcal{I} = \mathcal{I}_4$ and f is root essential, then $RN(f) = RR(f)$ which can be computed from the Reidemeister sequence.

Note we have yet to prove $\text{Coin}(f_*^{x_k},g_*^{x_k}) \to \text{Coin}(\bar{f}_*^{x_k},\bar{g}_*^{x_k}) \to \mathcal{E}(\alpha_k\tilde{f},\tilde{g}) \to \mathcal{E}(f,g)$ is exact.

Proof. It is well known that if $A \subset \Phi(f,g)$ and $\tilde{A} \subset \tilde{\Phi}(\alpha_k\tilde{f},\tilde{g})$ are Nielsen classes with $q_1(\tilde{A}) = A$, then for $t = 1,2,3$ we have that $\mathcal{I}_t(\tilde{A}) = \iota(A)\cdot\mathcal{I}_t(A)$ for some integer $\iota(A)$ dependent only on A (see [34, 33, 45]). In the case of semi- index, $\iota(A)$ can be zero, but in any case, if some $\tilde{A}$ is essential, then this $\iota(A) \neq 0$, and so $A(=\tilde{q}_1^{\alpha_k}(\tilde{A}))$ and every other $\tilde{A}'$ with $q_1(\tilde{A}') = A$ is essential too. So, when A is essential either all Nielsen classes in $q_1^{-1}(A)$ are essential, or none of them are. Thus when $Im\ \delta^{x_k} \cap Im\ \tilde{\rho} \neq \emptyset$ all $\tilde{A}$ with $q_1(\tilde{A}) = A = [x_k]$ are essential. And this means all Reidemeister classes in $Im\ \delta^{x_k}$ are essential. For the last part see [4]. □

Corollary 3.5. *If $Im\ \delta^{x_k} \cap Im\ \tilde{\rho} \neq \emptyset$ there are $[Coin(\bar{f}_*^{x_k},\bar{g}_*^{x_k}) : \hat{j}_{1*}^{x_k}(Coin(f_*^{x_k},g_*^{x_k}))]$ coincidence classes of $\mathcal{E}(\alpha_k\tilde{f},\tilde{g})$ that coalesce to $q_1^{\alpha_k}([\tilde{x}_k]) = [x_k]$ in $\mathcal{E}_k(f,g)$. In particular we have that the $[Coin(\bar{f}_*^{x_k},\bar{g}_*^{x_k}) : \hat{j}_{1*}^{x_k}(Coin(f_*^{x_k},g_*^{x_k}))]$ are independent of x_k in its class in $\mathcal{E}(f,g)$.* □

As we shall soon see, the Nielsen analogue of the equation $R(f,g) = \sum_{\alpha\in\tilde{\chi}} \overline{R}(\tau_\alpha f',g')$ in section 2 can fail. We give below necessary and sufficient conditions for it to hold. But first we define our terms. Let $\tilde{A},\tilde{B} \in \mathcal{E}(\alpha_k\tilde{f},\tilde{g})$, we say $\tilde{A}$ and $\tilde{B}$ are equivalent if they both lie in the image of $\tilde{\delta}^{x_k}$ for some $\tilde{x}_k \in \Phi(\alpha_k\tilde{f},\tilde{g})$ (see Proposition 3.4). We use $\overline{\mathcal{E}}(\alpha_k\tilde{f},\tilde{g})$ to denote the set of equivalence classes with cardinality $\overline{E}(\alpha_k\tilde{f},\tilde{g})$. We adopt some notions from [43].

Definition 3.6. (c.f. Moh'D [43]). For $\mathcal{I} \in \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ let $\mathcal{E}_{ED}(f,g)$ denote the set of essential defective classes $[y] \in \mathcal{E}(f,g)$ for which $|\hat{j}_{1*}^{y}(\text{Coin}(f_*^y,g_*^y)|$ is even.

We call $N_{ED}(f,g) := \#(\mathcal{E}_{ED}(f,g))$ the *Essential Defective Nielsen number* of f and g. It is a homotopy invariant lower bound for $\#(\mathcal{E}_{ED}(f,g))$ ([43]). We use the symbol $N_L^{\mathcal{I}}(f,g)$ (L for the linear part of $N^{\mathcal{I}}(f,g)$ - [43]) to denote the number

$$N_L^{\mathcal{I}}(f,g) := N^{\mathcal{I}}(f,g) - N_{ED}(f,g)$$

The proof of the next Lemma is trivial from the definitions.

Lemma 3.7. $N_L^{\mathcal{I}}(f,g) = \sum_{\alpha_k\in\tilde{\chi}} \overline{E}(\alpha_k\tilde{f},\tilde{g})$. □

Proposition 3.8. *Let $\mathcal{I} \in \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$. The following are equivalent:-*

(i) $N^{\mathcal{I}}(f,g) = N_L^{\mathcal{I}}(f,g)$.,
(ii) the $\tilde{q}_1^{\alpha_k} : \mathcal{E}(\alpha_k\tilde{f},\tilde{g}) \to \mathcal{E}_k(f,g)$ are surjective for all k,
(iii) for all $\tilde{A}$ and A with $q_1(\tilde{A}) = A$, we have that $\tilde{A}$ is essential if and only if A is essential.
(iv) there are no essential defective classes $[y] \in \mathcal{E}(f,g)$ for which $|\hat{j}_{1*}^{y}(Coin(f_*^y,g_*^y)|$ is even.
(v) the number $N_{ED}(f,g) = 0$,

These equivalent conditions are fulfilled if one or more of the following hold:-

(vi) there are no defective classes, in particular if $\mathcal{I}=\mathcal{I}_1$ or $\mathcal{I}_2$,

(vii) all the $|j_{1*}^{\alpha_k}(Coin(\alpha_k\tilde{f}_*,\tilde{g}_*))|$ are odd, and in particular

(viii) $Coin(\bar{\alpha}\bar{f}_*,\bar{g}_*)=1$ for all $\bar{\alpha}$.

Under these conditions $Coin(f_*^{x_k},g_*^{x_k})\to Coin(\bar{f}_*^{x_k},\bar{g}_*^{x_k})\to\mathcal{E}(\alpha_k\tilde{f},\tilde{g})\to\mathcal{E}(f,g)$ is exact.

Remark 3.9. The crucial enabling ingredient. As McCord observed in 1992 in [42, Page 355], either all classes that cover an essential class in $\tilde{\Phi}(f,g)$ are essential, or all are inessential. Proposition 3.8 tell us exactly when they are essential. As we shall see, when this is the case, we can essentially replace Reidemeister numbers in section 3 with the appropriate Nielsen numbers. So Proposition 3.8 gives the crucial enabling ingredient for this subsection. In terms of root theory, when f is root essential we compute $NR(f)$ using the Reidemeister formulas of section 2.

Proof of Proposition 3.8. As in the proof of Proposition 3.4, for $t=1,2,3$ when $q_1(\tilde{A})=A$, we have that $\mathcal{I}_t(\tilde{A})=\iota(A)\cdot\mathcal{I}_t(A)$ for some integer $\iota(A)$ depending only on A. Using the same ideas we see that each of the five conditions holds iff whenever $\mathcal{I}_t(A)\neq 0$ in such a formula, then $\iota(A)\neq 0$. The condition fails only if there is an essential defective class $A=[x]$ for which $|\hat{j}_{1*}^{x}(\mathrm{Coin}(f_*^x,g_*^x)|$ is even (see Moh'D's modification [43, Proposition 3.12], of Vendrúscolo's analysis [45, Theorem 3.7]). And A can only be defective when $\mathcal{I}_t=\mathcal{I}_3$. The result follows. $\square$

An example where the equivalent conditions of Proposition 3.8 do not hold (i.e. when $N_{ED}(f,g)\neq 0$) is given by Moh'D in [43, Example 3.18].

Let $\Theta\alpha_k\subseteq\{\theta\in H_2|q_{2*}^{x_k}(\theta\alpha_k)=q_{2*}^{x_k}(\alpha_k)\}$ be a set of representatives for $Im\ q_{2*}^{x_k}$ chosen as in Lemma 2.10, and let $\mathcal{E}\Theta\alpha_k:=\{\theta\in\Theta\alpha_k\mid[\theta]\in Im\ \tilde{\rho}\}$. We use $C_{\bar{\alpha}_k}$ and $\hat{j}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})$ to abbreviate $Coin(\bar{\alpha}_k\bar{f}_*,\bar{g}_*)$ respectively $\hat{j}_1^{\theta\alpha_k}(Coin(\theta\alpha_k\tilde{f}_*,\tilde{g}_*)$. We come to the analogy of Corollary 2.11

Lemma 3.10. *Let $\mathcal{I}\in\{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$, $\tilde{\chi}$ a set of Reidemeister lifts, and $\alpha_k\in\tilde{\chi}$. If $N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})\neq 0$*

$$N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})=\sum_{\theta\in\mathcal{E}\Theta\alpha_k}[C_{\bar{\alpha}_k}:\hat{j}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})].$$

In particular when $N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})\neq 0$, all the $[C_{\bar{\alpha}_k}:\hat{j}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})]$ are finite. Furthermore

$$\frac{N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Min_{\theta\in\mathcal{E}\Theta\alpha_k}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])}\geq\overline{E}(\alpha_k\tilde{f},\tilde{g})\geq\frac{N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Max_{\theta\in\mathcal{E}\Theta\alpha_k}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])}.$$

Proof. By Proposition 3.4, the injective image of $\tilde{\rho}:\mathcal{E}(\alpha_k\tilde{f},\tilde{g})\to\mathcal{R}(\tilde{f}_*^{\tilde{x_k}},\tilde{g}_*^{\tilde{x_k}})$ is a union of equivalence classes under the relation that defines $\overline{\mathcal{R}}(\tilde{f}_*^{\tilde{x_k}},\tilde{g}_*^{\tilde{x_k}})$. By definition there are $\#(\mathcal{E}\Theta\alpha_k)$ of them, and the cardinality of the equivalence class indexed by θ is $[C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})]$. These numbers are finite whenever $N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})\neq 0$ by the first part of the Lemma. The proof of the last part is identical to that of Corollary 2.11, once we replace $\Theta\alpha$ there with $\mathcal{E}\Theta\alpha_k$. $\square$

