Poly T-Complexes

By David W. Jones

Abstract: A simplicial set may be defined as a contravariant functor from the simplicial *model category* to the category of sets. This thesis develops a class Γ of polyhedral model categories. For a category M in Γ, an M-set is a functor $M^{\text{op}} \to \mathbf{Set}$. The development of Γ is motivated by the possibility of studying *T-complex* structures on M-sets.

In order to define Γ we introduce the category of *cone-complexes*, which are regular CW-complexes made more rigid. The addition of a structure of *marked faces* to a closed cell of a cone-complex gives a *polycell*, which is analogous to an ordered simplex of the simplicial model category. We take Γ to be a class of full subcategories of the category of polycells.

A *shellability* condition on polycells is used to define a subclass $E\Gamma$ of Γ. Each member of $E\Gamma$ is isomorphic to a category of posets with extra structure and is thus combinatorial in nature.

A *simplicial T-complex* is a simplicial set K with special elements (referred to as *thin*) in each dimension satisfying Dakin's axioms:

(T1) All degenerate elements of K are thin.

(T2) Every box has a unique thin filler.

(T3) If all faces but one of a thin element are thin, then so is the remaining face.

For M a member of Γ, an MT-complex may be defined using these axioms. We prove that, for M in $E\Gamma$, there is an equivalence of categories $MT-$complexes $\to$ simplicial $T-$complexes. Since $E\Gamma$ is infinite, this gives a rare example of an infinite class of non-trivially equivalent algebraic categories. Ashley has constructed an equivalence between simplicial T-complexes and the important category of *crossed complexes*, studied recently by Brown and Higgins.

We also show that a T-complex structure on an M-set defines a canonical degeneracy structure. This is of use in defining a functor from simplicial T-complexes to cubical T-complexes which we claim is an equivalence of categories.

Keywords: Categories, Model Categories, Cone-complexes, Polycells, Shelling, Collapsing, Equivalences of Categories, T-complexes, degeneracy structures.

POLY T-COMPLEXES

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DECLARATION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

R. Brown

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ABSTRACT

A simplicial set may be defined as a contravariant functor from the simplicial *model category* to the category of sets. This thesis develops a class Γ of polyhedral model categories. For a category M in Γ, an M-*set* is a functor $M^{op} \to$ Set. The development of Γ is motivated by the possibility of studying T-*complex* structures on M-sets.

In order to define Γ we introduce the category of *cone-complexes*, which are regular CW-complexes made more rigid. The addition of a structure of *marked faces* to a closed cell of a cone-complex gives a *polycell*, which is analogous to an ordered simplex of the simplicial model category. We take Γ to be a class of full subcategories of the category of polycells.

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CONTENTS

INTRODUCTION

The use of simplicial methods in algebraic topology, and in many of its applications, is well known (Gabriel-Zisman [37], May [32], Lamotke [40]). There is also some use in the literature of cubical methods, particularly the singular cubical complex (Massey [31], Federer [23], Kan [38]; also Adams-Hilton [1], Chen [17], Kamps [29]). The object of this thesis is to *develop as wide a generalization as seems reasonable of these methods, by considering as basic models members of a class of polyhedra which includes cubes, simplices and products of these.*

Partial motivation is provided by the problem of axiomatising those aspects of simplices and cubes which make them satisfactory as basic models in algebraic topology. As we shall see (Chapter II), this leads to a class of posets which we call C-*posets*. Remarkably, such posets have been studied independently by A. Bjorner [5], who was led to them for purely combinatorial reasons.

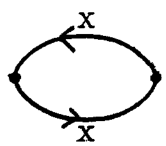
Previous extensions of simplicial or cubical theory of which we are aware are as follows:

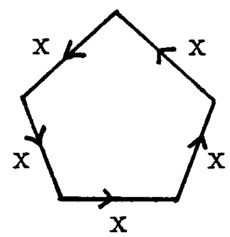
(i) Gug enheim [25] considers 'supercomplexes', for which the models are products $\Delta^{p_1} \times \Delta^{p_2} \times \ldots \times \Delta^{p_r}$ of simplices.

(ii) Hintze's 'polysets' [26] have elements modelled on the objects of the minimal geometric category which includes both the join of an object to a point and the product of an object with the unit interval.

(iii) Evrard [21] considers 'Γ-sets' for which the models are essentially triangulated cubes.
However, as far as we know, this thesis is the first wide study of its type.

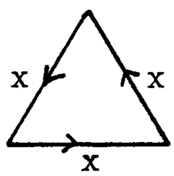
Initial motivation came also from combinatorial group theory, particularly from the notion of a *van Kampen diagram* (Chapter V). Generators of a group can be modelled by edges, and this gives a *Cayley diagram*. For modelling relations, faces are required and in particular an n-gon is needed to model $x^n = 1$ $(n \geq 2)$. But once diagrams such as





are considered there arises the problem of what range of geometric gadgets should be allowed. We are led to what we call *cone-complexes* and, among these, the *cone-cells*. However, these gadgets are sometimes too general. We are interested in certain kinds of Kan complexes, so that our models must have certain collapsing properties. The required properties turn out to be conveniently described in terms of *shellability*.

A further point is that $x^3 = 1$ is modelled by



which, while it is a 2-simplex, is not an *ordered* 2-simplex. This raises the question of generalizing the ordering of vertices which is so successful in simplicial complexes. A structure of *marked faces* in a cone-complex is found to work well. We define a category Poly of *polycells*, that is, cone-cells with marked faces. There is an infinite class Γ of subcategories of Poly which can be used as *model categories*. The members of Γ give rise to categories of 'poly-sets'.

Our main test-bed for this theory is to give a satisfactory notion of 'poly T-complex'. A *simplicial (cubical)* T-*complex* is a simplicial (respectively cubical) Kan complex K with special elements in each dimension ≥ 1 . These elements are called *thin* and satisfy the following axioms:

(T1) Every degenerate element of K is thin.
(T2) Every box in K has a unique thin filler.
(T3) If all faces but one of a thin element of K are thin then so is the remaining face.

The notion of a simplicial T-complex was found by M.K. Dakin [19]. The cubical version was taken up by R. Brown and P.J. Higgins [10, 11, 12] and plays an essential part in their proof of a higher-dimensional form of the Seifert-van Kampen theorem. Together with *crossed complexes* [3, 10; also 8, 14, 15] , ω-*groupoids* [10], and ∞-*groupoids* [13], simplicial and cubical T-complexes make up a set of five non-trivially equivalent algebraic categories. In fact, each structure may be regarded as a version of 'higher-dimensional group theory' [7]. The question arises: can the axioms above be used to define poly T-complexes such that there is an equivalence poly T-complexes $\rightarrow$ simplicial T-complexes? We construct an infinite class of such T-complexes.

Poly-sets have no degeneracies. Thus poly-sets form a generalization not of simplicial and cubical set but of the 'Δ-sets' and the '□-sets' of Rourke and Sanderson [33] and Hintze [26]. We consider how to introduce degeneracies and give one possible solution to the problem. Further, we show that certain poly T-complexes have a canonical degeneracy structure. This sheds some light on the problem of constructing a direct equivalence of categories simplicial T-complexes → cubical T-complexes.

The thesis is laid out as follows.

Chapter I introduces the categories of cone-complexes and polycells. The class Γ of model categories is defined.

In Chapter II we use a shellability condition on polycells to define a subclass $E\Gamma$ of Γ used in Chapter III. The categories in $E\Gamma$ are shown to be isomorphic to categories of C-posets with extra structure.

Chapter III contains the definition of poly T-complexes and the proof of the equivalences poly T-complexes → simplicial T-complexes.

Degeneracy structures in poly T-complexes are studied in Chapter IV. We define a functor poly T-complexes → cubical T-complexes and a pair of functors simplicial T-complexes ⇄ cubical T-complexes which we claim are equivalences of categories.

Chapter V considers areas which require further work.

Finally, there is an Appendix devoted to shelling in cone-complexes.

CHAPTER I
A CLASS OF MODEL CATEGORIES

This chapter introduces the class Γ of model categories. The notion of a model category occurs in the Appelgate-Tierney theory of categories with models [2] . We first recount some basic ideas from this theory then proceed to develop the geometric category Poly and to define Γ as a class of sub-categories of Poly.

The first step in the development of Poly is to define the category of *cone-complexes*. A cone-complex is a regular CW-complex equipped with a *cone structure* on each cell. This structure is analogous to the affine structure on a simplex and cone-complex maps are rigid in the same way as simplicial maps.

On providing a cone-complex with a structure of *marked faces* we obtain a *marked cone-complex*. A *polycell* (a Poly-object) is a marked cone-cell. Choosing a marked face structure for a polycell is analogous to ordering the vertices of a simplex. Poly-morphisms preserve marked faces and are comparable to the vertex-order preserving maps of the geometric version of the usual simplicial model category Δ . However, we allow only injective morphisms in Poly so that the model categories in Γ are actually analogous to Δ_I , the wide subcategory of Δ with injective maps.

§1 Categories with models

Let M be a small category, that is, a category whose class of objects is a set.

<u>1.1 Definition</u> [2] A category A together with a functor $I: M \to A$ is called a *category with models.* M is called the *model category* for A .

For example, the simplicial category Δ with objects the sets $[m] = \{0, 1, \ldots, m\}$ and morphisms the increasing functions $[m] \to [n]$ is a model category for Top (the category of topological spaces and continuous maps). Define $I: \Delta \to \text{Top}$ by $I([n]) = \Delta^n$ where Δ^n is the standard geometric n-simplex, and, for $\alpha: [m] \to [n]$, $I(\alpha)$ = the uniquely determined affine map $\Delta^m \to \Delta^n$.

The geometric simplices may be thought of as local objects which can be pasted together by homeomorphisms to create global objects. (Compare this with the notion of triangulation of a manifold.) The objects of Δ therefore act as models for the local building blocks in Top .

<u>1.2 Definition</u> [2] Given a category A with models, the functor I defines a *singular functor* $s: A \to \text{Set}^{M^{op}}$ as follows. For X an object of A , $sX: M^{op} \to \text{Set}$ is given by

$$sX(m) = \text{Hom}_A(Im, X) \qquad m \in \text{Ob}(M) ,$$
$$sX(\alpha) = (I\alpha, A) \qquad \alpha \text{ a morphism in } M .$$

(If $\alpha: m \to n$, $(I\alpha, A)$ denotes the map $\text{Hom}_A(In,X) \to \text{Hom}_A(Im,X)$ defined by $u \to u \circ I\alpha$.)
For a morphism $f: X \to X'$ of A , $sf: sX \to sX'$ is the natural transformation induced by composition with f .

