

# Exact sequences of fibrations of crossed complexes, homotopy classification of maps, and nonabelian extensions of groups\*

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Bangor Math. Preprint 08.04

## Abstract

The classifying space of a crossed complex generalises the construction of Eilenberg-Mac Lane spaces. We show how the theory of fibrations of crossed complexes allows the analysis of homotopy classes of maps from a free crossed complex to such a classifying space. This gives results on the homotopy classification of maps from a *CW*-complex to the classifying space of a crossed module and also, more generally, of a crossed complex whose homotopy groups vanish in dimensions between 1 and  $n$ . The results are analogous to those for the obstruction to an abstract kernel in group extension theory.

## Introduction

A major result in the theory of crossed complexes is the homotopy classification theorem of [BH91]:

$$[\Pi X_*, C] \cong [X, BC] \quad (\text{homclass})$$

where:

- on the left,  $\Pi X_*$  is the fundamental crossed complex of the skeletal filtration of the *CW*-complex  $X$ , and square brackets denote homotopy classes in the category  $\text{Crs}$  of crossed complexes, and
- on the right,  $BC$  is the classifying space of the crossed complex  $C$ , and square brackets denote homotopy classes of maps of topological spaces.

Results of Eilenberg–Mac Lane, including the case when  $C$  is just a group, and also the case of a local coefficient system, are obtained by giving special cases for the crossed complex  $C$ . A new application is to the case  $C$  is a crossed module, and this is exploited in, for example, [FK08, Far08, PT07].

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\*MSClassification2000: 13D02,18G50,20J05,55S37,55S45; KEYWORDS: homotopy classification, crossed complexes, fibrations, exact sequences.

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The purpose of this paper is to show how this homotopy classification can be analysed using the notion of exact sequence of a fibration of crossed complexes. In this way we show how crossed complexes give a convenient common setting for the homotopy classification of maps to an  $n$ -aspherical space  $Y$  (i.e. one in which  $Y$  is path connected and  $\pi_i Y = 0$  for  $1 < i < n$ ) as originally found by Olum in [Olu53], and for the classical theory of nonabelian extensions of groups and abstract kernels, [ML63].

An example of the convenience of setting of crossed complexes in these areas is that the traditional Schreier notion of factor set for an extension  $1 \rightarrow K \rightarrow E \rightarrow G \rightarrow 1$  of groups is conveniently formulated as a morphism of crossed complexes  $F^{st}(G) \rightarrow \text{AUT}(K)$  from the standard free crossed resolution  $F^{st}(G)$  of  $G$  to the crossed module  $\text{AUT}(K) := (K \rightarrow \text{Aut}(K))$ , and equivalence of factor sets is just homotopy of morphisms. By replacing  $F^{st}(G)$  by an equivalent free crossed resolution, we can get calculations, as exploited in [BP96]. That exposition also shows the use of extensions of the type of a crossed module, due originally to Dedecker, [Ded58]. This relates to work of [BM94].

The techniques involve the closed monoidal structure on the category of crossed complexes from [BH87], the exact sequences of a fibration of crossed complexes established by Howie in [How79], and the model category properties of crossed complexes from [BG89, BH91]. For more on model categories, see [Hov99].

This paper should also be seen as a modern development of pioneering work of J.H.C. Whitehead in [Whi49], which in some ways goes further than the work of Olum published later. Whitehead uses the term ‘homotopy system’ for what we call ‘reduced, free crossed complex’. These ideas are exploited by Ellis in [Ell88]. Although Whitehead does not have the notion of classifying space, he does explain his view that, in our terminology, crossed complexes have better realisation properties than chain complexes with a group of operators.

We are thus confirming that crossed complexes, though only a linear model of homotopy types, are convenient in that border between homology and homotopy which includes the Relative Hurewicz Theorem, [BH81], the Homotopy Addition Lemma [BS07], computation with crossed modules [BW95], and this level of the homotopy classification of maps.