Theorem 3.11. *Let $\mathcal{I}\in\{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$, $\mathcal{E}\Theta\alpha_k$ and $\tilde{\chi}$ and $\mathcal{E}\Theta\tilde{\chi}$ be as above, Let $\mathcal{E}\Theta\tilde{\chi}:=\cup_{\tilde{\chi}}\mathcal{E}\Theta\alpha_k$, then*

$$\sum_{\alpha_k\in\tilde{\chi}}\frac{N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Min_{\theta\in\mathcal{E}\Theta\tilde{\alpha}_k}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])}\geq N_L^{\mathcal{I}}(f,g)\geq\sum_{\alpha_k\in\tilde{\chi}}\frac{N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Max_{\theta\in\mathcal{E}\Theta\tilde{\alpha}_k}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])},$$

and

$$\frac{\sum_{\alpha_k\in\tilde{\chi}}N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Min_{\theta\in\mathcal{E}\Theta\tilde{\chi}}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])}\geq N_L^{\mathcal{I}}(f,g)\geq\frac{\sum_{\alpha_k\in\tilde{\chi}}N^{\mathcal{I}}(\alpha_k\tilde{f},\tilde{g})}{Max_{\theta\in\mathcal{E}\Theta\tilde{\chi}}([C_{\bar{\alpha}_k}:\hat{q}_{1*}^{\theta\alpha_k}(C_{\theta\alpha_k})])}.$$

Of course if any of the conditions of Proposition 3.8 are satisfied, then $N_L^{\mathcal{I}}(f,g)=N^{\mathcal{I}}(f,g)$.

Proof. Note that $\sum_{\alpha_k\in\tilde{\chi}}\overline{E}(\alpha_k\tilde{f},\tilde{g})=N^{\mathcal{I}}(f,g)-N_{ED}(f,g)$. The proof now follows the pattern of Theorem 2.12 using Lemma 3.10 instead of Corollary 2.11, and replacing $\Theta\tilde{\chi}$ by $\mathcal{E}\Theta\tilde{\chi}$. $\square$

Our first Corollary generalizes (from $\bar{G}_1$ and $\bar{G}_2$ finite) Jezierski's fixed point result ([33, Theorem 4.2]) and Moh'D's generalization to coincidences in [43, 44]. To see that the equations in the Corollary are simply different formulations of those in these references, we refer the reader to [33, Corollary 3.11] and [43, Proposition 3.9]. The generalization to cases where $\bar{G}_1$ and $\bar{G}_2$ may not be finite is useful when dealing with Mod K Nielsen classes of the fibres in the Nielsen theory of fibre-preserving maps in the next subsection (see also Proposition 3.22).

Corollary 3.12. *(Generalizing [33, 44, 43]) Let $\mathcal{I} \in \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, and $\tilde{\chi}$ be a set of Reidemeister lifts for Diagram (21). Let $\mathcal{E}\tilde{\chi}$ be the set of $\alpha_k \in \tilde{\chi}$ for which $N^{\mathcal{I}}(\alpha_k \tilde{f}, \tilde{g}) \neq 0$. Then if for any $\alpha_k \in \mathcal{E}\tilde{\chi}$ the $[C_{\bar{\alpha}_k} : \hat{j}_{1*}^{\theta\alpha_k}(C_{\alpha_k})]$ are independent of θ in $\mathcal{E}\Theta\alpha_k$, then*

$$N^{\mathcal{I}}(f,g) - N_{ED}(f,g) = \sum_{\alpha_k \in \mathcal{E}\tilde{\chi}} \frac{N^{\mathcal{I}}(\alpha_k \tilde{f}, \tilde{g})}{[C_{\bar{\alpha}_k} : \hat{j}_{1*}^{\alpha_k}(C_{\alpha_k})]}, \quad \textit{while} \quad N^{\mathcal{I}}(f,g) = \sum_{\alpha_k \in \mathcal{E}\tilde{\chi}} \frac{N^{\mathcal{I}}(\alpha_k \tilde{f}, \tilde{g})}{[C_{\bar{\alpha}_k} : \hat{j}_{1*}^{\alpha_k}(C_{\alpha_k})]}$$

if any of the conditions of Proposition 3.8 are satisfied. If f is root essential then we have that $[Ker\ \bar{f} : \hat{q}_{1*}(Ker\ f)]NR(f) = RR(\tilde{f})RR(\bar{f})$ ($q_1 : \pi_1(X_1) \to \pi_1(X_1)/H_1$, and $\bar{f}$ is induced by f).

We refer the reader to [33] and [43] for examples of the Corollary.

Proof. Under the given hypothesis we have from Lemma 3.11 that the $\overline{E}(\alpha_k \tilde{f}, \tilde{g})$ are equal to $N(\alpha_k \tilde{f}, \tilde{g})/[C_{\bar{\alpha}_k} : \hat{j}_{1*}^{\alpha_k}(C_{\alpha_k})]$. Lemma 3.10, Proposition 3.8 and Corollary 2.14 give the result. □

Corollary 3.13. *If in Corollary 3.12, we have that $[C_{\bar{\beta}} : \hat{j}_{1*}^{\beta}(C_\beta)] = 1$ for all $\beta \in \mathcal{E}\Theta\tilde{\chi}$ then*

$$N^{\mathcal{I}}(f,g) - N_{ED} = \sum_{\alpha \in \mathcal{E}\chi} N^{\mathcal{I}}(\alpha \tilde{f}, \tilde{g}). \quad \textit{In particular} \quad N^{\mathcal{I}}(f,g) = \sum_{\alpha \in \tilde{\mathcal{E}}\chi} N^{\mathcal{I}}(\alpha \tilde{f}, \tilde{g})$$

if $\mathcal{I} = \mathcal{I}_1$ or $\mathcal{I}_2$, or all $Coin(\bar{\alpha}\bar{f}_*, \bar{g}_*) = 1$, or all the $|\hat{j}_{1*}^{\beta}(C_\beta)|$ are odd. □

Example 3.14. Let $q : S^2 \to \mathbb{R}P^2$ be the quotient map, and $f = g : \mathbb{R}P^2 \to \mathbb{R}P^2$ the identity. Now $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 = \{1, \beta\}$, where β is the antipodal map. From Proposition 2.1 we have that $Fix(f_*) = \mathcal{R}(f_*) = \mathbb{Z}_2$. But $\pi_1(S^2) = 1$, so $j_* : \mathcal{R}(f_*^x) \to \mathcal{R}(\bar{f}_*^x)$, and $j_*^x : Fix(f_*^x) \to Fix(\bar{f}_*^x)$ are isomorphisms for any $x \in \Phi(f)$ by Theorem 2.4. This gives $[Fix(\bar{f}_*^x) : j_*(Fix(f_*^x))] = 2/2 = 1$ for any $x \in \Phi(f)$. Let $\tilde{f} : S^2 \to S^2$ be the identity lift of the identity. Now $\beta\tilde{f} = \beta$ and $\Phi(\beta) = \emptyset$, so $N(\beta\tilde{f}) = N(\beta) = 0$. Also $L(\tilde{f}) = 2$ so $N(\tilde{f}) = R(\tilde{f}) = 1$. Note that $\text{Coin}(\bar{\alpha}\bar{f}_*, \bar{g}_*) \neq 1$ and $|\hat{j}_{1*}^{\beta}(C_\beta)| = 2$ is even, but $N_{ED}(f,g) = 0$ since $\mathcal{I} = \mathcal{I}_1$. Therefore $N(f) = \sum_{\alpha \in \mathcal{E}\tilde{\chi}} N(\alpha\tilde{f}) = 1$.

We come now to the Nielsen counterpart of our extensions and generalizations of the Reidemeister bounds on averaging formulas (Theorem 2.16).

Theorem 3.15. *Suppose that $\pi_1(X_1)/H_2$ and $\pi_1(X_2)/H_2$ are finite. Let $\Xi \subset \pi_1(X_2)$ be a set of lifts for j_2 (Diagram (21), Theorem 2.16). Then for each of the essentialities of Definition 3.1*

$$\frac{Max_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{[\pi_1(X_2) : H_2]} \sum_{\alpha \in \mathcal{E}\Xi} N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}) \geq N_L(f,g) \geq \frac{Min_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{[\pi_1(X_2) : H_2]} \sum_{\alpha \in \mathcal{E}\Xi} N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}).$$

where $N_L(f,g) := N^{\mathcal{I}}(f,g) - N_{ED}(f,g)$, and $\mathcal{E}\Xi$ is the subset of Ξ for which $N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}) \neq 0$. In particular, if any of the conditions of Proposition 3.8 are satisfied, then

$$\frac{Max_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{[\pi_1(X_2) : H_2]} \sum_{\alpha \in \mathcal{E}\Xi} N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}) \geq N^{\mathcal{I}}(f,g) \geq \frac{Min_{\beta\in\pi_1(X)}(|\hat{j}_1^{\beta}(C_\beta)|)}{[\pi_1(X_2) : H_2]} \sum_{\alpha \in \mathcal{E}\Xi} N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}).$$

Proof. The proof mimics that of Theorem 2.16 replacing $R(\tau_\alpha f', g')$ by $E(\tau_\alpha\tilde{f}, \tilde{g})$ and $\overline{R}(\tau_\alpha f', g')$ by $\overline{E}(\tau_\alpha\tilde{f}, \tilde{g})$. Since $\sum_{\theta\in\mathcal{E}\Theta\alpha} 1 = \overline{E}(\tau_\alpha f', g')$ we also replace $\Theta\alpha$ by $\mathcal{E}\Theta\alpha$, and use the equality $\sum_{\alpha\in\tilde{\chi}} \overline{E}(\tau_\alpha\tilde{f}, \tilde{g}) = N_L(f,g)$ from Lemma 3.7. □

As we shall soon see, our first Corollary essentially contains a number of averaging formulas (see Remark 3.18) as well as an extension and two generalizations (Proposition 3.17).