Going back to the example of $I: \Delta \to \text{Top}$, $\text{Set}^{\Delta^{op}}$ is the category of simplicial sets and $s: \text{Top} \to \text{Set}^{\Delta^{op}}$ is the usual singular functor of homology theory.

The pasting together mentioned above of local objects in the category A with models is carried out by a *realization functor* $\mathrm{Set}^{M^{op}} \to A$.

Let $F: M^{op} \to \mathrm{Set}$ and consider the category (Y,F) whose objects are pairs (m,x) where $m \in \mathrm{Ob}(M)$, $x \in F(m)$, and whose morphisms $(m,x) \to (m',x')$ are morphisms $\alpha: m \to m'$ in M such that $F\alpha(x') = x$. There is a functor $\partial_o:(Y,F) \to M$ given by $\partial_o(m,x) = m$, $\partial_o\alpha = \alpha$. Take the composite of ∂_o with I, $(Y,F) \xrightarrow{\partial_o} M \xrightarrow{I} A$, and put $rF = \varinjlim I \circ \partial_o$ (assuming A has small colimits). It can be shown that r is a functor $\mathrm{Set}^{M^{op}} \to A$.

<u>1.3 Definition</u> [2] The *realization functor* $\mathrm{Set}^{M^{op}} \to A$ is defined to be r.

The basic point about realization is the following:

<u>1.4 Proposition</u> [2] *The functors* r *and* s *are adjoint.* □

If A has a colimit - preserving underlying set functor $U: A \to \mathrm{Set}$ there is an explicit description of $U(rF) = \varinjlim U \circ I \circ \partial_o$. Consider the set $\bar{F}$ of all triples (m, x, k) where $(m,x) \in \mathrm{Ob}(Y,F)$ and $k \in U \circ I(m)$. Let $\equiv$ be the equivalence relation on $\bar{F}$ generated by the relation $(m,x,k) \sim (m',x',k')$ if and only if there exists $\alpha: (m,x) \to (m',x')$ in (Y,F) such that $U \circ I\alpha(k) = k'$. (Thus $(m, F\alpha(x'), k) \sim (m',x', U \circ I\alpha(k))$.) Let $|m,x,k|$ denote the equivalence class containing (m,x,k). The set $\tilde{F}$ of equivalence classes together with the family of functions

$$i(m,x) : U \circ I(m) \to \tilde{F}$$

given by $k \to |m,x,k|$

is a colimit of $U \circ I \circ \partial_0$.

This means that, in our simplicial example, $r\colon \mathrm{Set}^{\Delta^{op}} \to \mathrm{Top}$ is the well-known geometric realization of Milnor (see May [32] , p.55). The standard result that Milnor's realization functor is adjoint to the singular functor $\mathrm{Top} \to \mathrm{Set}^{\Delta^{op}}$ is thus a special case of Proposition 1.4.

§2 Cone-complexes

We follow Massey [31] in defining a CW-complex to be *regular* if, for each cell $\bar{e}_\lambda$, there exists a characteristic map $h_\lambda\colon B^{n_\lambda} \to \bar{e}_\lambda$ which is a homeomorphism. (Some authors, Lundell and Weingram [30] for example, merely suppose that there is *some* homeomorphism $B^{n_\lambda} \to \bar{e}_\lambda$.)

Essentially, we wish to construct model categories whose objects are regular complexes which are balls and have one top-dimensional cell. However, the regular complex structure is not combinatorial enough : in the category Reg of regular complexes and regular cellular maps ([30] , p.27) there are too many isomorphisms $X \to Y$ for isomorphic objects X and Y .

We can make Reg more combinatorial by rigidifying the morphisms. One way of doing this is to choose a particular characteristic map for each cell of a complex and require that morphisms preserve characteristic maps. A problem with this approach is that characteristic maps of a complex in various dimensions need not be related. Given regular complexes X and Y which are isomorphic in Reg , choices of characteristic maps can be made such that no Reg - isomorphism $X \to Y$ preserves characteristic maps for all $(n-1)$ - cells as well as all n-cells of X . The extra structure therefore leads to too many isomorphism classes. We avoid this difficulty by modifying the definition of a characteristic map.

<u>2.1 Definition</u> A *cone-complex* $\{X,\{h_\lambda\}_{\lambda\in\Lambda}\}$ is a Hausdorff space X and a decomposition $X = \bigcup_{\lambda\in\Lambda} e_\lambda$ of X as a disjoint union of subspaces e_λ such that e_λ is an open n_λ - cell. Let $X^n = \bigcup_{\substack{\lambda\in\Lambda \\ n_\lambda\leq n}} e_\lambda$ and $\partial e_\lambda = \bar{e}_\lambda - e_\lambda$ We require for all $\lambda \in \Lambda$:

CC1) ∂e_λ is a union of a finite number of open cells belonging to $X^{n_\lambda -1}$ and is homeomorphic to $S^{n_\lambda -1}$;

CC2) h_λ is a homeomorphism $C\partial e_\lambda \to \bar{e}_\lambda$ which is the identity on ∂e_λ . (Ce_λ denotes the topological cone $(\partial e_\lambda \times I) / (\partial e_\lambda \times \{1\})$.)

We call the maps h_λ *characteristic maps.*

A cone-complex obviously has a regular cell structure. The complex $\{X,\{h_\lambda\}_{\lambda\in\Lambda}\}$ will generally be denoted simply by X . Throughout, we consider only finite cone-complexes so that the term *cone-complex* implies a finite number of cells. The theory could of course be easily extended to the infinite case by imposing the usual conditions on the topology of X .

For $\{X,\{h_\lambda\}_{\lambda\in\Lambda}\}$ a cone-complex let Y be a non-empty subspace of X and let Φ be the set of $\lambda\in\Lambda$ such that the image of h_λ is contained in Y . If the maps h_λ , $\lambda \in \Phi$, are the characteristic maps for a cone-complex structure on Y we say $\{Y,\{h_\lambda\}_{\lambda\in\Phi}\}$ is a *subcomplex* of X . It follows immediately from Theorem III 2.1 of [30] that each closed cell of X is a subcomplex of X . We refer to a cone-complex which is a ball and has one top-dimensional cell as a *cone-cell.* Each closed cell of a cone-complex is a cone-cell.

<u>2.2 Definition</u> A *map*

$$f\colon \{X, \{h_\lambda\}_{\lambda\in\Lambda}\} \to \{Y, \{k_\mu\}_{\mu\in M}\}$$

of cone-complexes is a homeomorphism into $f\colon X \to Y$ of spaces such that if e_λ is a cell of X then $f(e_\lambda)$ is a cell, say e'_μ, of Y and the cone structure of e_λ is preserved by f; that is, the diagram

$$\begin{array}{ccc} C\partial e_\lambda & \xrightarrow{Cf|\partial e_\lambda} & C\partial e'_\mu \\ \cong\downarrow h_\lambda & & k_\mu\downarrow\cong \\ \bar{e}_\lambda & \xrightarrow{f} & \bar{e}'_\mu \end{array}$$

commutes.

The category of cone complexes and cone-complex maps will be denoted by CC .

<u>2.3 Proposition</u> *Let* X , Y *be cone-complexes. If* $f\colon X \to Y$ *is a regular homeomorphism into then there is a cone-complex map* $f'\colon X \to Y$ *such that* $f'(\bar{e}_\lambda) = f(\bar{e}_\lambda)$ *for each cell* e_λ *of* X . □

<u>2.4 Proposition</u> *Let* $f\colon X \to Y$ *be a cone-complex map. Then*

(i) $f(X)$ *is a subcomplex of* Y *;*

(ii) f *restricts to a cone-complex isomorphism* $X \to f(X)$ *;*

(iii) f *is completely determined by specifying to which cell of* Y *each cell of* X *is mapped.* □

Clearly, the category CC is a more combinatorial version of Reg , with rigid maps analogous to injective simplicial maps of simplicial complexes. In fact, the underlying polyhedron $|K|$ of a simplicial complex K has a canonical cone-complex structure. Each simplex σ of K is a polyhedral cone on $\partial\sigma$ with cone point the barycentre of σ so there is an obvious (cone-complex) characteristic map $C\partial\sigma \to \sigma$. We identify K with the cone-complex $|K|$. It is easily seen that, for simplicial complexes K and L, the injective simplicial maps $K \to L$ are precisely the cone-complex maps. We therefore have:

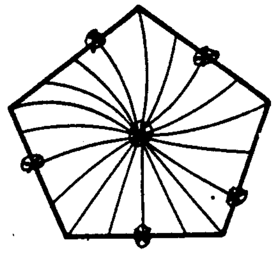
2.5 Proposition *The category of simplicial complexes and injective simplicial maps is a full subcategory of* CC . □

Let X be a cone-complex and let e_λ be a cell of X with characteristic map $h_\lambda\colon C\,\partial e_\lambda \to \bar{e}_\lambda$. Following the simplicial case, we call h_λ (cone point) the *barycentre* of e_λ. There is a notion of barycentric subdivision of X which coincides with the usual polyhedral definition when X is a simplicial complex.

2.6 Definition The *barycentric subdivision* SdX of X is the cone-complex defined as follows.

Let $SdX^o = X^o$. Assume $SdX^{(n-1)}$ has been given and let e_λ be an n-cell of X. Take the barycentre of e_λ to be an 0-cell of SdX^n and, for each k-cell e_μ of $Sd\partial e_\lambda$, let $h_\lambda(C\,\bar{e}_\mu)$ be a closed (k+1)-cell of SdX^n.

There is a canonical characteristic map for $h_\lambda(C\,\bar{e}_\mu)$ using the map $h_{C_\mu}\colon C\partial C\bar{e}_\mu \to C\bar{e}_\mu$ given in the next section.



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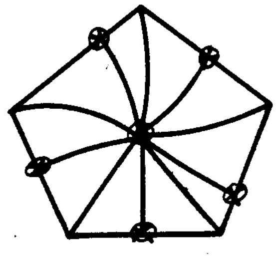


SdX

(barycentres denoted by ⊗)

The following result shows that a cone-complex is essentially only one subdivision away from a simplicial complex.