## 1 Crossed complexes

A crossed complex  $C$  is in part a sequence of the form

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

where all the  $C_n, n \geq 1$  are groupoids over  $C_0$ . The structure and axioms for a crossed complex are those universally satisfied by the main topological example, the *fundamental crossed complex*  $\Pi X_*$  of a filtered space  $X_*$ , where  $(\Pi X_*)_1$  is the fundamental groupoid  $\pi_1(X_1, X_0)$  and  $(\Pi X_*)_n$  is the family of relative homotopy groups  $\pi_n(X_n, X_{n-1}, x_0)$  for all  $x_0 \in X_0$ .

The fundamental groupoid of the crossed complex  $C$  is  $\pi_1 C = \text{Coker } \delta_2$  and the homology groups of  $C$  are for  $n \geq 2$  the families of abelian groups

$$H_n(C, x) = (\text{Ker } \delta_n(C_n(x) \rightarrow C_{n-1}(x)))/(\delta_{n+1} C_{n+1}(x)) \quad x \in C_0.$$

The crossed complex  $C$  is *aspherical* if it is connected (i.e. the groupoid  $C_1$  is connected (also called transitive)) and  $H_n(C, x) = 0$  for all  $n \geq 2, x \in C_0$ .

We assume the basic facts on crossed complexes as surveyed in for example [Bro04, Bro99]. In particular, we will use the monoidal closed structure on the category  $\text{Crs}$  with an exponential law of the form

$$\text{Crs}(A \otimes B, C) \cong \text{Crs}(A, \text{CRS}(B, C)),$$

for crossed complexes  $A, B, C$ , and also in the pointed form

$$\text{Crs}_*(A \otimes_* B, C) \cong \text{Crs}_*(A, \text{CRS}_*(B, C))$$

for pointed crossed complexes  $A, B, C$ , both of which are constructed in [BH87]. Here the elements of  $\text{CRS}(B, C)_0$  are morphisms  $B \rightarrow C$ , the elements of the groupoid  $\text{CRS}(B, C)_1$  are (left) homotopies between these morphisms, and the further elements of  $\text{CRS}(B, C)$  are forms of higher homotopies.

We shall assume the notion of a relatively free morphism  $i : A \rightarrow F$  of crossed complexes from [BH91] and the result there that if  $i$  as above is relatively free, and  $C$  is free, then  $i \otimes 1_C : A \otimes C \rightarrow F \otimes C$  is relatively free. See also [BG89].

A full exposition of the theory of crossed complexes will be given in [BHS08].

## 2 Fibrations of crossed complexes

A model category structure on the category  $\text{Crs}$  has been studied by Brown and Golasiński in [BG89], exploiting the notion of fibration of crossed complexes as defined in [How79]. We recall some of this material, but with a slightly different emphasis.

Recall that a morphism  $p : E \rightarrow B$  of groupoids is a *fibration (covering morphism)* if it is star surjective (bijective) [Bro06]. The extension to crossed complexes is quite simple (covering morphisms of crossed complexes are applied in [BRS99]).

**Definition 2.1** A morphism  $p : E \rightarrow B$  of crossed complexes is a *fibration* if

- (i) the morphism  $p_1 : E_1 \rightarrow B_1$  is a *fibration* of groupoids;
- (ii) for each  $n \geq 2$  and  $x \in E_0$ , the morphism of groups  $p_n : E_n(x) \rightarrow B_n(px)$  is surjective.

The morphism  $p$  is a *trivial fibration* if it is a fibration, and also a weak equivalence, by which is meant that  $p$  induces a bijection on  $\pi_0$  and isomorphisms  $\pi_1(E, x) \rightarrow \pi_1(B, px)$ ,  $H_n(E, x) \rightarrow H_n(B, px)$  for all  $x \in E_0$  and  $n \geq 2$ .  $\square$

We now follow model category ideas as in [BG89].