Corollary 3.16. *If $|\hat{j}_1^{\beta}(C_\beta)|$ is independent of those $\beta \in \pi_1(X_1)$ for which $N^{\mathcal{I}_t}(\beta\tilde{f}, \tilde{g}) \neq 0$, and if in addition either $\mathcal{I} = \mathcal{I}_1$ or $\mathcal{I}_2$, or $\mathcal{I} = \mathcal{I}_3$ and $|\hat{j}_1^{\beta}(C_\beta)|$ is odd, then with $\mathcal{E}\Xi$ as above*

$$N^{\mathcal{I}}(f,g) = \frac{|\hat{j}_1^{\beta}(C_\beta)|}{[\pi_1(X_2) : H_2]} \sum_{\alpha\in\mathcal{E}\Xi} N^{\mathcal{I}}(\alpha\tilde{f}, \tilde{g}). \textit{ In particular } N^{\mathcal{I}_t}(f,g) = \frac{1}{[\pi_1(X_2) : H_2]} \sum_{\alpha\in\mathcal{E}\Xi} N^{\mathcal{I}_t}(\alpha\tilde{f}, \tilde{g})$$

if $t = 1, 2, 3$, and $Coin(\tau_\alpha f, g) \subseteq H_1$ for all α for which $N^{\mathcal{I}_t}(\alpha\tilde{f}, \tilde{g}) \neq 0$.

Proof. In each case the Max and Min in Theorem 3.15 coincide. In the second case this number is 1. So as appropriate the right hand side of each equation in the Corollary is equal to $N^{\mathcal{I}_t}(f,g) - N_{ED}(f,g)$. In both cases however, $N_{ED}(f,g) = 0$ from Proposition 3.8 . $\square$

In addition to illustrating Corollary 3.13, Example 3.14 also illustrates the first formula in Corollary 3.16. This very same example was used in [38, Example 4.3] to show the averaging formula in 3.16 need not hold when $Coin(\alpha\tilde{f},\tilde{g}) \not\subset H_1$ for all α.

Corollary 3.17. *(Generalizing both [39, Theorem 4.9] and [41, Theorem 4.1]) Suppose, in the context of this subsection that X_1 and X_2 are orientable infra-nilmanifolds of the same dimension, or smooth non orientable infra-nilmanifolds of the same dimension, or $X_1 = X_2$ is a smooth infra-solvmanifold of type R (see [41, p. 119]). Let $A = q_1(\Phi(\alpha\tilde{f},\tilde{g}))$ be essential and $N^{\mathcal{I}}(\alpha\tilde{f},\tilde{g}) \neq 0$, then $Coin(\tau_\alpha f_*, g_*) = 1$. So then, in the usual notation*

$$N^{\mathcal{I}_3}(f,g) = \frac{1}{[\pi_1(X_1):\pi_1(\tilde{X}_1)]} \sum_{\alpha\in\mathcal{E}\Xi} N^{\mathcal{I}_3}(\alpha\tilde{f},\tilde{g}).$$

Proof. When $N^{\mathcal{I}}(\alpha\tilde{f},\tilde{g}) \neq 0$, then $Coin(\alpha\tilde{f}_*,\tilde{g}_*) = 1$. For infra-nilmanifolds this comes from [14], and for solvmanifolds of type R from the coincidence analogue of the proof that weakly Jiang maps on solvmanifolds are “essentially fix trivial” (see [27])). From 2.4, when $Coin(\alpha\tilde{f}_*,\tilde{g}_*) = 1$ then $Coin(\tau_\alpha f_*, g_*)$ is essentially a subgroup of $Coin(\bar{\alpha}\bar{f}_*,\bar{g}_*) \subset \pi_1(X_1)/H_1$ (a finite group). But $Coin(\tau_\alpha f_*, g_*) \subset \pi_1(X_1)$ which is torsion free, so $Coin(\tau_\alpha f_*, g_*)$ must be trivial.

For the formula, in each case the spaces $\tilde{X}_1$ and $\tilde{X}_2$ are appropriate quotients by torsion free discrete cocompact subgroups of simply connected nilpotent respectively solvable of type R, Lie groups. Each case gives rise to a diagram of the form of the one given in Diagram (21). As above $Coin(\tau_\alpha f_*, g_*) = 1$ when $N^{\mathcal{I}}(\alpha\tilde{f},\tilde{g}) \neq 0$. The result follows by Corollary 3.16. $\square$

Remark 3.18. The history of averaging formulas. Corollary 3.17 includes the following Nielsen averaging formulas:- firstly [9, Theorem 2.5] (orientable double cover), next [38, Theorem 3.5] (fixed point version on infra-nilmanifolds), then [39, Theorem 4.9] (coincidence version for orientable infra-nilmanifolds), and in the same year [41, Theorem 4.2] (fixed point version for infra-solvmanifolds of type R). This formula for Nielsen numbers first appeared in [42, Corollary 5.10] with no connection to the sequences. The Reidemeister version appeared in [22] in 2012. The two non orientable cases presented in Corollary 3.17, are new.

In our illustration of the non orientable case, which cost me a finite number of beers in Daejeon Korea, we use the identification of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ with $\mathrm{GL}_{n+1}(\mathbb{R})$ defined using the injective homomorphism $\Delta : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{GL}_{n+1}(\mathbb{R})$ given on $(a, A) \in \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ by

$$\Delta(a,A) := \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}. \tag{23}$$

Example 3.19. Non-orientable infra-nilmanifold example. For $k = 1, 2, 3$ let e_k denote the standard basis elements of $\mathbb{R}^3$, $t_k := \Delta(e_k, I)$, where I is the 3×3 identity matrix, and let $\beta,\ \gamma \in Im(\Delta)$ be given by

$$\beta := \Delta(b,B) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma := \Delta(c,C) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where B and C (used below) and b and c are determined by these equations. Let M be the 3-dimensional non-orientable (because $det(B) = det(C) = -1$) infra-nilmanifold $M := \Pi\backslash\mathbb{R}^3$, where $\Pi \subset \mathrm{Aff}(\mathbb{R}^n)$ is generated by t_1, t_2, t_3, β and γ. We identify the integer lattice of pure translations in Π with $\mathbb{Z}^3$, and this gives rise to the exact sequence

$$1 \to \mathbb{Z}^3 \to \Pi \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1.$$

To see that the holonomy group is indeed $\mathbb{Z}_2 \times \mathbb{Z}_2$ observe that $\beta^2 = t_3$, $\gamma^2 = t_1$, $\beta\gamma = \gamma^{-1}\beta$ and $(\beta\gamma)^2 = \beta\gamma\gamma^{-1}\beta\ = \beta^2 = t_3$, so that $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\beta\gamma}$ each have order 2.

Next, since M is a $K(\pi, 1)$, the map $f : M \to M$ is determined up to homotopy by defining f_* on Π by $f_*(\beta) = t_1^{2p} t_3^{\ell} \beta$, $f_*(\gamma) = t_1^n t_2^{2q} \gamma$, $f_*(t_1) = t_1^{1+2n}$, $f_*(t_2) = t_2^m$ and $f_*(t_3) = t_3^{1+2\ell}$ for arbitrary integers ℓ, m, n, p and q. Let $g : M \to M$ be determined by $g_*(\beta) = t_1^x t_2^y t_3^z$, $g_*(\gamma) = 1$, $g_*(t_1) = 1$, $g_*(t_2) = 1$ and $g_*(t_3) = t_1^{2x} t_2^{2y} t_3^{2z}$ for arbitrary integers x, y, and z.

The formulas give rise to lifts $\tilde{f}$ and $\tilde{g}$ of f and g to the 3-torus with respective linearizations

$$F = \begin{pmatrix} 1+2n & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1+2\ell \end{pmatrix} \text{ and } G = \begin{pmatrix} 0 & 0 & 2x \\ 0 & 0 & 2y \\ 0 & 0 & 2z \end{pmatrix}.$$

By Corollary 3.17 and Proposition 3.3 (maps of tori) we have that

$$N^{\mathcal{I}_3}(f,g) = \frac{1}{4}(|det(F-G)| + |det(BF-G)| + |det(CF-G)| + |det(BCF-G)|) = |(1+2n)m(1+2\ell-2z)|.$$

As an alternative way to see this, note that $Coin(\bar{f}, \bar{g}) = 1$, so $R(\bar{f}, \bar{g}) = 1$ by Proposition 2.1. Corollary 3.13 gives that $N^{\mathcal{I}_3}(f,g) = N^{\mathcal{I}_3}(\tilde{f}, \tilde{g})$, which is also equal to $|(1+2n)m(1+2\ell-2z)|$.

The following root theory example also illustrates weakly Jaing possibilities (see Remark 3.21).

Example 3.20. Root theory example. Let $\Pi \subset \mathrm{Aff}(\mathbb{R}^2)$ be generated by t_1, t_2 and β, where for $k = 1, 2$ the $t_k = \Delta(e_k, I)$, are the 2 dimensional versions of the t_k in Example 3.19, and β is

$$\beta := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad F := \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}.$$

Then $\beta^2 = t_2$, $\Pi = < t_1, \beta \mid \beta t_1 = t_1^{-1}\beta >$ and $X := \Pi \backslash \mathbb{R}^2$ is the Klein Bottle, and is covered by $\tilde{X} := T^2$, the 2-torus. Since X is a $K(\pi, 1)$ we define a self map f up to homotopy by putting $f_*(t_1) = t_1^a t_2^b$, and $f_*(\beta) = \beta^d$ (so that $f_*(t_2) = t_2^d$) where a is odd, and $b, d \in \mathbb{Z}$ arbitrary. Now f lifts to $\tilde{X}$ with linearization F given above, so for any essentiality $\mathcal{I}$ for which f is root essential we have that $NR^{\mathcal{I}}(f) = RR(\tilde{f}) = |det(F)| = |ad|$.