2.7 Proposition *If* X *is a cone-complex there is a simplicial complex* τX *which is* CC - *isomorphic to* SdX .

Proof (See [30], p. 80) Form the abstract simplicial complex whose vertex set is the set of barycentres of cells of X and whose k-simplices ($k \geq 0$) are members of the set

$$\{(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_k) \mid \hat{a}_i = \text{barycentre of the i-cell } a_i \text{ of } X ; \; a_i \neq a_j \text{ for } i \neq j ; \; \bar{a}_0 \subset \bar{a}_1 \subset \ldots \subset \bar{a}_k \}$$

Let τX be a geometric realization of this complex. A CC - isomorphism $\tau X \to SdX$ can be constructed by induction on the skeleta of X . □

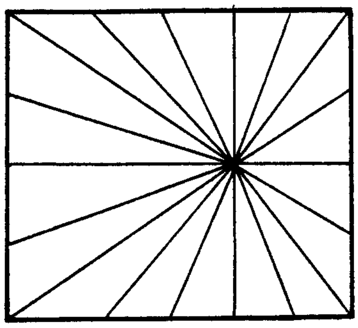
§3 Three standard constructions

Let X be a cone-complex. As for a CW-complex, we need to define standard cone-complex structures for $X \times I$ and CX . This is done below, where the more specialized dome construction is also considered.

In order to define the cone-complex characteristic maps for $X \times I$ we have to choose a homeomorphism $C\partial(\bar{e}_\lambda \times I) \to \bar{e}_\lambda \times I$ for each cell e_λ of X . Now $\bar{e}_\lambda$ has a cone structure with barycentre $\hat{e}_\lambda$. There is a canonical cone structure on $\bar{e}_\lambda \times I$ with cone point $(\hat{e}_\lambda , \frac{1}{2})$ such that:

(i) the rays containing $(\hat{e}_\lambda ,0)$ and $(\hat{e}_\lambda , 1)$ are $\hat{e}_\lambda \times [0, \frac{1}{2}]$ and $\hat{e}_\lambda \times [\frac{1}{2}, 1]$ respectively;

(ii) if r is any other ray and $p_0 : \bar{e}_\lambda \times I \to \bar{e}_\lambda$, $p_1 : \bar{e}_\lambda \times I \to I$ are the projection maps then $p_0(r)$ is contained in a ray of $\bar{e}_\lambda$ and p_1 maps r linearly.



We therefore have a canonical homeomorphism
$h_{\lambda I}\colon C\partial(\bar{e}_\lambda \times I) \to \bar{e}_\lambda \times I$.

<u>3.1 Definition</u> The *cylinder* $X \times I$ on the cone-complex X is defined to be the space $X \times I$ with the following cone-complex structure:

(i) the structure on $(X \times \{0\}) \cup (X \times \{1\})$ is induced by the maps $i_0\colon X \to X \times \{0\}$, $i_1\colon X \to X \times \{1\}$
$x \mapsto (x,0)$ $\quad x \mapsto (x,1)$;

(ii) for each k-cell e_λ of X , $\bar{e}_\lambda \times I$ is a closed $(k+1)$-cell of $X \times I$ with characteristic map
$h_{\lambda I}\colon C\partial(\bar{e}_\lambda \times I) \to \bar{e}_\lambda \times I$.

For each cell e_λ of a cone-complex X there is a canonical homeomorphism $h_{C\lambda}\colon C\partial C\bar{e}_\lambda \to C\bar{e}_\lambda$ whose definition is similar to that of $h_{\lambda I}$.

<u>3.2 Definition</u> The (CC-) *cone* on X is the space CX with the following cone-complex structure:

(i) the cone point is an 0-cell ;

(ii) the structure of $X \hookrightarrow CX$ is inherited ;

(iii) for each k-cell e_λ of X , $C\bar{e}_\lambda$ is a closed $(k+1)$-cell of CX with characteristic map $h_{C\lambda}\colon C\partial C\bar{e}_\lambda \to C\bar{e}_\lambda$.

<u>3.3 Remark</u> The definition of barycentric subdivision (2.6) essentially uses CC- cones. An alternative to (2.6) is to construct SdX inductively by replacing each cell of X by the CC- cone on its subdivided boundary.

<u>3.4 Definition</u> Let X be an n-dimensional cone-cell. For $n \geq 1$, the *dome* DX on X is the space CX with the following cone-complex structure:

(i) the structure of $X \hookrightarrow CX$ is inherited ;

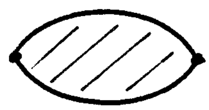
(ii) $C\partial X$ is a closed n-cell with the identity map $C\partial X \to C\partial X$ as characteristic map ;

(iii) CX is the single closed $(n+1)$ - cell and has $h_{CX}: C\partial CX \to CX$ as characteristic map.

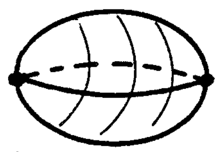
For $n = 0$, DX is taken to be the CC- cone CX .

When $n \geq 1$, DX can be thought of as an $(n+1)$ - cell whose boundary consists of two copies of X glued along ∂X . Some examples are shown below.

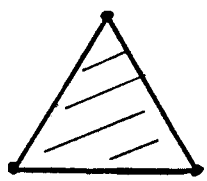
X



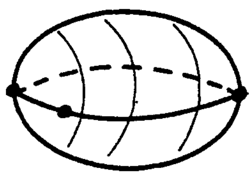
DX



D(DX)



X



DX

We make a convention that the copy of X in DX consisting of $X \times \{0\} \subset CX$ is denoted by X^- and that the other copy of X is X^+ .

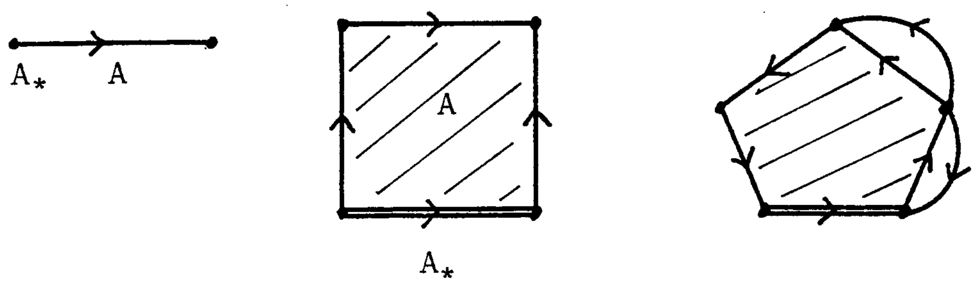
§4 Marked cone-complexes and the category Poly

The introduction by Eilenberg of a vertex-ordering on simplices in singular theory was an important step in the development of simplicial theory. Vertex-orderings 'tie down' maps in the sense that if $f: \Delta^m \to \Delta^n$, $g: \Delta^m \to \Delta^n$ are order-preserving simplicial maps with $f(\Delta^m) = g(\Delta^m)$ then $f = g$; that is, an order-preserving simplicial map is determined by its set image. We have to impose extra structure on cone-complexes so that if X and Y are cone-cells a CC - map $f: X \to Y$ preserving the structure is determined by its set image.

4.1 Definition A *marked cone-complex* is a cone-complex X together with, for $k \geq 1$ and each closed k-cell A of X , a choice of a closed (k-1) - cell A_* (called the *marked face* of A) in the boundary of A .

The category $\overrightarrow{CC}$ has marked cone-complexes as objects, and a morphism $f: X \to Y$ of $\overrightarrow{CC}$ is a cone-complex map preserving marked faces. That is, $f(A_*) = (f(A))_*$ for each closed cell A of X with $\dim A \geq 1$.

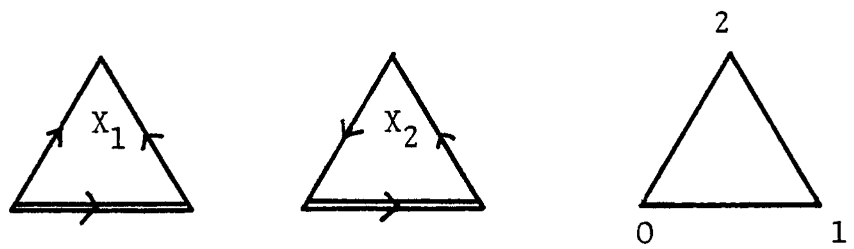
Some simple marked cone-complexes are pictured below. We adopt the convention that the marked face of a 2-cell is always represented by a double edge and that an arrow on a 1 - cell A points away from the vertex A_* .



<u>4.2 Definition</u> A marked cone-cell is called a *polycell*. We take Poly to be the full subcategory of $\overrightarrow{CC}$ whose objects are polycells.

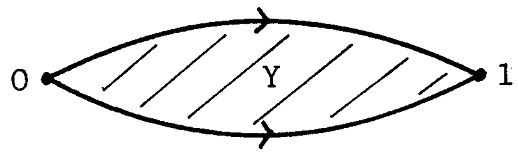
A subcomplex of a marked cone-complex X inherits a structure of marked faces. Each closed cell of X is thus a polycell. Marked cone-complexes are analogous to ordered simplicial complexes while polycells correspond to ordered simplices. We have two main reasons for using marked faces rather than, say, a vertex-ordering in the definition 4.1 .

First (see next section), there may be more marked face structures on a cone-cell X than there are orderings of the vertices of X . Consider $X = \Delta^2$.

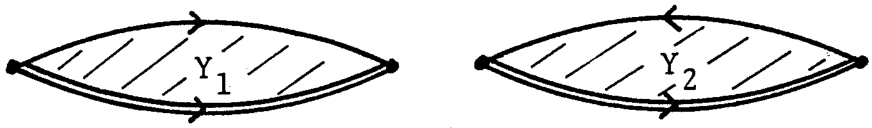


Any two vertex-ordered 2-simplices are isomorphic but there are two non-isomorphic polycell structures X_1 and X_2 on Δ^2 . If polycells are to be used to model van Kampen diagrams, as suggested in the Introduction, then both X_1 and X_2 are required: X_1 to represent relators of the form a^2b^{-1}, X_2 to represent a^3 .

Secondly, vertex-orderings fail to tie down morphisms involving certain cone-cells. For example, an ordering of the vertices of the cone-cell Y below does not differentiate between the 1 - cells of Y . There are thus two CC - isomorphisms $Y \to Y$ preserving the ordering.



On the other hand, each of the two marked face structures Y_1 and Y_2 on Y differentiate between 1-cells and there are unique Poly - isomorphisms $Y_1 \to Y_1$, $Y_2 \to Y_2$.



We proceed to show that any Poly morphism $f: X \to Y$ is determined by its set image $f(Y)$.

One or two preliminaries are necessary. By a *face* of a marked cone-complex X we will always mean a closed cell of X so that, in particular, a face of a polycell is also a polycell.

Define inductively, for an n-dimensional polycell X and $1 \le r \le n$,

$$X_{r*} = (X_{(r-1)*})_*$$
$$= \text{the marked face of } X_{(r-1)*} \,.$$

We have a sequence $X_{n*} \subset X_{(n-1)*} \subset \ldots \subset X_* \subset X$, where $\dim X_{r*} = (n-r)$, of faces of X . This is reminiscent of the notion of a *flag* in an n-dimensional vector space V , namely a sequence $0 \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1}$ of subspaces of V such that $\dim F_r = r$. We therefore state:

<u>4.3 Definition</u> The *flag* in an n-polycell X is the sequence $X_{n*} \subset X_{(n-1)*} \subset \ldots \subset X$ of faces of X .