**Definition 2.2** Consider the following diagram.

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow i & \nearrow & \downarrow p \\ C & \longrightarrow & B. \end{array}$$

If given  $i$  the dotted completion exists for all morphisms  $p$  in a class  $\mathcal{F}$ , then we say that  $i$  has the *left lifting property (LLP)* with respect to  $\mathcal{F}$ . We say a morphism  $i : A \rightarrow C$  is a *cofibration* if it has the LLP with respect to all trivial fibrations. We say a crossed complex  $C$  is *cofibrant* if the inclusion  $\emptyset \rightarrow C$  is a cofibration.  $\square$

Here is an important example of a cofibration, [BG89, Corollary 2.4]. We give the proof for the convenience of the reader.

**Proposition 2.3** *Let  $i : A \rightarrow F$  be a relatively free crossed complex. Then  $i$  is a cofibration.*

**Proof** We consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & E \\ i \downarrow & \nearrow g & \downarrow p \\ F & \xrightarrow{f} & B \end{array}$$

in which  $p$  is supposed a trivial fibration, and the morphisms  $f, \alpha$  satisfy  $fi = p\alpha$ . We construct the regular completion  $g$  on a relatively free basis  $X$  of  $F$  by induction.

For  $n = 0$ , we just lift a point in  $B$  to a point in the corresponding component in  $E$ . This defines  $g^0$  on  $A \cup X^0$ .

For the case  $n = 1$ , consider a basis element  $x \in X^1(a, b)$ . Then  $f(x) \in B(fa, fb)$ . Then  $g^0a, g^0b$  belong to the same component of  $E$ , by the condition on  $\pi_0$ , and so there is an element  $e \in E(g^0a, g^0b)$ . So  $fx - pe$  is a loop in  $B_1(g^0a)$ . By the condition for  $p$  on  $\pi_1$ , there is a loop  $e_1 \in E_1(g^0a)$  such that  $pe_1$  is equivalent to  $fx - pe$ , i.e.  $pe_1 = fx - pe + \delta_2b_2$  for some  $b_2 \in B_2(fa)$ . By the fibration condition,  $b_2 = pe_2$  for some  $e_2 \in E_2(g^0a)$ . Then  $p(e_1 + \delta_2e_2 + e) = fx$ . So we can choose  $g^1x = e_1 + \delta_2e_2 + e$  to obtain an extension on  $x$ .

Suppose  $n \geq 2$  and  $g$  is defined on  $X^{n-1}$ . Consider an element  $x$  of the free basis in dimension  $n$ . Then  $g\delta x$  is defined and  $pg\delta x = f\delta x$ .

By the fibration condition, we can choose  $e_n \in E_n$  such that  $pe_n = fx$ . Let  $w = g\delta x - \delta e_n \in E_{n-1}$ . Then  $pw = 0, \delta w = 0$ . By the triviality condition,  $w$  is a boundary, i.e.  $w = \delta z$  for some  $z \in E_n$ . Then  $\delta(z + e_n) = g\delta x$ . So we can extend  $g$  by defining it on  $x$  to be  $z + e_n$ .  $\square$

**Definition 2.4** We write  $\mathbb{C}(n)$  for the free crossed complex on one generator  $c^n$  of dimension  $n$ , and  $\mathbb{S}^{n-1}$  for the subcomplex of  $\mathbb{C}(n)$  generated by the elements of dimension  $(n - 1)$ . Thus  $\mathbb{C}(1)$  is essentially a groupoid, often written  $\mathcal{I}$ , which, with the inclusions  $0, 1 \rightarrow \mathbb{C}(1)$ , is a unit interval object in the category of crossed complexes, [KP97];  $\mathbb{C}(n)$  is a model of the  $n$ -disc, and  $\mathbb{S}^{n-1}$  is a model of the  $(n - 1)$ -sphere.

The following is [BG89, Proposition 2.1.] with a different proof.