Remark 3.21. Various root essentialities. The maps in root theory can be thought of as weakly Jiang (the Nielsen number is zero or equal to the Reidemeister number). When the latter case holds the Nielsen number can be computed from the algebra. Though we defined $\mathcal{I}_4$ explicitly, there are many ways of defining root essentiality, and each has the weakly Jiang property. In particular, in addition to $\mathcal{I}_4$ we can add to the list simply by putting g equal to a constant map in either $\mathcal{I}_2$ or $\mathcal{I}_3$. In addition we could replace $\mathbb{Z}$ coefficients in these last two essentialities with $\mathbb{Z}_2$ coefficients, or we could use the non-empty homotopy invariance root essentiality (see [1]).

We close the subsection with a result which connects Mod H Nielsen numbers with ordinary Nielsen numbers. Our special interest is in the Mod K Nielsen numbers on the fibres of fibre preserving maps. Though needed in the fibre space contex, its proof belongs here. It is here where (unlike [33, 44, 43]) we need the possibility that $\bar{G}_1$ and $\bar{G}_2$ are infinite (Corollary 3.12). In the context of Diagram of (5), suppose that $x \in \Phi(f,g)$, and $b = p_1(x)$. Let $F_b = p_1^{-1}(b)$, $F_{f(b)} = p_1^{-1}(f(b))$ and $f_b,\ g_b : F_b \to F_{f(b)}$ be the restrictions of f and g. Then, with the obvious notation, we have the following special case (ignore $f, g : E_1 \to E_2$ for now) of Diagram (21)

$$\begin{array}{ccccccc} F_b & \to & E_1 & K_1 := Ker\ i_{1*} & \longrightarrow & \pi_1(F_b) & \stackrel{j_{1*}}{\longrightarrow} & \pi_1(F_b)/K_1 \\ f_b \downarrow\downarrow g_b & & f \downarrow\downarrow g & f^x_{bK*} \downarrow\downarrow g^x_{bK*} & & f^x_{b*} \downarrow\downarrow g^x_{b*} & & \bar{f}^x_{b*} \downarrow\downarrow \bar{g}^x_{b*} \\ F_{f(b)} & \to & E_2 & K_2 := Ker\ p_{2*} & \longrightarrow & \pi_1(F_{f(b)}) & \stackrel{j_{2*}}{\longrightarrow} & \pi_1 F_{f(b)})/K_2. \end{array} \tag{24}$$

Note that instead of writing $\tilde{f}_b$ we have written f_{bK}. Since we are dealing with fibre spaces here, each ordinary Nielsen class contained in a mod K Nielsen class has the same index ([47, 25, 31]). So suppose further, that x lies in an essential Nielsen class, then in situations like this, we can modify and extend the sequences in Equation (8) to give sequences of the form

$$\mathrm{Coin}(f^x_{b*}, g^x_{b*}) \stackrel{q_*}{\to} \mathrm{Coin}(\bar{f}^x_{b*}, \bar{g}^x_{b*}) \stackrel{\delta}{\to} \mathcal{R}(f_{bK*}, f_{bK*}) \stackrel{\mathcal{E}q_1}{\to} \mathcal{E}(f_b, g_b) \to \mathcal{E}_K(f_b, g_b) \to 1.$$

This easily gives rise the the following result, stated in terms of the various mod K Nielsen numbers $N^{\mathcal{I}}_K(f_b, g_b)$ which are useful in Theorems 3.30 and 3.32.

Proposition 3.22. Mod K Nielsen numbers on the fibres. *If, in the situation just described we have that $\pi_1(F_{f(b)})$ is abelian, then for any $x \in \Phi(f_b, g_b)$ we have*

$$N_K^{\mathcal{I}}(f_b, g_b) = \frac{[Coin(\bar{f}^x_{b*}, \bar{g}^x_{b*}) : q_*(Coin(f^x_{b*}, g^x_{b*})] N^{\mathcal{I}}(f_b, g_b)}{R(f^x_{bK*}, g^x_{bK*})}.$$

3.3 Nielsen theories of Fibre preserving maps

Though the fibre space context of our applications were the subject of [26], there are results here that are new. These include upper and lower bounds on the various Nielsen numbers we are seeking to compute, but also in our way of using the bounds to prove formulas. In addition we prove a number of results stated without proof in [26]. These including the algebraic results out of which the Nielsen bounds presented here fall. In addition we take the opportunity to prove Theorem 3.34 stated, but again not proved, in [26]. This result was expected to appear in a joint publication with Keppelmann, but this is now not going to happen. As a Corollary of this results we give an alternative, much shorter and simpler proof of a result by Dobreńko and Jezierski (Corollary 3.35). The original proof gives a prototype of the averaging formula.

Using the notation and notions explained in diagram (24), and combining the two left hand sides of Diagrams in (5) and (24), we produce (for each x) the following example of Diagram (1)

$$\begin{array}{cccccccc} 1 \to & \pi_1(F_b, x)/K_1 & \longrightarrow & \pi_1(E_1, x) & \xrightarrow{p_{1*}} & \pi_1(B_1, b) & \to 1 \\ & \downarrow\downarrow & & f_* \downarrow\downarrow g_* & & \bar{f}* \downarrow\downarrow \bar{g}_* & \\ 1 \to & \pi_1(F_{f(b)}, f(x))/K_2 & \longrightarrow & \pi_1(E_2, f(x)) & \xrightarrow{p_{2*}} & \pi_1(B_2, \bar{f}(b)) & \to 1 \end{array} \tag{25}$$

(we assume throughout that the Topological spaces in Diagrams (5) and (24) are all path connected). We will use this to deduce subsequence (6) in Lemma 3.26 below.

The functions i_1 and p_1 in Diagrams (5) and (24) induce the following sequence of finite sets

$$\tilde{\Phi}_K(f_b, g_b) \xrightarrow{i_1} \tilde{\Phi}(f, g) \xrightarrow{p_1} \tilde{\Phi}(\bar{f}, \bar{g}).$$

Clearly different choices of $b \in \Phi(\bar{f}, \bar{g})$ will give different "fibres" $\tilde{\Phi}_K(f_b, g_b)$ for this sequence, and p_1 need not be surjective. However each non-empty Nielsen class A in $\tilde{\Phi}(f, g)$ determines such a sequence. In particular each Nielsen class A of f and g has "components" denoted $\tilde{A}$ in $\tilde{\Phi}_K(f_b, g_b)$ and $\bar{A} := p_1(A)$ in $\Phi(\bar{f}, \bar{g})$, where $i_1(\tilde{A}) = A$, and $b \in \bar{A}$. There are of course in general many choices of $\tilde{A}$ in this scenario.

Definition 3.23. For f and g as above, and for each of the essentialities of Definition 3.1 we say that f *and* g *satisfy the product rule for a given essentiality* if for all Nielsen classes A in $\tilde{\Phi}(f, g)$, A is essential if and only if $\bar{A} = p_1(A)$ and all choices of the component $\tilde{A}$ of A are essential.

Proposition 3.24. *Let $\mathcal{I}$ one of the essentialities of Definition 3.1, p_1 and p_2 be locally trivial fibre bundles, and all associated maps and spaces lie in the category that defines $\mathcal{I}$. Suppose further that f and g are fibre preserving maps from p_1 to p_2 (see Diagram (5). Then f and g satisfy the product rule for each essentiality $\mathcal{I}$ in the following situations:*

(a) $\mathcal{I} = \mathcal{I}_1,\ \mathcal{I}_2$,

(b) $\mathcal{I} = \mathcal{I}_3$ and for all $A \in \mathcal{E}(f, g)$ and all $\tilde{A}$ with $q_1(\tilde{A}) = A$ then either (i) $\mathcal{I}_3(\bar{A}) \cdot \mathcal{I}_3(\tilde{A}) \le 1$, or (ii) neither $\bar{A}$ $(= p_1(A))$ nor any choice of $\tilde{A}$ is defective.

(c) $\mathcal{I} = \mathcal{I}_4$ and f is root essential.

Using the same identifications as in the proof of [4, Theorem 6.1] (also in the proof below), we can use Example 3.1 of [3] to show that f (i.e. the pair f, $*$) need not satisfy the product rule when f is not root essential. The point of the product rule is to ensure the surjectivity of the $\mathcal{E}\tilde{i}_1 : \mathcal{E}_K(f_b, g_b) \to \mathcal{E}p_1^{-1}(\bar{A})$ in Lemma 3.27 below.

Proof. Part (a) for $\mathcal{I}_1$ for compact ANR's is found in [47, Theorem 4.1]. For manifolds [31, Lemma 5.7] serves for both $\mathcal{I}_1$ and $\mathcal{I}_2$. Part (b) is [32, Theorem 3:13, Lemma 4.2]. Part (c) follows from [3, Theorem 2.3] once (as seen in the proof of [4, Theorem 6.1]) we replace $NR(g)$ (their g) with $NR(f_b)$ and $NR(h)$ with $NR(\bar{f})$ (see [4] for their g and h). $\square$

Corollary 3.25. *Let $F_1 \to E_1 \to B_1$ and $F_2 \to E_2 \to B_2$ be locally trivial fibre bundles in the smooth category with matching dimensions in both fibre and base. Suppose further that F_1, F_2, B_1 and B_2 are orientable, then all fibre preserving maps f and g in the smooth category, satisfy the product rule for $\mathcal{I}_3$ (or $\mathcal{I}_2$ if appropriate). In particular this is true under the additional conditions that E_1 and E_2 are solvmanifolds, and p_1 and p_2 Mostow fibrations (i.e. [37]).*

Proof. Proposition 3.24(b)(ii) is satisfied, since F_1, F_2, B_1 and B_2 are orientable. $\square$

When f and g satisfy the product rule for an essentiality, then $p_1 : \tilde{\Phi}(f,g) \to \tilde{\Phi}(\bar{f},\bar{g})$ restricts to $\mathcal{E}p_1 : \mathcal{E}(f,g) \to \mathcal{E}(\bar{f},\bar{g})$. Since this may not be surjective (see [26, Example 2.10]), some of the $\mathcal{E}p_1^{-1}(\bar{A})$ in the Lemma below may be empty.