The vertex X_{n*} is a base-point specified by the marked face structure of X and the flag itself can be thought of as a generalized base-point.

The notion of a *pseudomanifold* is required.

<u>4.4 Definition</u> ([35], p.82) An *n-dimensional pseudomanifold* is an n-dimensional finite regular complex K which satisfies the following conditions (taking 'cell' to mean 'closed cell'):

(i) every cell of K is a face of some n-cell ;

(ii) every (n-1) - cell of K is a face of exactly two n-cells ;

(iii) if E and E' are n-cells of K there is a sequence $E = E_o, E_1, \ldots, E_k = E'$ of n-cells of K such that, for each i , E_i and E_{i+1} have an (n-1)-face in common.

It is a standard result ([35], p. 81) that any regular cell decomposition of S^n is an n-pseudomanifold. We therefore have:

<u>4.5 Proposition</u> *If* X *is an* n-*polycell then* BdX *is an* (n-1)-*pseudomanifold.* □

<u>4.6 Corollary</u> *For* X *an* n-*polycell and* $p \leq n$ *each cell of* X *of dimension* $<p$ *is contained in some closed* p-*cell* .

<u>Proof</u> Use downward induction on skeleta. □

<u>4.7 Proposition</u>

(i) *Let* $f: X \to Y$ *be a morphism of* Poly *. Then* f(X) *is a face of* Y *and* $f: X \to f(X)$ *is a* Poly *- isomorphism.*

(ii) *If* $f, g: X \to Y$ *are isomorphisms of* Poly *then* $f = g$.

(iii) *If* $f, g: X \to Y$ *are morphisms of* Poly *such that* $f(X) = g(X)$ *then* $f = g$.

<u>Proof</u>

(i) By Proposition 2.4, $f(X)$ is a face of Y and f is a CC - isomorphism $X \to f(X)$. Since f preserves distinguished faces f^{-1} does the same, and we have a Poly - isomorphism $X \to f(X)$.

(ii) We first prove the following.

<u>Claim</u> If A and B are q-polycells $(q \geq 1)$ and $f, g: A \to B$ are Poly - isomorphisms which agree on a $(q-1)$ - face F of A then $f = g$.

The proof is by induction on the common dimension of A and B .

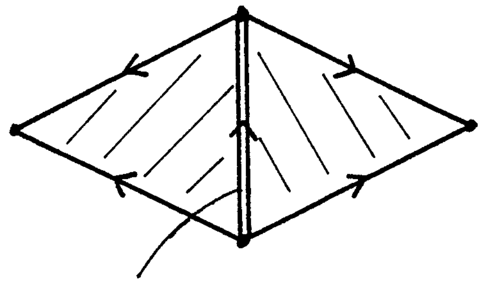
Assume the claim holds for polycells of dimension $q-1$ and consider $f, g: A \to B$. By Proposition 4.5, if F' is any $(q-1)$ - face of A other than F there is a sequence $F = F_o, F_1, \ldots, F_k = F'$ of $(q-1)$ - faces of A such that $F_i \cap F_{i+1} \supseteq$ a $(q-2)$ - face. Now $f|F = g|F$ implies that f and g agree on the $(q-2)$ - face contained in $F \cap F_1$. Thus, by the inductive assumption, f and g agree on the face F_1 . Similarly $f|F_2 = g|F_2, \ldots, f|F_k = g|F_k$, giving $f|F' = g|F'$.

We have shown that f and g agree on all $(q-1)$ - faces of A . Since BdA is a $(q-1)$ - pseudomanifold each of its cells is contained in some $(q-1)$ - face, so that $f|\text{BdA} = g|\text{BdA}$. The preservation of cone structure by CC - maps then ensures that $f = g$.

The claim follows, with the inductive process started by noting that an isomorphism $A \to B$ of 1-polycells is determined by the destination of one vertex of A .

To prove part (ii) of the proposition we note that, since f and g preserve distinguished faces, $f(X_{r*}) = g(X_{r*})$ for $r = 1, 2, \ldots, n$. Thus f and g agree on the vertex X_{n*} . For each r, X_{r*} is an $(n-r)$-face of the $(n-r+1)$-polycell $X_{(r-1)*}$ (taking $X = X_{o*}$). Hence, making repeated use of the <u>Claim</u> we can move up the flag in X to obtain $f = g$.
(iii) This follows immediately from (i) and (ii) . □

We remark that, in general, a $\overrightarrow{CC}$ morphism is not determined by its set image. For instance, there are two $\overrightarrow{CC}$ isomorphisms $X \to X$, where X is the following marked cone-complex.



marked face of both 2-cells

<u>§5 Consequences of the marked face structure of a polycell</u>

We now look at some combinatorial properties of a polycell which depend on its marked face structure and which strengthen the analogy between polycells and vertex-ordered simplices. Throughout this section X denotes an n-polycell.

<u>5.1 Proposition</u> *The system of marked faces defines an orientation of* X *and assigns to each* $(n-1)$*-face* A_i *of* X *a parity* $\varepsilon_i (\varepsilon_i = \pm 1)$ *relative to* X_* *(for* $n \geq 1$*)* .

<u>Proof</u> Now $H_n(X, X^{(n-1)}) \cong \mathbb{Z}$ and an orientation of X is defined to be a choice of generator of $H_n(X, X^{(n-1)})$.

There is a boundary map

$$H_n(X, X^{(n-1)}) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

which is the composite

$$H_n(X, X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

and is also part of the homology exact sequence of the triple $(X, X^{(n-1)}, X^{(n-2)})$. Since $H_n(X, X^{(n-2)}) = 0$ (this follows from the exact sequence of the pair $(X, X^{(n-2)})$ and the contractability of X), ∂ is injective.

Let X have $(n-1)$ - faces $X_* = A_o, A_1, \ldots, A_k$. The inclusion $A_i \to X$ induces an isomorphism

$$\phi: \sum_i H_{n-1}(A_i, A_i^{(n-2)}) \xrightarrow{\cong} H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

where, on the left, we have a direct sum of copies of $\mathbb{Z}$, one copy for each face A_i. By definition, an orientation of A_i induces an isomorphism $H_{n-1}(A_i, A_i^{(n-2)}) \cong \mathbb{Z}$. However, because each $(n-2)$ - face of X is a face of precisely two $(n-1)$ - faces (Proposition 4.5) there are only two elements, say $\pm\alpha$, in the left hand group such that $\phi(\pm\alpha) = \partial$ (generator of $H_n(X, X^{(n-1)})$.

Suppose an orientation $\theta(A_i)$ of each A_i has been chosen. Then we can write $\alpha = \sum_i \varepsilon_i \theta(A_i)$, $\varepsilon_i = \pm 1$, and α is determined up to sign. This can be fixed by insisting that $\varepsilon_o = +1$ or that $\varepsilon_o = -1$: we choose $\varepsilon_o = +1$. (The ε_i, $i \neq 0$, are then completely determined by the following rule: if B is an $(n-2)$ - face in $A_i \cap A_j$ and ε_B in $A_i = -\varepsilon_B$ in A_j then $\varepsilon_i = \varepsilon_j$, otherwise $\varepsilon_i = -\varepsilon_j$.) By injectivity of ∂ , once α is determined so also is a generator of $H_n(X, X^{(n-1)})$, that is, an orientation $\theta(X)$ of X . Abusing notation and regarding ϕ as the identity, we have $\partial\theta(X) = \sum_i \varepsilon_i \theta(A_i)$.

To start the inductive process, a 1-polycell can be oriented by ordering its two vertices. □

<u>5.2 Proposition</u> *The structure of marked faces of* X *determines a total ordering of the* $(n-1)$*-faces of* X $(n \geq 1)$.

<u>Proof</u> Let V be any set and let $f: \{0, 1, \ldots, k\} \longrightarrow V$ be a surjection. Then a total order is defined on V by letting $v_o = f(0)$ be the first element and, if $v_o, v_1, \ldots, v_r$ have been defined, letting v_{r+1} be $f(x)$ where x is the least element of $\{0, 1, \ldots, k\} \setminus \bigcup_{i=o}^{r} f^{-1}(v_i)$.

The ordering on the set of $(n-1)$ - faces of X can be constructed inductively.

Assume an order on the $(n-2)$ - faces of any $(n-1)$ - polycell Y and consider the n-polycell X . We label the marked face X_* by $X(0)$. Suppose $X(0)$, $X(1)$, ..., $X(q)$ have been labelled in order. Let $R(q)$ be the set of remaining $(n-1)$ - faces of X . Since BdX is an $(n-1)$ - pseudomanifold each $(n-2)$ - face of X is contained in exactly two $(n-1)$ - faces. Let $B(q)$ be the set of $(n-2)$ - faces Z such that Z is contained in one of $X(0)$, $X(1)$, ..., $X(q)$ and in one of the $(n-1)$ - faces in $R(q)$. Using the total order on the $(n-2)$ - faces of each $X(i)$ we can order $B(q)$ lexicographically. Since each element of $B(q)$ is contained in a unique element of $R(q)$ we get a function $B(q) \dashrightarrow R(q)$ and hence an order on the image of this function. The $(n-1)$ - faces in the image can now be labelled. This process is continued until all the $(n-1)$ - faces of X are labelled (as ensured by part (iii) of the definition of a pseudomanifold).

To start the induction, for Y a 1-polycell, take $Y_* = Y(0)$ and let the remaining vertex be $Y(1)$. □

<u>5.3 Proposition</u> *The marked face structure determines a total order* $\zeta(X)$ *on the set of all faces of* X .

<u>Proof</u> A modified lexicographic ordering using (5.2) is constructed.

For some $k \geq 1$, the $(n-1)$ - faces of X are $X(0), X(1), \ldots, X(k)$. The $(n-2)$ - faces of $X(i)$, $0 \leq i \leq k$, may be labelled $X(i,0), X(i,1), \ldots, X(i,m_i)$ for some m_i . We can continue in this way to give a label to each face of X . Any face of dimension $< n-1$ will have more than one label: an $(n-2)$ - face of Z of $X(1) \cap X(2)$ is both $X(1,p)$ and $X(2,q)$.