**Proposition 2.5** *The following are equivalent for a morphism  $p : E \rightarrow B$  in Crs:*

- (i)  $p$  is a trivial fibration:

- (ii)  $p_0$  is surjective; if  $e, e' \in E_0$  and  $b_1 \in B_1(p_0e, p_0e')$ , then there is  $e_1 \in E_1(e, e')$  such that  $p_1e_1 = b_1$ ; if  $n \geq 1$  and  $e \in E_n$  satisfies  $\delta^0e = \delta^1e$  for  $n = 1$ ,  $\delta e = 0$  for  $n \geq 2$ , and  $b \in B_{n+1}$  satisfies  $\delta b = p_n e$ , then there is

$$e' \in E_{n+1} \quad \text{such that} \quad p_{n+1}e' = b \quad \text{and} \quad \delta e' = e;$$

- (iii)  $p$  has the RLP with respect to  $\mathbb{S}(n-1) \rightarrow \mathbb{C}(n)$  for all  $n \geq 0$ ;
- (iv) if  $F$  is a crossed complex of free type then  $p$  has the RLP with respect to  $\mathbb{S}(n-1) \otimes F \rightarrow \mathbb{C}(n) \otimes F$  for all  $n \geq 0$ ;
- (v) if  $F$  is a crossed complex of free type then the induced morphism  $p_* : \text{CRS}(F, E) \rightarrow \text{CRS}(F, B)$  is a trivial fibration.

**Proof** The proof that (i), (ii), (iii) are equivalent follows easily from the definition and Proposition 2.3.

Further (iii) implies (iv) since we know that under the condition of (iv),  $\mathbb{S}(n-1) \otimes C \rightarrow \mathbb{C}(n) \otimes C$  is relatively free [BH91, Proposition 5.1]. Finally (iv) trivially implies (iii).  $\square$

We also need a pointed version of (iv) of the previous proposition.

**Proposition 2.6** *If  $F, E, B$  are pointed reduced crossed complexes with  $F$  of free type, and  $p : E \rightarrow B$  is a trivial fibration, then so also is*

$$p_* : \text{CRS}_*(F, E) \rightarrow \text{CRS}_*(F, B).$$

**Proof** This relies on the pointed exponential law, [BH87], and the clear fact that  $\mathbb{S}^{n-1} \otimes_* F \rightarrow \mathbb{C}(n) \otimes_* F$  is relatively free, which follows from methods analogous to those of [BH91].  $\square$

**Definition 2.7** If  $G$  is a groupoid, we write  $\mathbb{K}(G, 1)$  for the crossed complex which is  $G$  in dimension 1 and trivial elsewhere. Thus  $\mathbb{K}(G, 1)$  is certainly aspherical.  $\square$

**Proposition 2.8** *Let  $F, C$  be reduced crossed complexes with  $F$  free and  $C$  aspherical. Let  $G = \pi_1(C)$ . Then there are bijections*

$$\pi_0 \text{CRS}_*(F, C) = [F, C]_* \cong [F, \mathbb{K}(G, 1)]_* \cong \text{Hom}(\pi_1 F, \pi_1 C),$$

and for all  $f : F \rightarrow C$  and  $n \geq 1$  we have

$$\pi_n(\text{CRS}_*(F, C), f) = 0.$$

**Proof** Let  $G = \pi_1 C$ . Since  $C$  is aspherical, the natural morphism  $p : C \rightarrow \mathbb{K}(G, 1)$  is not only a fibration but also a weak equivalence of crossed complexes. It follows that  $p_* : \text{CRS}_*(F, C) \rightarrow \text{CRS}_*(F, \mathbb{K}(G, 1))$  is a trivial fibration and so a weak equivalence. In particular,  $p_*$  induces a bijection of  $\pi_0$ . This gives the first result, since  $\pi_0 \text{CRS}_*(F, \mathbb{K}(G, 1))$  is clearly bijective with  $\text{Hom}(\pi_1 F, G)$ .

The second result follows, since all homotopies and higher homotopies  $F \rightarrow \mathbb{K}(G, 1)$  are trivial.  $\square$

**Remark 2.9** The classical homological algebra type of inductive proof is somehow hidden in this proof.  $\square$

Howie states in [How79] that a fibration  $p : E \rightarrow B$  of crossed complexes yields a family of exact sequences involving the  $H_n$ ,  $\pi_1$  and  $\pi_0$ , akin to the well known family of homotopy exact sequences of a fibration of spaces, or of groupoids; for the latter see [Bro06, Chapter 7]. The standard properties of these sequences are:

- dependency on base points;
- non abelian features in dimension 1;
- sets with base point in dimension 0 but with some useful information obtainable from operations.