Lemma 3.26. *Suppose that f and g satisfy the product rule for an essentiality $\mathcal{I}$, then*

$$\mathcal{E}(f,g) = \bigsqcup_{\bar{A}\in\mathcal{E}(\bar{f},\bar{g})} \mathcal{E}p_1^{-1}(\bar{A}) \text{ and } N^{\mathcal{I}}(f,g) = \sum_{\bar{A}\in\mathcal{E}(\bar{f},\bar{g})} \#(\mathcal{E}p_1^{-1}(\bar{A})). \qquad \square$$

The top row of the diagram below is exact (apply Theorem 2.4 to Diagram (25)).

$$\begin{tikzcd}
\mathrm{Coin}(f^x_*, g^x_*) \arrow[r, "\hat{p}^x_{1*}"] & \mathrm{Coin}(\bar{f}^b_*, \bar{g}^b_*) \arrow[r, "\delta^x"] \arrow[dr, dotted, "\tilde{\delta}^x"'] & \mathcal{R}_K(f^x_{b*}, g^x_{b*}) \arrow[rr, "i^x_{2*}"] & & \mathcal{R}(f^x_*, g^x_*) \\
 & & \mathcal{E}_K(f_b, g_b) \arrow[u, "\rho_K"] \arrow[rr, dotted, "\mathcal{E}i_1"] \arrow[dr, dotted, "\mathcal{E}\tilde{i}_1"'] & & \mathcal{E}(f,g) \arrow[u, "\rho"'] \\
 & & & \mathcal{E}p_1^{-1}(\bar{A}) \arrow[ur, dotted] &
\end{tikzcd}$$

Here ρ_K and ρ are the restrictions (to essential classes) of the usual inclusions.

Lemma 3.27. *Suppose that f and g satisfy the product rule for an essentiality, $\bar{A} \in \mathcal{E}(\bar{f},\bar{g})$, $b \in \bar{A}$ and $x \in \Phi(f_b, g_b)$. If $Im\ \delta^x \cap Im\ \rho_K \neq \emptyset$ then δ^x factors through $\mathcal{E}_K(f_b,g_b)$ as indicated. Moreover $i_1 : \tilde{\Phi}_K(f_b,g_b) \to \tilde{\Phi}(f,g)$ restricts to $\mathcal{E}i_1$, and $\mathcal{E}i_1$ factors through $\mathcal{E}p_1^{-1}(\bar{A})$ as shown. Furthermore $\mathcal{E}\tilde{i}_1 : \mathcal{E}_K(f_b,g_b) \to \mathcal{E}p_1^{-1}(\bar{A})$ is surjective, and the diagram is commutative.*

Proof. Note first that $\tilde{\Phi}(f_b,g_b) \to \tilde{\Phi}(f,g)$ factors surjectively through $\tilde{\Phi}_K(f_b,g_b)$. Let $A \in \tilde{\Phi}(f,g)$. When $\bar{A} \in \mathcal{E}(\bar{f},\bar{g})$ then the product rule for essentiality implies that either all classes $\tilde{A} \in \tilde{\Phi}(f_b,g_b)$ that coalesce to A are essential or all are inessential. So then over each essential $\bar{A}$, a class $\tilde{A}_K$ in $\tilde{\Phi}_K(f_b,g_b)$ is essential if and only if all of the classes in $\tilde{\Phi}(f_b,g_b)$ that coalesce to it, are essential. In particular over each essential $\bar{A}$, the composition $\tilde{\Phi}(f_b,g_b) \to \tilde{\Phi}_K(f_b,g_b) \to \tilde{\Phi}(f,g)$ restricts to $\mathcal{E}(f_b,g_b) \to \mathcal{E}_K(f_b,g_b) \to \mathcal{E}(f,g)$ (though the first two sets may be empty). But in any case $\mathcal{E}_K(f_b,g_b) \to \mathcal{E}p_1^{-1}(\bar{A})$ is surjective if and only if $\mathcal{E}(f_b,g_b) \to \mathcal{E}p_1^{-1}(\bar{A})$ is. The rest of the proof now mimics that of Proposition 3.4. $\square$

Corollary 3.28. *Under the conditions of Lemma 3.27, if $Im\ \delta^{x_k} \cap Im\ \tilde{\rho} \neq \emptyset$, then there are $[Coin(\bar{f}^b_*, \bar{g}^b_*) : \hat{p}^x_{1*}(Coin(f^x_*, g^x_*))]$ mod K coincidence classes of $\mathcal{E}_K(f_b,g_b)$ that coalesce to $[x]$ in $\mathcal{E}p_1^{-1}(\bar{A})$. In particular the $[Coin(\bar{f}^b_*, \bar{g}^b_*) : \hat{p}^x_{1*}(Coin(f^x_*, g^x_*))]$ are independent of x in its class in $\mathcal{E}(f,g)$. Furthermore $Coin(f^x_*, g^x_*) \to Coin(\bar{f}^b_*, \bar{g}^b_*) \to \mathcal{E}_K(f_b,g_b) \to \mathcal{E}(f,g) \to \mathcal{E}(\bar{f},\bar{g})$ is exact.* $\square$

Following the by now familiar pattern, we define an equivalence relation on $\mathcal{E}_K(f_b,g_b)$ by equating those $\tilde{A}$ and $\tilde{B}$ which lie in the image of $\tilde{\delta}^x$ for some $x \in \Phi(f_b,g_b)$. We use $\overline{\mathcal{E}}_K(f_b,g_b)$ to denote the set of equivalence classes with cardinality $\overline{E}_K(f_b,g_b)$. A set $\mathcal{E}\chi \subset \Phi(\bar{f},\bar{g})$ is said to be a *set of essential representatives for $\bar{f}$ and $\bar{g}$* provided $\mathcal{E}\chi$ contains exactly one b from each of the essential classes of $\bar{f}$ and $\bar{g}$ (note we are using $\mathcal{E}\chi$ to distinguish it from $\mathcal{E}\tilde{\chi} \subset \Phi(f,g)$ used earlier). For each $b \in \mathcal{E}\chi$ we say that $\mathcal{E}\Theta_b \subset \Phi(f_b,g_b)$ is a *Mod K set of essential representatives for f_b and g_b* if $\mathcal{E}\Theta_b$ contains exactly one representative of each equivalence class of $\overline{\mathcal{E}}_K(f_b,g_b)$. Note that $\mathcal{E}\Theta_b$ may be empty, and that $\#(\mathcal{E}\Theta_b) = \overline{E}_K(f_b,g_b)$. We use $\mathcal{E}\Theta\chi$ to denote the union (over $\mathcal{E}\chi$) of the $\mathcal{E}\Theta_b$, and abbreviate C_b respectively $\hat{p}^x_{1*}(C_x)$ by $\mathrm{Coin}(\bar{f}^b_*, \bar{g}^b_*)$ respectively $\hat{p}^x_{1*}(\mathrm{Coin}(f^x_*, g^x_*))$.

The right hand equality in the Lemma below generalizes [30, Theorem 3.3] (from the fixed point case). The proof mimics that of the first part of Lemma 3.10. Note that, under the given conditions, the analogue of $N_{ED}(f,g)$ from subsection 3.2 is always zero.

Lemma 3.29. *Let f and g satisfy the product rule for an essentiality $\mathcal{I}$, and let $\mathcal{E}\chi$ and $\mathcal{E}\Theta\chi$ be chosen. Then for each $b \in \mathcal{E}\chi$ either $N_K^{\mathcal{I}}(f_b, g_b) = 0$ or*

$$N_K^{\mathcal{I}}(f_b, g_b) = \sum_{x \in \mathcal{E}\Theta_b} [C_{p_1(x)} : \hat{p}_{1*}^x(C_x)]. \quad \textit{Furthermore} \quad N^{\mathcal{I}}(f, g) = \sum_{b \in \mathcal{E}\chi} \overline{E}_K(f_b, g_b). \qquad \square$$

Once we observe, from Definition 3.23, that there can be no essential classes of f and g that lie over inessential Reidemeister classes of $\bar{f}$ and $\bar{g}$, then the proof of the following brand new Theorem follows previous patterns of section 2 and subsection 3.2 exactly.

Theorem 3.30. *Let f, g, $\mathcal{I}$, $\mathcal{E}\chi$ and $\mathcal{E}\Theta\chi$ be as in Lemma 3.29. Then*

$$\sum_{b \in \mathcal{E}\chi} \frac{N_K^{\mathcal{I}}(f_b, g_b)}{Min_{x \in \mathcal{E}\Theta_b}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])} \geq N^{\mathcal{I}}(f, g) \geq \sum_{b \in \mathcal{E}\chi} \frac{N_K^{\mathcal{I}}(f_b, g_b)}{Max_{x \in \mathcal{E}\Theta_b}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])},$$

and

$$\frac{\sum_{b \in \mathcal{E}\chi} N_K^{\mathcal{I}}(f_b, g_b)}{Min_{x \in \mathcal{E}\Theta\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])} \geq N^{\mathcal{I}}(f, g) \geq \frac{\sum_{b \in \mathcal{E}\chi} N_K^{\mathcal{I}}(f_b, g_b)}{Max_{x \in \mathcal{E}\Theta\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)])}. \qquad \square$$

We remark that the relationship between the $N_K^{\mathcal{I}}(f_b, g_b)$ and the $N_K^{\mathcal{I}}(f_b, g_b)$ can be studied using the techniques of Proposition 3.22.

The bounds in Theorem 3.30 need not be whole numbers. When this is the case we can, of course, improve the bounds. We illustrate this with an example from [30].