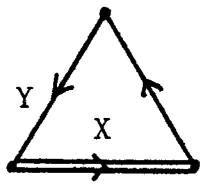
The ordering $\zeta(X)$ is defined as follows. The top-dimensional face X is taken as least element. Then the $(n-1)$ - faces are ordered $X(0) < X(1) < \ldots < X(k)$. Next the $(n-2)$ - cells are ordered lexicographically

$$X(0,0) < X(0,1) < \ldots < X(0,m_0) < X(1,0) < \ldots$$

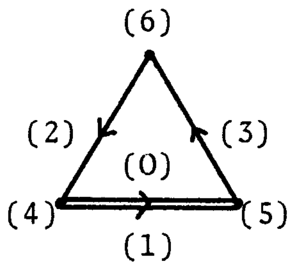
but omitting a face if it has appeared previously under a different label. We proceed in order of decreasing dimension, using the modified lexicographic ordering within each dimension. □

The ordering $\zeta(X)$ plays a crucial role in Chapter III, where it is used to specify certain collapses of polycells.

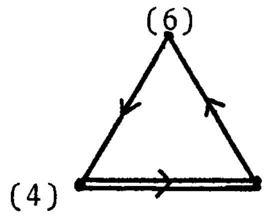
Note that $\zeta(X)$ induces a total order $\zeta_o(X)$ on the set of vertices of X. In turn, $\zeta_o(X)$ induces an ordering $\zeta_o(X)Y$ of the vertices of each face Y of X. Since Y is itself a polycell there is also an ordering $\zeta_o(Y)$ determined by the marked face structure of Y. In general $\zeta_o(Y)$ and $\zeta_o(X)Y$ do not agree. For example, consider the following polycell structure on Δ^2.



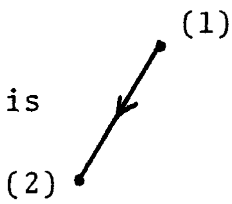
The ordering $\zeta(X)$ is



which gives $\zeta_o(X)Y$ =



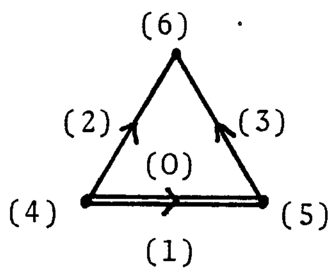
The ordering $\zeta_o(Y)$ is



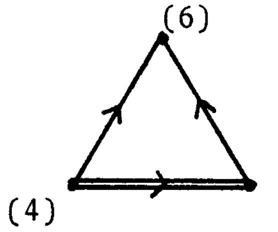
which disagrees with $\zeta_o(X)Y$.

In contrast, if we take the other possible polycell structure on Δ^2 we have:

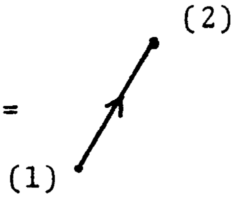
$\zeta(X')$ =



giving $\zeta_o(X')Y'$ =



which agrees with $\zeta_o(Y')$ =



Intuitively, the fact that $\zeta_o(B)$ and $\zeta_o(A)B$ do not agree in general explains why marked face structures are not equivalent to vertex-orderings on cone-cells. We can think of the polycell X' as corresponding to a vertex-ordered 2-simplex but, because $\zeta_o(Y) \neq \zeta_o(X)Y$, there can be no vertex-ordering carrying the same amount of information as the marked face structure of X . An interesting question is whether, given a vertex-ordering ζ_v on a cone-cell X , there is a marked face structure on X such that $\zeta_o(X) = \zeta_v$. This seems likely, but we have no proof at present.

We remark that the extra strength of the marked face structure allows us to do homotopy theory with polycells, rather than just homology theory.

§6 Subcategories of Poly appropriate as model categories

We have seen that although the category Poly is very general its objects have a rich structure and its morphisms are restricted. It follows that certain subcategories of Poly can be used as model categories. We consider a class of such subcategories.

6.1 Definition The members of the class Γ are full subcategories M of Poly satisfying:

(i) for each $n \geq 0$, $Ob(M)$ contains an n-polycell ;

(ii) for each face A of an object of M there is an object of M which is Poly-isomorphic to A ;

(iii) M is skeletal ; that is, any two isomorphic objects of M are identical.

Useful model categories are obtained if condition (iii) is omitted. However, we wish to look at the theory of 'M-T - complexes' (Chapter III) and this seems to be neater if (iii) is included. Moreover, the skeleta of a full subcategory M of Poly satisfying (i) and (ii) are members of Γ and, for M' a skeleton of M, the functor categories $Set^{M'^{op}}$ and $Set^{M^{op}}$ are equivalent.

Using the notation of §1 we make each category $M \in \Gamma$ into a model category for Top by defining $I: M \to Top$ to be the forgetful functor which sends each polycell to its underlying space. Then, for each M, there is a singular functor $s: Top \to Set^{M^{op}}$ (Definition 1.2) and an adjoint realization functor $r: Set^{M^{op}} \to Top$ (1.3, 1.4). The functor category $Set^{M^{op}}$ is called the category of *M - sets* .

Let $K: M^{op} \to Set$ be an M-set. For X an n-dimensional object of M, each $x \in K(X)$ is said to be an *n-cell* of K .

By conditions (ii) and (iii) of Definition 6.1 , for each face A of X, there is precisely one M-object A' isomorphic to A. Proposition 4.7 (iii) ensures that there is a unique M-morphism $\delta_A: A' \to X$ such that $\delta_A(A') = A$. There is thus a unique map $\partial_A = K(\delta_A): K(X) \to K(A')$ corresponding to the face A of X .

<u>6.2 Definition</u> We call ∂_A a *face map*. For each cell $x \in K(X)$, $\partial_A x$ is a *face* of x.

We will now give some examples of members of Γ. First, three standard constructions are required.

<u>6.3 Definition</u> Let X be a marked cone complex. The $(\overrightarrow{CC}\text{-})$ *cylinder* $X \times I$ on X is the CC - cylinder with the following marked face structure: $X \times \{0\}$, $X \times \{1\}$ inherit the marked faces of X and $(A \times I)_* = A \times \{0\}$ for each face A of X.

For each face A, $A \times I$ is clearly a polycell, and $X \times I$ is a marked cone-complex. If X is itself a polycell then so is $X \times I$. The situation is the same for the following construction.

<u>6.4 Definition</u> Let X be a marked cone-complex. The $(\overrightarrow{CC}\text{-})$ *cone* CX on X is the CC - cone with the following marked face structure: $X \subset CX$ retains its marked faces and, for each face A of X, $(CA)_* = A$.

Recall that the dome construction (3.4) was defined on cone-cells only.

<u>6.5 Definition</u> Let X be an n-polycell. For $n \geq 1$, the $(\overrightarrow{CC}\text{-})$ *dome* DX on X is the CC - dome with the marked face structures of X^+, X^- inherited from X and $(DX)_* = X^-$. For $n = 0$, DX is the $\overrightarrow{CC}$ - cone CX.

Note that marked face structures other than those given above may be imposed on CC - cylinders, cones and domes. We reserve the terms $\overrightarrow{CC}$ - cylinder, cone and dome for the particular choice of structure made in each case.

Let $\underset{\sim}{0}$ denote the 0-polycell $(0, 0, \ldots) \in \mathbb{R}^{\infty}$. We use $\underset{\sim}{0}$ whenever a standard choice of 0-polycell is required, as in the definitions of the skeletal categories below.

<u>6.6 Definition</u> The category G is the full subcategory of Poly with objects G^o , G^1, ... where $G^o = \underset{\sim}{0}$ and, for each $n \geq 1$, G^n is the $\vec{CC}$ -dome DG^{n-1} .

A polycell isomorphic to G^n is called an *n-globe* . Note that, as a CW-complex , G^n is the standard n-cell with its commonly used cell structure

$$G^n = e^o_{\pm} \cup e^1_{\pm} \cup \ldots \cup e^n_{\pm}$$

Up to isomorphism, G may be considered the simplest category in Γ.

<u>6.7 Definition</u> The category Δ_I is the full subcategory of Poly with objects Δ^o, Δ^1, ... where $\Delta^o = \underset{\sim}{0}$ and, for each $n \geq 1$, Δ^n is the $\vec{CC}$-cone $C\Delta^{n-1}$.

A polycell isomorphic to Δ^n is referred to as an *n-simplex* .

The order $\zeta_o(\Delta^n)$ defined on the set of vertices of Δ^n by the marked face structure (§5) can be described using the flag $\Delta^n_{n*} \subset \Delta^n_{(n-1)*} \subset \ldots \subset \Delta^n$ in Δ^n : for $j = 1, 2, \ldots, n$ vertex j is the unique vertex of $\Delta^n_{(n-j)*}$ not contained in $\Delta^n_{(n-j+1)*}$; vertex 0 is Δ^n_{n*} . The marked face structure of Δ^n is completely determined by $\zeta_o(\Delta^n)$, using the following rule: for $k \geq 1$ and each k-face X of Δ^n , X_* is the unique (k-1)-face of X not containing the greatest vertex of X . Thus the marked face structure of Δ^n is equivalent to a vertex ordering. There is an obvious canonical isomorphism between Δ_I and the wide subcategory

Δ_I with injective morphisms of the usual simplicial category. In future we identify the two categories Δ_I .

<u>6.8 Definition</u> The category $\Box_I$ is the full subcategory of Poly with objects $I^0, I^1, \ldots$ where $I^0 = \underset{\sim}{Q}$ and, for $n \geq 1$, I^n is the $\vec{CC}$-cylinder $I^{n-1} \times I$.

A polycell isomorphic to I^n is called an *n-cube*. $\Box_I$ is isomorphic to the wide subcategory $\Box_I$ with injective morphisms of the usual cubical model category $\Box$. Again we identify the two isomorphic categories.

Combinations of the dome, cone and cylinder constructions can be used to build four other members of Γ. For instance, CP is the full subcategory of Poly defined inductively as follows: the single 0-dimensional object of CP is $\underset{\sim}{Q}$; the n-dimensional objects of CP are the $\vec{CC}$-cones and cylinders on the (n-1)-dimensional objects of CP (identifying $C\underset{\sim}{Q}$ with $\underset{\sim}{Q} \times I$). Here the notation CP indicates that cone and product (that is, cylinder) operations are used in the definition. Similarly, we have categories DC , DP and DCP . There is an isomorphism between CP and the category **P** of Hintze [26].

Bigger categories in Γ can easily be constructed. The skeleta of Poly are maximal and we can obtain other members of Γ from these by imposing extra conditions on polycells. Examples of such categories are given in the next chapter.

CHAPTER II

POSETS AND SHELLABILITY

This chapter is devoted to a subclass $E\Gamma$ of the class Γ of model categories. It will be shown in the next chapter that, for any category $\mathbb{M}$ in $E\Gamma$, the categories of $\mathbb{M}T$-complexes and simplicial T-complexes are equivalent. The members of $E\Gamma$ are categories $\mathbb{M}$ in Γ such that $\mathbb{M}$ has nice objects and $Ob(\mathbb{M})$ contains certain specified polycells.