Let  $x \in E_0$  and let  $\mathcal{F}_x = p^{-1}(px)$  be the sub crossed complex of  $E$  consisting of all elements of  $E_0$  which map by  $p$  to  $x$  and of all elements of some  $E_n, n \geq 1$ , which map by  $p$  to the identity at  $px$ .

**Theorem 2.10** *There is an exact sequence*

$$\begin{aligned} \cdots \rightarrow H_n(\mathcal{F}_x, x) &\xrightarrow{i_n} H_n(E, x) \xrightarrow{p_n} H_n(B, px) \xrightarrow{\partial_n} \cdots \\ \cdots \rightarrow \pi_1(\mathcal{F}_x, x) &\xrightarrow{i_1} \pi_1(E, x) \xrightarrow{p_1} \pi_1(B, px) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_x) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B). \end{aligned}$$

Here the terms of the sequence are all groups, except the last three which are sets with base points the classes  $x_{\mathcal{F}}, x_E, x_B$  of  $x, x, px$  respectively.

- (i) *There is an operation of the group  $\pi_1(E, x)$  on the group  $\pi_1(\mathcal{F}_x, x)$  making the morphism*

$$i_1 : \pi_1(\mathcal{F}_x, x) \rightarrow \pi_1(E, x)$$

*into a crossed module.*

- (ii) *There is an operation of the group  $\pi_1(B, px)$  on the set  $\pi_0(\mathcal{F}_x)$  such that the boundary  $\pi_1(B, px) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_x)$  is given by  $\partial_1(\alpha) = \alpha \cdot x_{\mathcal{F}}$ .*

*Further we have additional exactness at the bottom end as follows:*

- (a)  $\partial_1\alpha = \partial_1\beta$  *if and only if there is a  $\gamma \in E(x)$  such that  $p_1\gamma = -\beta + \alpha$ ;*  
 (b) *if  $\bar{u}$  denotes the component in  $\mathcal{F}_x$  of an object  $u$  of  $\mathcal{F}_x$ , then  $i_*\bar{u} = i_*\bar{v}$  if and only if there is an  $\alpha \in B(y)$  such that  $\alpha_{\#}\bar{u} = \bar{v}$ ;*  
 (c) *if  $\hat{y}$  denotes the component of  $y$  in  $B$  then*

$$i_*[\pi_0\mathcal{F}_x] = p_*^{-1}[\hat{y}].$$

**Proof** The proof of this theorem is a development of the part of the theorem which deals with fibrations of groupoids and which is given for example by Brown in [Bro06]. We leave the details as an exercise.  $\square$

### 3 Homotopy classification of maps to an $n$ -aspherical space

We have already analysed a simple case of  $[F, C]_*$  in Proposition 2.8. We now proceed to some slightly more complicated examples, using the exact sequences of a fibration. We also concentrate on the pointed homotopy classification, since this is nearest to classical results on group extensions, but the unpointed case is handled similarly.

**Definition 3.1** A space  $X$  is called  $n$ -aspherical if it is path-connected and  $\pi_i(X) = 0$  for  $1 < i < n$ . Thus any path-connected space is 2-aspherical. Analogous terms are applied also to crossed complexes.

**Definition 3.2** For a group or groupoid  $Q$ ,  $Q$ -module  $A$ , and  $n \geq 2$ , let  $\mathbb{K}(Q, 1; A, n)$  denote the crossed complex which is  $Q$  in dimension 1,  $A$  in dimension  $n$ , with the given action of  $Q$ , and all boundaries are trivial.  $\square$

**Remark 3.3** Let  $F$  be a crossed complex. We will usually write  $\phi : F_1 \rightarrow \pi_1 F$  for the quotient morphism. Then for any morphism  $\theta : \pi_1 F \rightarrow Q$  the composite  $\theta\phi$  is completely determined by  $\theta$ . However sometimes we start with a group (or groupoid)  $\Phi$  and choose what we have called a free crossed resolution  $F$  of  $\Phi$ . This is an aspherical free crossed complex together with a choice of isomorphism  $\pi_1 F \rightarrow \Phi$ , or, equivalently, with a quotient morphism  $\phi : F_1 \rightarrow \Phi$  with kernel the image of  $\delta_2 : F_2 \rightarrow F_1$ . In such case  $\phi$  is not determined by  $F$ .