Example 3.31. The Möbius pretzel ([30, Example 1.3]) Let $f : E \to E$ and $p : E \to B$ be as in [30, Example 1.3]. We sketch the example and refer the reader to that reference for the details. Each fibre of p is the figure eight denoted $\mathbf{8}$, and the restriction of f to each fibre is the self map of $\mathbf{8}$ that is, it is a map of degree -1 on each of the circles making up $\mathbf{8}$. So for each b we have that $N_K(f_b) = N(f_b) = 3$. On the other hand part of E is formed by rotating $\mathbf{8}$ about a central axis while also rotating $\mathbf{8}$ through 180 degrees. In this way the North pole of the initial $\mathbf{8}$ is connected by a line of fixed points to its South pole. What this means geometrically is that the North and South poles are coalesced in E. So if x is either of these points, then algebraically $[Fix\ f_*^{p(x)} : p_*(Fix\ f_*^x)] = 2$. On the other hand if y is taken at the wedge point of $\mathbf{8}$, then $[Fix\ f_*^{p(y)} : p_*(Fix\ f_*^y)] = 1$. Since $\bar{f} = 1$ and B is also a copy of $\mathbf{8}$, Theorem 3.30 gives us

$$\frac{N(f_b)}{1} \geq N(f) \geq \frac{N(f_b)}{2} \quad \text{that is} \quad \frac{3}{1} \geq N(f) \geq \frac{3}{2}. \quad \text{So} \quad 3 \geq N(f) \geq 2,$$

since $N(f)$ must be a whole number. In fact $N(f) = 2$.

The fixed point case ($\mathcal{I} = \mathcal{I}_1$) contained in the next result, was proved in [30], the two coincidence cases ($\mathcal{I} = \mathcal{I}_2$ and $\mathcal{I}_3$) as well as the root theory version ($\mathcal{I} = \mathcal{I}_4$) were stated but not proved in [26]. The root theory result is an analogue of [4, Theorem 6.1] (see [26, Section 11] for further discussion). We take this opportunity to point out that the necessary hypothesis, that f is root essential, is missing from the statements of Theorem 11.2 and Corollary 11.4 in [26].

Theorem 3.32. *Suppose under the hypothesis of Lemma 3.29, we have for each b in $\mathcal{E}\chi$ that $[C_b : \hat{p}_{1*}^x(C_x)]$ is independent (a) of $x \in \mathcal{E}\Theta_b$, or (b) of $x \in \Phi(f, g)$. Then*

$$\textit{(a)} \quad N^{\mathcal{I}}(f, g) = \sum_{b \in \mathcal{E}\chi} \frac{N_K^{\mathcal{I}}(f_b, g_b)}{[C_b : \hat{p}_{1*}^x(C_x)]} \quad \textit{respectively (b)} \quad N^{\mathcal{I}}(f, g) = \frac{\sum_{b \in \mathcal{E}\chi} N_K^{\mathcal{I}}(f_b, g_b)}{[C_b : \hat{p}_{1*}^x(C_x)]}.$$

In case (b) if f and g are fibre uniform (all the $N_K^{\mathcal{I}}(f_b, g_b)$ all equal), then $[C_b : \hat{p}_{1*}^x(C_x)]N^{\mathcal{I}}(f, g) = N_K^{\mathcal{I}}(f_b, g_b)N^{\mathcal{I}}(\bar{f}, \bar{g})$. If f is root essential then $[Ker\ \bar{f} : \hat{p}_1(Ker\ f)]NR(f) = NR_K(f_b)NR(\bar{f})$.

Proof. The proof is dual to 3.12 and requires the dual of the last part of Lemma 3.10. We have left this to the reader. For f root essential use Corollary 2.14. $\square$

The Corollary below puts together the various naïeve product/addition formulas from [47, 30, 31, 33] and a roots version stated in [26]. The bounds on $N^{\mathcal{I}}(f, g)$ given in Theorem 3.30 allow us to furnish a simple rigorous proof of necessity.

Corollary 3.33. *(Naïeve addition formulas). Under the hypothesis of Lemma 3.29,*

$$N^{\mathcal{I}}(f,g) = \sum_{b\in\mathcal{E}\chi} N^{\mathcal{I}}(f_b,g_b) \quad (= N^{\mathcal{I}}(f_b,g_b)N^{\mathcal{I}}(\bar{f},\bar{g}) \text{ if } f,\ g \text{ are fibre uniform})$$

if and only if (i) $[Coin(\bar{f}_*^{p_1(b)}, \bar{g}_*^{p_1(b)}) : \hat{p}_{1*}(Coin(f_*^x, g_*^x)] = 1$ *for all* $x \in \mathcal{E}\Theta\chi$*, and (ii)* $N_K^{\mathcal{I}}(f_b,g_b) = N^{\mathcal{I}}(f_b,g_b)$ *for all* $b \in \mathcal{E}\chi$*.*

Proof. If the $[\mathrm{Coin}(\bar{f}_*^{p_1(b)}, \bar{g}_*^{p_1(b)}) : \hat{p}_{1*}(\mathrm{Coin}(f_*^x, g_*^x)] = 1$ and $N_K^{\mathcal{I}}(f_b,g_b)$ are as given, then the equality follows trivially from Theorem 3.32. Conversely, suppose $N^{\mathcal{I}}(f,g) = \sum_{b\in\mathcal{E}\chi} N^{\mathcal{I}}(f_b,g_b)$. We show first that $N^{\mathcal{I}}(f_b,g_b) = N_K^{\mathcal{I}}(f_b,g_b)$ for all $b \in \mathcal{E}\chi$. So suppose that $N^{\mathcal{I}}(f_b,g_b) > N_K^{\mathcal{I}}(f_b,g_b)$ for some $b \in \mathcal{E}\chi$. Since $Min_{x\in\mathcal{E}\Theta\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)]) \geq 1$ (clearly), then from the left hand inequality in Theorem 3.30 and our assumption we have that

$$\sum_{b\in\mathcal{E}\chi} N_K^{\mathcal{I}}(f_b,g_b) \geq Min_{x\in\mathcal{E}\Theta\chi}([C_{p_1(x)} : \hat{p}_{1*}^x(C_x)]) \sum_{b\in\mathcal{E}\chi} N^{\mathcal{I}}(f_b,g_b) \geq \sum_{b\in\mathcal{E}\chi} N^{\mathcal{I}}(f_b,g_b) > \sum_{b\in\mathcal{E}\chi} N_K^{\mathcal{I}}(f_b,g_b),$$

a contradiction. So $N^{\mathcal{I}}(f_b,g_b) = N_K^{\mathcal{I}}(f_b,g_b)$ for all $b \in \mathcal{E}\chi$. Next, suppose $[C_{p_1(x)} : \hat{p}_{1*}^x(C_x)] > 1$ for some $x \in \mathcal{E}\Theta\chi$. Let $b = p(x)$, then $N_K^{\mathcal{I}}(f_b,g_b) > \overline{E}_K(f_b,g_b)$ and $\sum_{b\in\mathcal{E}\chi} N^{\mathcal{I}}(f_b,g_b) = \sum_{b\in\mathcal{E}\chi} N_K^{\mathcal{I}}(f_b,g_b) > \sum_{b\in\mathcal{E}\chi} \overline{E}_K(f_b,g_b) = N^{\mathcal{I}}(f,g)$, again a contradiction. □

Corollary 3.33 lends itself supremely to a pair of self maps of a solvmanifold. First we remind the reader of a number of facts concerning solvmanifolds, path lifting functions for fibrations and linearization. Details can be found, for example, in [27, 28, 29], [23] and [37] respectively. A *solvmanifold S is a coset space of a connected simply connected solvable Lie group by a closed uniform subgroup.* Solvmanifolds are compact, and have the property that every self map $f : S \to S$ is homotopic to a fibre preserving with respect to it's minimal Mostow fibration denoted $N_r \to S \xrightarrow{p} T^t$, where N_r is a nilmanifold of dimension r and T^t a torus of dimension t.

We describe the linearized gluing data for p. Let $\lambda : a \to b$ be a path in T^t, the path lifting function defines a homotopy equivalence $\tau_\lambda : p^{-1}(a) \to p^{-1}(b)$, which up to homotopy is functorial (i.e. [23, Corollary 7.7]). Let $\ell(\tau_\lambda)$ be the linearization of τ_λ (see for example [28, Definition 2.1]). Then $\ell(\tau_\lambda)$ is an $r \times r$ integer valued matrix, which is invertible over $\mathbb{Z}$ (it's inverse is $\ell(\tau_{\lambda^{-1}})$). In particular $det(\ell(\tau_\lambda)) = \pm 1$. The linearization process is functorial, so if $\mu : b \to c$ is a path, then $\ell(\tau_\mu)\ell(\tau_\lambda) = \ell(\tau_\mu\tau_\lambda) = \ell(\tau_{\mu\lambda})$ (since $\tau_\mu\tau_\lambda \sim \tau_{\mu\lambda}$ i.e. [23, Lemma 7.4]). The *linearized gluing data for p* is the homomorphism $A : \pi_1(T^t) \to Aut(\mathbb{Z}^r)$ defined by $A([\lambda]) = \ell(\tau_\lambda)$. Note that it is sufficient to define A on a basis for $\pi_1(T^t) = \mathbb{Z}^t$.

A self map $f : S \to S$ of a solvmanifold S can also be linearized. Without loss we assume that f is fibre-preserving, and let $\bar{f}$ denote the induced map on the base T^t of the Mostow fibration. We assume that 0 is a fixed point of $\bar{f}$, and use f_0 to denote the restriction of f to the fibre over 0. We linearize both f_0 and $\bar{f}$. If $\ell(f_0) = X$ and $\ell(\bar{f}) = Y$, then (X, Y) *is the linearization of* f.

The fixed point case of the next result is well known. The coincidence version given below is a special case (of a pair of self maps) of one that was announced in [26]. This was to appear in a future joint publication by Ed Keppelmann and the author, but this will not now happen. Note that our result includes the non-orientable solvmanifolds.