We specify niceness by means of a *shellability* condition. A polycell satisfying this condition has tamely embedded faces and can be given a combinatorial description. As a result, a maximal category $\mathbb{P}$ in $E\Gamma$ is isomorphic to a category $\mathbb{P}'$ of posets with extra structure. That is, $\mathbb{P}$ is combinatorial in nature, which is desirable in a model category. The properties of $\mathbb{P}$ are important because each member of $E\Gamma$ is isomorphic to a full subcategory of $\mathbb{P}$.

Terminology and results from the theory of simplicial complexes and PL topology are used without comment in this chapter. The reader is referred to Hudson [27].

§1 Shelling

All simplicial complexes are taken to be finite. We say that an n-dimensional simplicial complex K is *pure* if each face of K is contained in an n-face.

1.1 Definition (see [20], p. 34) The n-dimensional simplicial complex K is *shellable* if K is pure and the n-simplices of

K can be given a linear order $F_1, F_2, \ldots, F_t$ such that the following conditions hold for $1 < k \leq t$ except that (ii) may fail when $k = t$:

(i) for each $i < k$ there exists $j < k$ such that $F_j \cap F_k$ is an $(n-1)$-simplex and $F_i \cap F_k \subseteq F_j \cap F_k$;

(ii) there is an $(n-1)$-face of F_k not contained in F_i for any $i < k$.

In other words the n-simplex F_k is required to intersect the complex $\bigcup_{i=1}^{k-1} F_i$ in a non-empty union of maximal proper faces of F_k which does not include every such face of F_k . An ordering of n-simplices which satisfies (i) and (ii) is called a *shelling* of K . A shelling represents an especially nice and useful way of assembling K from its component n-simplices.

If the n-simplices of K can be given a linear order $F_1, F_2, \ldots, F_t$ satisfying condition (i) then K is said to be *semishellable* and the order $F_1, F_2, \ldots, F_t$ is referred to as a *semishelling* of K . Some authors (in particular, Bjorner and Wachs [4, 5, 6] take the term *shelling* to mean what we call a semishelling.

The concept of shellability has been widely studied and there are definitions similar to 1.1 for finite convex cell complexes and finite regular complexes (see the survey paper [20] and Chapter V). There is also a notion of shellability for posets, which plays an important part in sections 3 and 4 of this chapter.

<u>1.2 Definition</u> A *weak* n- *pseudomanifold with boundary* is a pure n-dimensional simplicial complex K such that every (n-1) - simplex of K lies in at most two n-simplices. The *boundary* BdK of K is the (n-1) - dimensional complex consisting of all (n-1) - simplices that lie in one n-simplex.

If the boundary of K is empty, K is said to be a *weak* n- *pseudomanifold.*

Note that if Definition I 4.4 is restricted to simplicial complexes and condition (iii) omitted the definition of a weak pseudomanifold is obtained. Our terminology is not standard since the term *pseudomanifold (with boundary)* is sometimes used [20] for our weak pseudomanifold (with boundary).

The following results are well known.

<u>1.3 Proposition</u> [20, p. 41] *For* K *a semishellable weak pseudomanifold, the semishellings of* K *are identical to shellings.* □

<u>1.4 Proposition</u> [20, p. 35] *If* $F_1, F_2, \ldots, F_t$ *is a shelling of a weak* n- *pseudomanifold* K *with boundary then* K *is a combinatorial* n- *sphere or combinatorial* n-*ball depending on whether condition (ii) of Definition 1.1 fails or holds when* $k = t$ *and whether* BdK *is empty or non-empty.* □

Recall that a simplicial complex has a canonical cone-complex structure. We refer to a cone-complex Z which is CC - isomorphic to a simplicial complex Z' as a *simplicial cone-complex.* The isomorphism $Z' \to Z$ is a triangulation of Z . There are obvious analogues of Definitions 1.1 and 1.2 for triangulated spaces so we have a notion of shelling and of a weak pseudomanifold with boundary for simplicial cone-complexes.

The triangulation $Z' \to Z$ makes Z a PL space. Recall that a PL space is a PL ball or sphere if it is triangulated as a combinatorial ball or sphere. Following from Proposition 1.4 we have:

<u>1.5 Proposition</u> *Let* Z *be a simplicial cone-complex. If* Z *is a weak* n- *pseudomanifold with boundary and has a shelling* $F_1, \ldots, F_t$ *then the space* Z *is a* PL n- *sphere or a* PL n- *ball depending on the failure of condition (ii) when* $k = t$ *and whether* BdZ *is empty.* □

By Proposition I 2.7 , if X is a cone-complex the barycentric subdivision SdX is a simplicial cone-complex.

<u>1.6 Definition</u> The cone-complex X is S- *shellable* if SdA is shellable for each face A of X .

<u>1.7 Proposition</u> *For* $k \geq 0$ *, every* k- *face of an* S- *shellable cone-complex* X *is a* PL k- *ball.*

<u>Proof</u> For $k \geq 1$ and each k-face A of X , SdBdA is a regular cell decomposition of S^{k-1} and is therefore a (k-1)-pseudomanifold. It follows that SdA is a weak k-pseudomanifold with boundary. Since SdA is shellable the result follows from Proposition 1.5 and the fact that the spaces A and SdA coincide. □

<u>1.8 Proposition</u> *If* B *is a* $(k - 1)$- *face of the* k-*face* A *in an* S- *shellable cone-complex then there is a homeomorphism of pairs* $(A, B) \cong (I^k, I^{k-1})$.

<u>Proof</u> The inclusion i: BdB $\to$ BdA is a PL embedding of the PL (k-2) - sphere BdB into the PL (k-1) - sphere BdA . By Rushing [34] , Theorem 1.7.2 , such an embedding is locally flat. The generalized Schoenflies Theorem [34, p. 48] states that a locally flat embedding $S^{k-2} \to S^{k-1}$ is flat. The flatness of BdB in BdA implies the desired result. □

The propositions above show that an S-shellable cone-complex has nice faces nicely embedded. The following result is useful for checking the S-shellability of cone-complexes.

<u>1.9 Proposition</u> *For* $k \geq 1$ *and any* k- *face* A *of a cone-complex,* SdA *is shellable if and only if* SdBdA *is shellable.*

<u>Proof</u>

=> Let dim A = k . Since SdA is isomorphic to the CC-cone on SdBdA a shelling $F_1, F_2, \ldots, F_t$ of SdA induces an ordering $B_1, B_2, \ldots, B_t$ on the (k-1) - faces of SdBdA . The ordering $B_1, B_2, \ldots, B_t$ is clearly a semishelling of SdBdA , which is a weak pseudomanifold. Thus, by Proposition 1.3 , we have a shelling of SdBdA .

<= is obvious. □

For $n \leq 2$ all triangulations of n-balls are shellable so all 1- and 2- dimensional cone-complexes are S-shellable. For $n \geq 3$ there exist combinatorial triangulations of n-balls which are not shellable [20]. However, all triangulations of 2-spheres are shellable and it is not known if 3- and 4- spheres are shellable. Hence, by Proposition 1.9 , all 3-dimensional cone-complexes are S-shellable and it is unknown whether all 4- and 5- dimensional cone-complexes are S-shellable.

Edwards [36] (see also [20]) has shown that for $n \geq 5$ there exist non-combinatorial triangulations of n-spheres. Thus, by 1.4, there exist non-shellable triangulations of n-spheres for $n \geq 5$. It is therefore likely that there are cone-complexes of dimension ≥ 6 which are not S-shellable, although we do not have an example at present.

The dome, cone and cylinder constructions (Chapter I, §3) and other cone-complex constructions preserve S-shellability. Proofs of S-shellability tend to be rather tedious so such results are gathered into an Appendix.

We shall be concerned with S-shellable marked cone-complexes and, in particular, S-shellable polycells. These are referred to as $\overrightarrow{SC}$- *complexes* and S- *polycells* respectively.

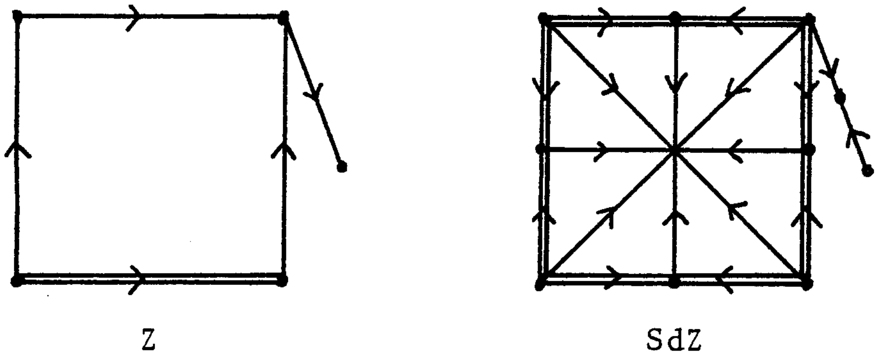
<u>1.10 Definition</u> We let $\overrightarrow{SC}$ be the full subcategory of $\overrightarrow{CC}$ whose objects are $\overrightarrow{SC}$- complexes.

SPoly is the full subcategory of Poly with S-polycells as objects.