**Proposition 3.4** *Let  $p : E \rightarrow B$  be a morphism of crossed complexes. Then  $p$  is a fibration if and only if for any free crossed complex  $F$ , the induced morphism  $p_* : \text{CRS}(F, E) \rightarrow \text{CRS}(F, B)$  is a fibration. If further  $p$  and  $F$  are pointed, then the induced morphism of pointed internal homs  $p_* : \text{CRS}_*(F, E) \rightarrow \text{CRS}_*(F, B)$  is a fibration of crossed complexes.*

**Proof** The forward implication follows easily from the exponential law, as in the proof of Proposition 2.5.

To prove it in the other direction, one takes again  $F$  to be  $\mathbb{C}(n)$ , the free crossed complex on one generator of dimension  $n$ .  $\square$

**Definition 3.5** Let  $F, C$  be crossed complexes, let  $A$  be a subcomplex of  $F$ , with inclusion  $i : A \rightarrow F$  and let  $f : A \rightarrow C$  be a morphism. We write  $[F, C; f]$  for the set of homotopy classes rel  $A$  of morphisms  $F \rightarrow C$  which extend  $f$ . We write similarly  $[F, C; f]_*$  for the pointed homotopy classes in the case  $A, F, C, i$  are pointed.

**Theorem 3.6** *Let  $F$  be a reduced free crossed complex and let  $\Phi = \pi_1 F$ . Then  $[F, \mathbb{K}(Q, 1; A, n)]_*$  is the disjoint union of sets  $[F, \mathbb{K}(Q, 1; A, n) : \theta\phi]_*$  one for each morphism  $\theta : \Phi \rightarrow Q$ , namely those homotopy classes inducing  $\theta$ . Further, the morphisms  $F \rightarrow \mathbb{K}(Q, 1; A, n)$  inducing  $\theta : \Phi \rightarrow Q$  may be given the structure of abelian group which is inherited by homotopy classes.*

**Proof** The morphism  $q : \mathbb{K}(Q, 1; A, n) \rightarrow \mathbb{K}(Q, 1)$  which is the identity in dimension 1 and 0 elsewhere is a fibration inducing a fibration  $q_* : \text{CRS}_*(F, \mathbb{K}(Q, 1; A, n)) \rightarrow \text{CRS}_*(F, \mathbb{K}(Q, 1))$ . The induced map on  $\pi_0$  is surjective since every morphism  $f : F \rightarrow \mathbb{K}(Q, 1)$  may be lifted by 0 to a morphism





**Proof** Consider the morphisms of crossed complexes  $C \xrightarrow{j} \xi C \xrightarrow{p} \zeta C$  as shown in the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & & C \\
& & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow j \\
\cdots & \longrightarrow & A & \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & & \xi C \\
& & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow p \\
\cdots & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & Q & & \zeta C
\end{array}$$

Then  $\xi C$  is aspherical,  $\zeta C = \mathbb{K}(Q, 1; A, n + 1)$ , and  $p : \xi C \rightarrow \zeta C$  is a fibration of crossed complexes.

Since  $F$  is a free reduced crossed complex, we have an induced fibration of crossed complexes

$$p_* : \text{CRS}_*(F, \xi C) \rightarrow \text{CRS}_*(F, \zeta C). \quad (2)$$

On applying  $\pi_0$  to this we get, considering previous identifications, a map of sets

$$p_* : \text{Hom}(\Phi, Q) \rightarrow \bigsqcup_{\theta \in \text{Hom}(\Phi, Q)} H_{\theta\phi}^{n+1}(F, A). \quad (3)$$

**Lemma 3.11** *A morphism  $\theta : \Phi \rightarrow Q$  maps to 0 in  $H_{\theta\phi}^{n+1}(F, A)$  if and only if  $\theta$  is induced by a morphism  $F \rightarrow C$ .*

**Proof** Suppose  $\theta$  is induced by a morphism  $f : F \rightarrow C$ . Then  $f$  factors through  $pj$  and is therefore 0 in  $H_{\theta\phi}^{n+1}(F, A)$ .