Theorem 3.34. *(Heath Keppelmann unpublished) Let S be a solvmanifold with Mostow fibration $N_r \to S \to T^t$, and linearized gluing data $A : \mathbf{Z}^t \to Aut(\mathbf{Z}^r)$. Let $\mathcal{I} \in \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, and suppose that $f, g : S \to S$ are self maps with linearizations (X, Y) and (W, Z) respectively. If $det(Y - Z) = 0$ then $N^{\mathcal{I}}(f,g) = 0$. If $det(Y - Z) \neq 0$ then*

$$N^{\mathcal{I}}(f,g) = \sum_{[\bar{\alpha}]\in\mathcal{R}(\bar{f}_*,\bar{g}_*)} |\det(A(\bar{\alpha})X - W)|.$$

Furthermore for any root essentiality (see Remark 3.21) for which f is root essential we have that

$$NR(f) = |det(Y)det(X)|.$$

Proof. If $det(Y - Z) = 0$ then $\bar{f}$ can be homotoped so that $\Phi(\bar{f}, \bar{g}) = \emptyset$. In this case f can also be homotoped so that $\Phi(f,g) = \emptyset$, so clearly $N^{\mathcal{I}}(f,g) = 0$ for all essentialities $\mathcal{I}$. When

$det(Y-Z) \neq 0$ we may assume without loss, that $\#(\Phi(\bar{f},\bar{g})) = |det(Y-Z)|$, and we can replace $\mathcal{E}\chi$ with $\Phi(\bar{f},\bar{g})$ in Corollary 3.33. Furthermore the homomorphism $Y-Z : \mathbb{Z}^t \to \mathbb{Z}^t$ is injective, so from Proposition 2.1, and the fact that $\mathbb{Z}^t$ is Abelian, we have that $\mathrm{Coin}(\bar{f}_*^{p_1(b)}, \bar{g}_*^{p_1(b)})$ is trivial for all for all $b \in \Phi(\bar{f},\bar{g})$. Thus $[\mathrm{Coin}(\bar{f}_*^{p_1(b)}, \bar{g}_*^{p_1(b)}) : \hat{p}_{1*}(\mathrm{Coin}(f_*^x, g_*^x)] = 1$ for all $b \in \Phi(\bar{f},\bar{g})$. Finally $\pi_2(T^t)$ is also trivial, so $N^{\mathcal{I}}(f_b,g_b) = N^{\mathcal{I}}_K(f_b,g_b)$ for all $b \in \Phi(\bar{f},\bar{g})$. Thus the conditions of Corollary 3.33 are fulfilled and we can use the naïeve addition formula given there.

The next step is to show for $b \in \Phi(\bar{f},\bar{g})$, that $N^{\mathcal{I}}(f_b,g_b) = |\det(A(\bar{\alpha})X - W)|$, where $[\bar{\alpha}] = \rho([b]) = [\bar{g}(\lambda)f(\lambda^{-1})]$ and $\lambda : 0 \to b$ is any path. As in [25, Lemma 4.5] (see also [23]) path lifting functions define τ_λ, $\tau_{\bar{f}(\lambda)}$ and $\tau_{\bar{g}(\lambda)}$ below, and yield homotopy commutative diagrams

$$\begin{array}{ccc} F_0 & \xrightarrow{f_0} & F_0 \\ \tau_\lambda \downarrow & & \downarrow \tau_{\bar{f}(\lambda)} \\ F_b & \xrightarrow{f_b} & F_{\bar{f}(b)} \end{array} \quad \text{and} \quad \begin{array}{ccc} F_0 & \xrightarrow{g_0} & F_0 \\ \tau_\lambda \downarrow & & \downarrow \tau_{\bar{g}(\lambda)} \\ F_b & \xrightarrow{g_d} & F_{\bar{f}(b)}. \end{array}$$

So then $f_b \simeq \tau_{\bar{f}(\lambda)} f_0 \tau_{\lambda^{-1}}$ and $g_b \simeq \tau_{\bar{g}(\lambda)} g_0 \tau_{\lambda^{-1}}$, where $\simeq$ denotes homotopy. Since homotopic maps have the same linearization we have by Proposition 3.3, that

$$\begin{aligned} N^{\mathcal{I}}(f_b,g_b) &= \left|det\left(\ell(\tau_{\bar{f}(\lambda)} f_0 \tau_{\lambda^{-1}}) - \ell(\tau_{\bar{g}(\lambda)} g_0 \tau_{\lambda^{-1}})\right)\right| \\ &= \left|det(\ell(\tau_{\bar{g}(\lambda)}) \left(\ell(\tau_{\bar{g}(\lambda^{-1})})\ell(\tau_{\bar{f}(\lambda)} f_0) - \ell(g_0)\right) \ell(\tau_{\lambda^{-1}}))\right| \\ &= |det(\ell(\tau_{\bar{g}(\lambda)}))|\left|det\left(\ell(\tau_{\bar{g}(\lambda^{-1})}\tau_{\bar{f}(\lambda)})\ell(f_0) - \ell(g_0)\right)\right||det(\ell(\tau_{\lambda^{-1}}))| \\ &= \left|det\left(\ell(\tau_{\bar{g}(\lambda^{-1})\bar{f}(\lambda)})\ell(f_0) - \ell(g_0)\right)\right|, \end{aligned}$$

since $|det(\ell(\tau_{\bar{g}(\lambda)}))| = |det(\ell(\tau_{\lambda^{-1}}))| = 1$, and $\tau_{\bar{g}(\lambda^{-1})}\tau_{\bar{f}(\lambda)} \simeq \tau_{\bar{g}(\lambda^{-1})\bar{f}(\lambda)}$. Now $\ell(\tau_{\bar{g}(\lambda^{-1})\bar{f}(\lambda)}) = A(\bar{\alpha})$, where $\bar{\alpha}$ represents $[\bar{g}(\lambda^{-1})\bar{f}(\lambda)]$ in $\mathcal{R}(\bar{f},\bar{g})$, and $\ell(f_0) = X$, and $\ell(g_0) = W$, so

$$\sum_{b \in \Phi(\bar{f},\bar{g})} N^{\mathcal{I}}(f_b,g_b) = \sum_{b \in \Phi(\bar{f},\bar{g})} \left|det\left(\ell(\tau_{\bar{g}(\lambda^{-1})\bar{f}(\lambda)})\ell(f_0) - \ell(g_0)\right)\right| = \sum_{[\bar{\alpha}] \in \mathcal{R}(\bar{f}_*,\bar{g}_*)} |\det(A(\bar{\alpha})X - W)|.$$

By Corollary 3.25 if $\mathcal{I} \in \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, then $\mathcal{I}$ satisfies the product rule for essentialities, and this last term is equal to $N^{\mathcal{I}}(f,g)$ by Proposition 3.24 and Corollary 3.33.

If f is root essential for any root essentiality $\mathcal{I}$, then $N^{\mathcal{I}}(f,g) = RR(f_*)$ which by above and by Corollary 2.14 is $RR(f_{0*})RR(\bar{f}_*) = |det(X)||det(W)| = |det(X)det(W)|$ as required. □

For our final result we use Theorem 3.34 to give a fibre space proof of [9, Theorem 3.8], which was proved there using a prototype of the averaging formula, that is in the dual context to this section. Let $S^1 \hookrightarrow K^2 \xrightarrow{p} S^1$ be the fibering of the Klein bottle K^2 presented in Example 2.7. The gluing data $A : \mathbb{Z} \to Aut(\mathbb{Z})$, is given by $A(z) = (-1)^z$ (see [29]). Each homotopy class of a self map f has a representative induced by a linear map on $\mathbb{R}^2$ given by $(s,t) \to (ds+bt, at)$ where a, b and d are integers, and either a is odd or $d = 0$. Such a representative is fibre preserving, and has linearization (d,a) (see [29]).

Corollary 3.35. *([9, Theorem 3.8]) Let f and g be arbitrary self maps of K^2 with linearizations (d,a) and (d',a') respectively. Then*

$$N^{\mathcal{I}_3}(f,g) = |a-a'|max(|d|,|d'|).$$

Proof. By Proposition 3.3 we have that $N^{\mathcal{I}_3}(\bar{f},\bar{g}) = |a-a'|$. Now $N^{\mathcal{I}_3}(\bar{f},\bar{g}) = N^{\mathcal{I}_3}(f,g) = 0$ if $a = a'$. Otherwise, by Proposition 2.2, we have $R(\bar{f},\bar{g}) = \mathbb{Z}_{|a-a'|}$. By Theorem 3.34 we have that $N^{\mathcal{I}_3}(f,g) = \sum_{[\bar{\alpha}] \in \mathbb{Z}_{|a-a'|}} |\det(A(\bar{\alpha})d - d')|$. If either d or d' is equal to 0, then (since $A(\bar{\alpha}) = \pm 1$) there are $|a-a'|$ terms in the sum, each equal to $max(|d|,|d'|)$, giving the result for these cases. If neither d nor d' is zero, then $a - a'$ is even, and half of the $A(\bar{\alpha})$ are +1, and half -1. So $N^{\mathcal{I}_3}(f,g) = 1/2(|a-a'|)\,(|(d-d'| + |-d-d'|) = |a-a'|max(|d|,|d'|)$, completing the proof. □

4 Proof of the main Theorem

In this section we give a rigorous proof of Theorem 2.4 using the theory of groupoids. Theorem 2.4 is a more precise statement of a version stated but not proved in [26, Theorem 9.18], and also of the special case of the fixed point version given in [25, Theorem 1.8] sketched there by a different method. Groupoid theory not only gives a proof of the Theorem, but unveils the inspiration for it. Our technique is to show that Diagram (1) gives rise to a fibration of groupoids and that the sequences in Theorem 2.4 are simply the exact sequences associated with this fibration.

4.1 Groupoid Preliminaries

A groupoid A can be defined as *a category in which every morphism is an isomorphism.* The fundamental groupoid $\pi(X)$ of a space X is a familiar example. The objects $Ob(\pi(X))$ of $\pi(X)$ are the points of X, and the morphisms are the path classes between points. If $\sigma \in A(x,y)$ is a morphism from x to y in a groupoid A, we use $\sigma(0) = x$, and $\sigma(1) = y$ to denote end points. A functor $p : A \to B$ between groupoids is said to be *a fibration of groupoids* if for every $x \in Ob(A)$ and $\bar{\gamma} : p(x) \to y \in B$ there is a $\gamma : x \to \gamma(1) \in A$ with $p(\gamma) = \bar{\gamma}$. As an example of a fibration of groupoids, if $p : E \to B$ is a topological fibration, then the induced functor $p_* : \pi(E) \to \pi(B)$ is a fibration of groupoids. If $\bar{x} \in Ob(B)$ we will use the symbol $F_{\bar{x}} := p^{-1}(\bar{x}) \subseteq A$ to denote the groupoid fibre. So then $Ob(F_{\bar{x}}) = \{x \in Ob(A) | p(x) = \bar{x}\}$, and $F_{\bar{x}}(x,y) = \{\lambda \in A | p(\lambda) = 1_{\bar{x}}\}$.