<u>§2 The class EΓ of model categories</u>

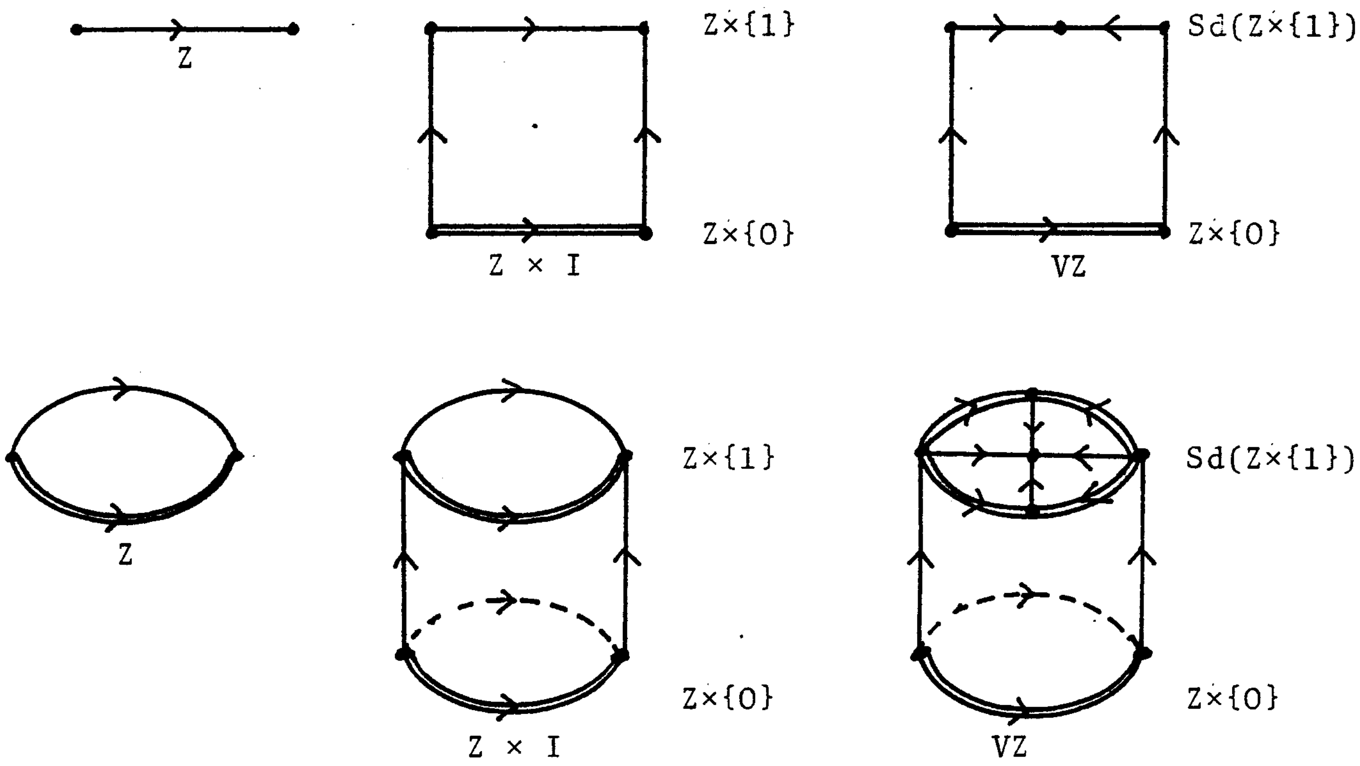
Before defining the class EΓ we need to give two constructions of marked cone-complexes.

<u>2.1 Definition</u> For Z a marked cone-complex the $(\overrightarrow{CC})$ *barycentric subdivision* SdZ of Z is the CC barycentric subdivision with the following marked face structure. For $k \geq 1$ and each k-face a of SdZ let $\hat{A}$ be the barycentre of the unique k-face A of Z containing a. Take a_* to be the (k-1)-face of a which does not contain $\hat{A}$.



<u>2.2 Definition</u> For an n-dimensional marked cone-complex Z the (n+1) - dimensional marked cone-complex VZ is formed by taking the $\overrightarrow{CC}$ cylinder Z × I and replacing Z × {1} by Sd(Z × {1}) .

Examples of VZ for low-dimensional Z are given below.



<u>2.3 Remarks</u>

(i) Each face a of SdZ is a Poly - simplex (that is, a is isomorphic to an object of the category Δ_I defined in I 6.7).

(ii) For each subcomplex Y of Z the marked cone-complex structure of VZ induces the structure of VY on $Y \times I$. Thus the faces of VZ are the faces of Z, the polycells VA for A a face of Z, and Poly-simplices of dimension $\leq \dim Z$. This fact is important later on.

We have from the Appendix that:

(i) an n-simplex is S-shellable for $n \geq 0$;

(ii) if the complex Z is S-shellable then so is VZ. The following definition is therefore meaningful.

2.4 Definition Let $E\Gamma$ be the class of categories M in Γ which satisfy:

(i) each M-object is S-shellable;

(ii) for $n \geq 0$, M has an object Poly-isomorphic to $\Delta^n \in Ob(\Delta_I)$;

(iii) for each object X of M there is an object of M which is Poly-isomorphic to VX.

2.5 Proposition *Each category in* $E\Gamma$ *has a full subcategory isomorphic to* Δ_I. □

In order to show that $E\Gamma$ is infinite we construct an infinite subset $SC = \{SC_1, SC_2, \ldots\}$ using the categories Δ_I, $\Box_I$ (See I §6).

2.6 Definition For $i \geq 1$, SC_i is the full subcategory of Poly with objects defined as follows. The single 0-dimensional object of SC_i is the standard polycell $\underset{\sim}{0}$. The set of j-dimensional objects of SC_i is

$\{\Delta^j, I^j\} \cup \{VX \mid X \in Ob(SC_i), \dim X = j-1\}$ for $j \leq i$ and

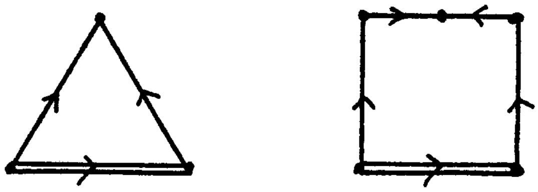
$\{\Delta^j\} \cup \{VX \mid X \in Ob(SC_i), \dim X = j-1\}$ for $j > i$, where $\Delta^j \in Ob(\Delta_I)$, $I^j \in Ob(\Box_I)$ and Δ^1, I^1 and $\underset{\sim}{VQ}$ are identified.

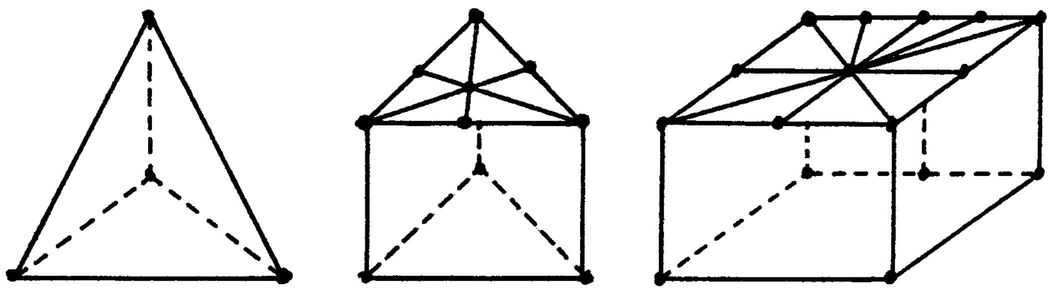
There is no difficulty in checking that SC_i is a member of $E\Gamma$ for each i .

The low dimensional objects of SC_1 are :

dim 0 

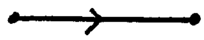
1 

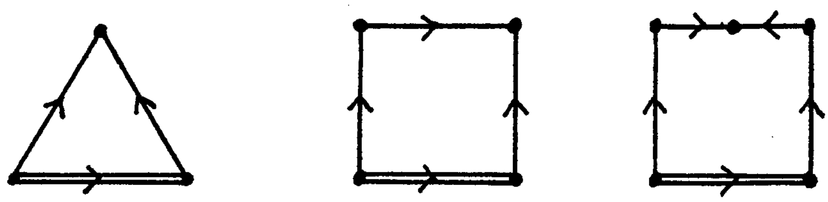
2 

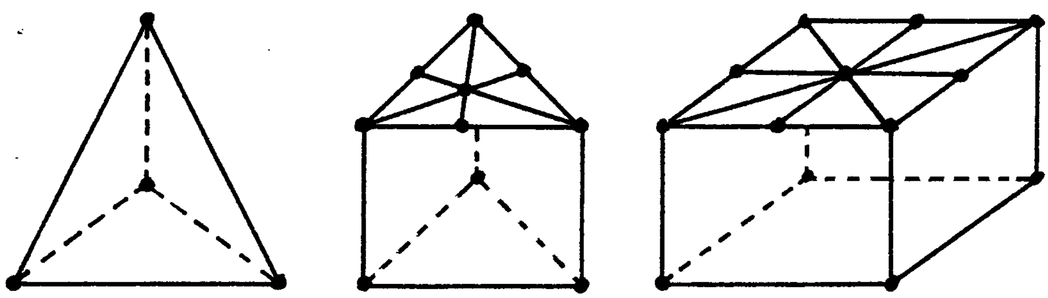
3 

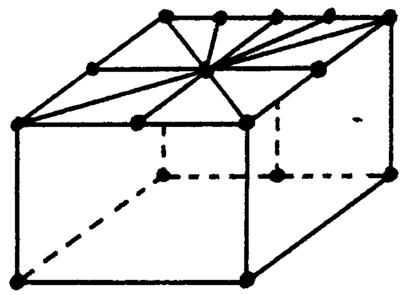
The objects of SC_2 in the same dimensions are:

dim 0 

1 

2 

3 



For $j < i$ there is no object of SC_j which is Poly-isomorphic to $I^i \in Ob(SC_i)$ (since all i-dimensional objects of SC_j have some simplicial faces). Therefore SC_i is not isomorphic to SC_j for $i \neq j$. This gives:

<u>2.7 Proposition</u> *The class* $E\Gamma$ *has an infinite number of non-isomorphic members.* □

The elements of SC are among the simplest in $E\Gamma$: SC_1 is the smallest category in $E\Gamma$ up to isomorphism and the SC_i , $i = 2, 3, \ldots$, lie between SC_1 and the least category in $E\Gamma$ having Δ^n and I^n (for all $n \geq 0$) among its objects.

The bigger members of $E\Gamma$ are perhaps the most interesting. For example, we can construct a category $Cv \in E\Gamma$ whose objects are based on convex polytopes.

For any point a in the interior of a convex polytope A , A is obviously a polyhedral cone on BdA with cone point a . There is thus a canonical cone-complex characteristic map CBdA → A associated with each point of Int A . All faces of A are themselves convex polytopes so there is a set of cone-cell structures for A . That is, there is a set of *convex cone-cells* corresponding to each polytope A .

<u>2.8 Definition</u> We denote by Conv the full subcategory of Poly whose objects are convex cone-cells with marked face structures. The category Cv is defined to be a skeleton of Conv.

<u>2.9 Proposition</u> *The category* Cv *is a member of* EΓ .

<u>Proof</u> Clearly $Cv \in \Gamma$. To show that $Cv \in E\Gamma$ we note:

(i) It follows easily from Proposition 5.2 of Bjorner [4] that the barycentric subdivision of any convex cone-cell is shellable. Each Cv -object is therefore S-shellable.

(ii) The standard geometric n-simplex is convex so Cv has an object Poly - isomorphic to Δ^n for $n \geq 0$.

(iii) Ewald and Shephard [22] have observed that the barycentric subdivision of the boundary complex of a convex polytope is isomorphic to the boundary complex of some simplicial convex polytope. Using this, we can show that if X is an object of Cv there exists a Cv - object Poly - isomorphic to VX . □

The skeleta of SPoly (1.10) are maximal in EΓ : if P is skeleton of SPoly then we have immediately.