Suppose conversely that  $\theta$  determines 0 in  $H_{\theta\phi}^{n+1}(F, A)$ . We know that  $\theta$  is induced by a morphism  $f' : F \rightarrow \xi C$ . Then  $pf'$  is homotopic to 0 and so by the fibration condition  $f'$  is homotopic to  $f''$  such that  $pf'' = 0$ . Hence  $f''$  determines  $f : F \rightarrow C$  such that  $jf = f''$ . Then  $f$  also induces  $\theta$ . This proves the lemma.  $\square$

Let  $\mathcal{F}(f)$  denote the fibre of  $p_*$  over  $pf$ . Then we have an exact sequence

$$\rightarrow \pi_1(\text{CRS}_*(F, \xi C), f) \rightarrow \pi_1(\text{CRS}_*(F, \zeta C), pf) \rightarrow \pi_0 \mathcal{F}(f) \rightarrow \pi_0 \text{CRS}_*(F, \xi C) \rightarrow \pi_0 \text{CRS}_*(F, \zeta C).$$

By Proposition 2.8,  $\pi_1(\text{CRS}_*(F, \xi C), f) = 0$ , the above sequence translates to

$$0 \rightarrow H_{\theta\phi}^n(F, A) \rightarrow [F, C; \theta\phi] \rightarrow \text{Hom}(\Phi, Q).$$

Further we have a free action of the abelian group  $H_{\theta\phi}^n(F, A)$  on the set  $[F, C]_{\theta\phi}$ . This completes the proof of the theorem.  $\square$

This result generalises the classical theory of extensions of groups and  $Q$ -kernels. To apply the theory to that case, the crossed complex  $F$  is taken to be a free crossed resolution of the group  $G$ . If  $F$  is the standard free crossed resolution of  $G$ , then the relation with factor systems is shown in [BP96]. The advantage of this approach is that it is clear that the standard free crossed resolution may be replaced by any free crossed resolution of  $G$ , and in many cases it is possible to construct small such resolutions. Then we may get a finite description of the classes of extensions.

## 4 Applications to spaces

For the applications to spaces we need to know when a space is of the homotopy type or homotopy  $n$ -type of  $BC$  for some crossed complex  $C$ . A key result is the following:

**Theorem 4.1** *Let  $Y_*$  be the skeletal filtration of a CW-complex  $Y$ . Then there is 1-equivalence  $q : Y \rightarrow B\Pi Y_*$  such that if  $y \in Y_0$  then the exact sequence of the homotopy fibre over  $y$  is equivalent to Whitehead's exact sequence:*

$$\cdots \rightarrow \Gamma_n(Y, y) \rightarrow \pi_n(Y, y) \xrightarrow{\omega} H_n(\tilde{Y}_y) \rightarrow \cdots \rightarrow \Gamma(\pi_2(Y, y)) \rightarrow \pi_3(Y, y) \xrightarrow{\omega} H_3(\tilde{Y}_y) \rightarrow 0$$

where  $\omega$  is the Hurewicz morphism.

**Proof** The original form of this comes from [BH81, section 8], but in essence used a cubical version of the classifying space. The simplicial version is dealt with in [Ash88].  $\square$

**Corollary 4.2** *If further  $Y$  is  $n$ -aspherical, then the homotopy fibre of  $q : Y \rightarrow B\Pi Y_*$  is  $n$ -connected.*

**Remark 4.3** Whitehead in [Whi49] gives an example of a 5-dimensional free crossed complex which is not isomorphic to  $\Pi Y_*$  for any CW-filtration  $Y_*$ . He also introduces the condition  $J_n$  on a CW-filtration; this generalises the condition of  $n$ -asphericity.  $\square$

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