If A is a groupoid and $x \in Ob(A)$, then the symbol $A\{x\}$ will denote the group at x, that is the subgroupoid of $A(x,x)$ whose morphisms begin and end at x. It is of course a group. If $A = \pi(X)$ is as above, then $\pi(X)\{x\} = \pi_1(X,x)$, the fundamental group of X at x. The symbol $\pi_0(A)$ denotes the quotient set of $Ob(A)$ under the equivalence relation that $x \sim y$ if and only if $A(x,y) \neq \emptyset$. If $A = \pi(X)$, then $\pi_0(\pi(X)) = \pi_0(X)$ in the usual sense.

Theorem 4.1. *(Ronald Brown [7]) Let $p : A \to B$ be a fibration of groupoids in which p is epi, and $x \in Ob(A)$. Then the sequence below is an exact sequence of groups (first four terms) and based sets (last four terms with the obvious base points)*

$$1 \to F_{p(x)}\{x\} \xrightarrow{i} A\{x\} \xrightarrow{p} B\{p(x)\} \xrightarrow{\delta} \pi_0(F_{\bar{x}}) \xrightarrow{i_*} \pi_0(A) \xrightarrow{p_{2*}} \pi_0(B) \to 1,$$

where $\delta(\bar{\beta}) = [\beta(1)]$ where β is a lift of $\bar{\beta}$ at x, and square brackets denote component.

Furthermore if β, $\theta \in B\{\bar{x}\}$ then $\delta(\beta) = \delta(\theta)$ iff there is a $\gamma \in A\{x\}$ with $p(\gamma) = \beta^{-1}\theta$. Moreover $\#(Im\ \delta) = [B\{p(x)\} : p(A\{x\})]$.

The proof of Theorem 4.1 is essentially a much simpler version of the proof of the exactness of the bottom end of the exact sequence of a topological fibration $p : E \to B$ (see [10] for the "Furthermore" part). In fact the long exact sequence of Topological fibration can be deduced from Theorem 4.1 by applying it iteratively to the fibration $p^{S^n} : E^{S^n} \to B^{S^n}$ (see [24]).

4.2 Proof of Theorem 2.4

The main point is that each Diagram of the form of (1) gives rise to a fibration of groupoids, and that Theorem 2.4 is then the corresponding special case of Theorem 4.1. We make this explicit.

Definition 4.2. Let $f, g : G_1 \to G_2$ be homomorphisms of groups. Then f and g determine a groupoid $G_1 \tilde{\times} G_2$ whose set of objects is G_2. Morphisms in $G_1 \tilde{\times} G_2$, are pairs $(\alpha, \beta) \in G_1 \times G_2$, where (α, β) takes the object β to the object $g(\alpha^{-1})\beta f(\alpha)$. That is $(\alpha,\beta)(1) = g(\alpha^{-1})\beta f(\alpha)$. The composite $(\alpha', \beta')(\alpha, \beta)$ takes β to $g((\alpha'\alpha)^{-1})\beta f(\alpha'\alpha)$, and is defined if and only if $\beta' = g(\alpha^{-1})\beta f(\alpha)$. The inverse of (α, β) is $(\alpha^{-1}, g(\alpha^{-1})\beta f(\alpha))$.

The proofs of the next two lemmas follow trivially from the definitions.

Lemma 4.3. *For f and g as in Definition 4.2, the group of $G_1 \tilde{\times} G_2\{1\}$ of $G_1 \tilde{\times} G_2$ at $1 \in G_2$ is the group $\{\alpha \in G_1 | 1 = g(\alpha) f(\alpha^{-1})\}$. It is canonically isomorphic to* Coin(f,g)*, and the set $\pi_0(G_1 \tilde{\times} G_2)$ is in canonical bijective correspondence with $\mathcal{R}(f,g)$.* □

Lemma 4.4. *In the context of Diagram (1) the functor $(p_1, p_2) : G_1 \tilde{\times} G_2 \to \bar{G}_1 \tilde{\times} \bar{G}_2$ defined on (α, β) by$(p_1,p_2)((\alpha,\beta) = (p_1(\alpha), p_2(\beta))$ is an epic fibration of groupoids. Furthermore the fibre of (p_1, p_2) over the base point $1 \in G_2$ is the groupoid $H_1 \tilde{\times} H_2$.* □

Remark 4.5. Lemma 2.10 shows that $R(\tau_\alpha f', g')$ is independent of $\alpha \in p_2^{-1}(\bar\alpha)$, but for Corollary 2.15 we need that it is independent of $\alpha \in (\hat p_2^\alpha)^{-1}([\bar\alpha]) \subset G_2$. A first principles proof is complex, and is absent from the literature. The conclusion of the Lemma below is a little more than we need for this paper, but we get it for free from the abstract homotopy theory of groupoids.

Lemma 4.6. *If $p : A \to B$ is a fibration of groupoids and $b_1, b_2 \in Ob(B)$ lie in the same component, then there is a natural equivalence $\mathcal{F} : F_{b_1} \to F_{b_2}$ of categories (in this case groupoids). In particular $\mathcal{F}$ induces bijections from $\pi_0(F_{b_1})$ to $\pi_0(F_{b_2})$, and for any $x \in Ob((F_{b_1})$ an isomorphism of groups from $F_{b_1}\{x\}$ to $F_{b_2}\{\mathcal{F}(x)\}$. In particular in the context of Diagram (14) the cardinalities of $\mathcal{R}(\tau_\alpha f', g')$ and $Coin(\tau_\alpha f', g')$ are independent of $\alpha \in (\hat p_2^\alpha)^{-1}([\bar\alpha]) \subset G_1$.*

Proof. An abstract homotopy equivalence in the category of groupoids is simply a natural equivalence of functors. The existence of such an abstract homotopy equivalence $\mathcal{F}$, follows from an abstract homotopy version of the topological result that any two fibres of a surjective fibration are of the same homotopy type (see for example [23]). An appropriate abstract homotopy version can be found in [35, Satz 5.4]. That the category of groupoids satisfies Kan condition DNE(2) (part of the hypotheses of [35, Satz 5.4]) can be found in [36, p. 157]. $\square$

Proof of Theorem 2.4. The facts that the sequences exist, are exact, and that the "Furthermore" part is true all follow from Theorem 4.1 using Lemmas 4.3 and 4.4. As to the interpretation of δ, let $\bar\beta \in Coin(\tau_{\bar\alpha}\bar f, \bar g) \cong \bar G_1 \tilde\times \bar G_2\{1\}$. Then a lift of $\bar\beta$ ($\cong (\bar\beta, 1)$) to $G_1 \tilde\times G_2$ at $1 \in Ob(G_1 \tilde\times G_2) = G_2$ is a pair $(\beta, 1)$, where $p_2(\beta) = \bar\beta$. By definition $\delta(\bar\beta) = [(1,\beta)(1)] = [g(\beta^{-1})\tau_\alpha f(\beta)]$. The independence of the cardinality of $R(\tau_\alpha f', g')$ comes from Lemma 4.6. Finally to see that the sequence is a sequence of groups when G_2 is Abelian, we note that when G_2 is Abelian so also are H_2 and $\bar G_2$. By Proposition 2.1 this gives the canonical group structures on the three Reidemeister sets and, with the exception of δ, that the sequence is an exact sequence of groups and homomorphisms. To see that δ is a homomorphism, let $\bar\beta, \bar\theta \in Coin(\tau_{\bar\alpha}\bar f, \bar g)$, and $\beta, \theta \in G_2$ be such that and $p_2(\beta) = \bar\beta$, and $p_2(\theta) = \bar\theta$. Then since G_2 is abelian we have that $\delta(\bar\beta)\delta(\bar\theta) = [g(\beta^{-1})\tau_\alpha f(\beta) g(\theta^{-1})\tau_\alpha f(\theta)] = [g(\beta^{-1})g(\theta^{-1})\tau_\alpha f(\beta)\tau_\alpha f(\theta)]$ $= [g((\beta\theta)^{-1})\tau_\alpha f(\beta\theta)] = \delta(\bar\beta\bar\theta)$ as required. $\square$

Remark 4.7. Note on the description of δ as a Bockstein. If we apply Proposition 2.1 to Diagram (1), we obtain the following grid

$$
\begin{array}{cccccccc}
1 \to & \mathrm{Coin}(f', g') & \xrightarrow{\hat i_1} & \mathrm{Coin}(f, g) & \xrightarrow{\hat p_1} & \mathrm{Coin}(\bar f, \bar g) & & \\
& \downarrow & & \downarrow & & \downarrow & & \\
1 \to & H_1 & \xrightarrow{i_1} & G_1 & \xrightarrow{p_1} & \bar G_1 & \to 1 & \\
& \downarrow f' \cdot^{-1} g' & & \downarrow f \cdot^{-1} g & & \downarrow \bar f \cdot^{-1} \bar g & & \\
1 \to & H_2 & \xrightarrow{i_2} & G_2 & \xrightarrow{p_2} & \bar G_2 & \to 1 & \\
& \downarrow & & \downarrow & & \downarrow & & \\
& \mathcal{R}(f', g') & \xrightarrow{\hat i_2} & \mathcal{R}(f, g) & \xrightarrow{\hat p_2} & \mathcal{R}(\bar f, \bar g) & \to 1, &
\end{array}
$$

in which both the vertical and horizontal sequences are short exact. That δ exists and is well defined is essentially a non-Abelian version of the well known "snake lemma" for Abelian groups and homomorphism (see [40]).

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