<u>2.10 Proposition</u>

(i) The category P *is a member of* EΓ .

(ii) Each category in EΓ *is isomorphic to a full subcategory of* P .

P is bigger than Cv because polycells such as the globes are objects of P but not of Cv . In fact P is very big because it contains isomorphs of all but the wildest polycells. It is thus of interest that P can be given a combinatorial description using posets with extra structure. The rest of the chapter is devoted to this topic.

§3 Face posets of S-polycells

In this section a poset with extra structure is associated to each S-polycell.

First we recall some terminology from the theory of posets. Further details may be found in, for example, Bjorner [4].

All posets are taken to be finite. A poset is said to be *bounded* if it has a least element and a greatest element.

3.1 Definition [4, p.160] The *length* of a chain c in a poset Q is one less than the number of elements in c . We say Q is *pure* if all maximal chains have the same length. If Q is bounded and pure it is called *graded*.

3.2 Proposition [4, p.160] *A pure poset satisfies the Jordan-Dedekind condition: all unrefinable chains between two comparable elements have the same length.* □

3.3 Definition [4, p. 160] Let 0 be the least element of a graded poset Q . For $x \in Q$ the *rank* $\rho(x)$ of x is the common length of all unrefinable chains from 0 to x in Q .

The fact that a rank can be assigned to each element of Q explains the use of the term *graded poset*.

3.4 Definition [4, pp.160,182] The *order complex* $\Delta(Q)$ of a poset Q is the abstract simplicial complex of all chains of Q .

The definitions of shellability, weak pseudomanifold with boundary and so on given in section 1 also apply to abstract simplicial complexes.

3.5 Definition The poset Q is *shellable* if its order complex $\Delta(Q)$ is shellable.

Our terminology here is not completely standard. A poset is commonly defined to be shellable if its order complex is what we call *semishellable*. However, we will be mainly concerned with posets Q such that $\Delta(Q)$ is a weak pseudomanifold , in which case shellings and semishellings of $\Delta(Q)$ are identical (Proposition 1.3) .

Bjorner and Wachs [4, 5, 6] have developed and applied various notions of *lexicographic shelling* of a poset. Their work provides a number of useful tools for proving that a poset is shellable.

The new ideas we need are simple.

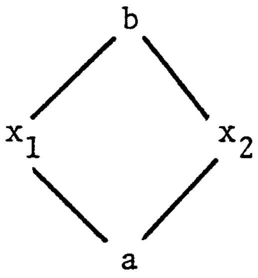
For $a \leq b$ in a poset Q , $[a,b]$ denotes the interval $\{x \in Q \mid a \leq x \leq b\}$.

3.6 Definition A graded poset Q is said to satisfy the *diamond condition* if for every pair of elements a, b in Q such that $a < b$ and $\rho(a) = \rho(b) - 2$ then:

(i) the interval $[a,b]$ contains exactly two elements x_1 , x_2 apart from a and b ;

(ii) each of x_1 , x_2 covers a and is covered by b .

In other words $[a,b]$ is of the form of the diamond



<u>3.7 Definition</u> A C-*poset* is a poset Q satisfying:

(i) Q is graded ;

(ii) Q satisfies the diamond condition ;

(iii) for each element a of Q the sub-poset $[0,a]$, where 0 is the least element of Q, is shellable.

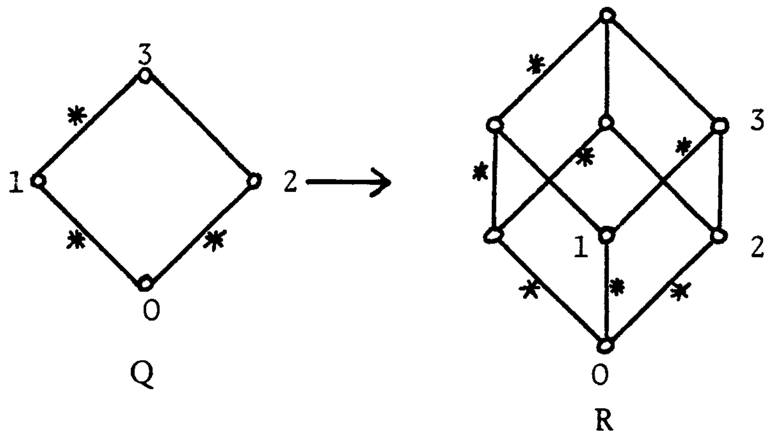
We can define a *tree* to be a poset R with least element 0_R such that the interval $[0_R, a]$ is a chain for each element a of R. If Q is a poset with least element 0, a maximal subtree R of Q is a sub-poset of Q such that if $a \in Q$ then $a \in R$ and the interval $[0_R, a]$ in R is an unrefinable chain $0 \to a$ in Q.

<u>3.8 Definition</u> An S-*poset* (Q,Q_*) is a C-poset Q together with a maximal subtree Q_* of Q.

The category SPos has S-posets as objects and a morphism $f: (Q,Q_*) \to (R,R_*)$ of SPos is an order-preserving map of pairs $(Q,Q_*) \to (R,R_*)$ which restricts to a poset isomorphism $Q \to [0,b]$ for some $b \in R$.

The S-poset (Q, Q_*) will often be denoted simply by Q.

An SPos-morphism is represented below. Edges of the Hasse diagram of Q which are marked by * belong to the diagram of Q_*.



(The image of Q is shown by numbered vertices.)

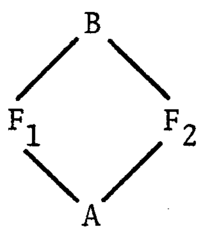
3.9 Definition The *face-poset* of a polycell X is the pair $(F(X), F(X)_*)$ where $F(X)$ is the set of faces of X (including the empty face) ordered by inclusion and $F(X)_*$ is the maximal subtree of $F(X)$ ordered as follows: for faces A and B of X, $A \leq_* B$ if A belongs to the flag in B.

3.10 Proposition *If X is an S-polycell then $F(X)$ is an S-poset.*

Proof

(i) The poset $F(X)$ is bounded, having least element $\emptyset$ and greatest element X. Each maximal chain in $F(X)$ is of length $n+1$ where $\dim X = n$. Hence $F(X)$ is graded.

(ii) Consider two elements $A < B$ in $F(X)$ such that $\rho(B) = k$ $(k \geq 2)$ and $\rho(A) = \rho(B) - 2$. The rank of an element in $F(X)$ is one greater than its dimension in X. Therefore A is a $(k-3)$-face of the $(k-2)$-pseudomanifold BdB, which implies that A is a face of exactly two $(k-2)$-cells of BdB. Thus the interval $[A,B]$ in $F(X)$ is of the form



(If $\rho(B) = 2$ then $A = \emptyset$ and BdB is an 0-pseudomanifold; that is, B is a 1-cell having two vertices.) It follows that $F(X)$ satisfies the diamond condition.

(iii) In the proof of Proposition I 2.7 the simplicial complex τX is defined together with a CC-isomorphism $\alpha: \tau X \to SdX$. There is a canonical isomorphism $\beta: |\Delta(F(X) - \emptyset)| \to \tau X$, where $|\Delta(F(X) - \emptyset)|$ denotes a geometric realization of the order complex. The definitions of α and β are such that there is an isomorphism $SdX \to |\Delta(F(X) - \emptyset)|$ which restricts to an isomorphism $SdA \to |\Delta(F(A) - \emptyset)|$ for each face A of X.

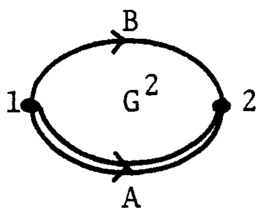
Now $|\Delta F(A)|$ is a cone on $|\Delta(F(A) - \emptyset)|$ so that shellability of $|\Delta(F(A) - \emptyset)|$ is inherited by $|\Delta F(A)|$. If $|\Delta F(A)|$ is shellable then the interval $[0,A]$ in $F(X)$ is shellable. Hence, since X is S-shellable, $[0,A]$ is shellable for each element A in $F(X)$.

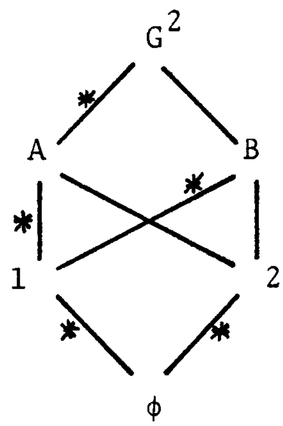
We have now shown that $F(X)$ is a C-poset. It follows that $(F(X), F(X)_*)$ is an S-poset. □

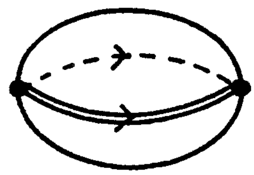
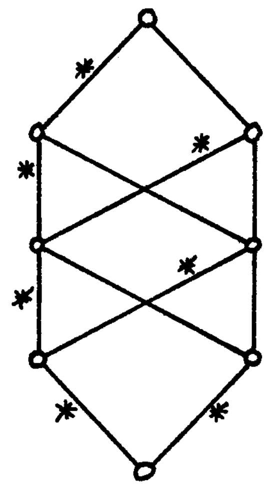
We have easily

<u>3.11 Proposition</u> F *defines a functor from the category* SPoly *of* S*-polycells to the category* SPos *of* S*-posets.* □

The face-poset of a polycell is analogous to the face-lattice of a convex polytope. Indeed, face-lattices of polytopes are C-posets. Not all face-posets of polycells are lattices: for instance $F(G^n)$, where G^n is an n-globe, is not a lattice for $n \geq 2$.

G^2 is 

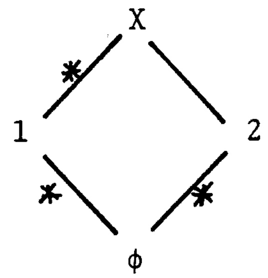
and $F(G^2)$ is 

G^3 is  so that $F(G^3)$ is 

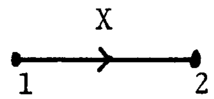
The form of $F(G^n)$ for $n \geq 2$ is clear. Any distinct elements of the same rank have two minimal upper bounds and so no least upper bound.

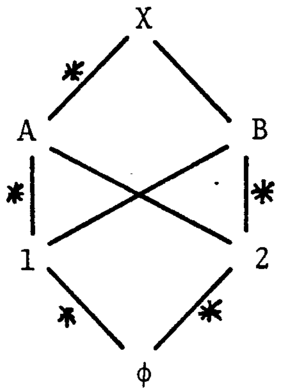
Note that $F(G^2)$ is one of the four simplest non-trivial S-posets up to isomorphism. The other three are

, the face-poset of an 0 - polycell X

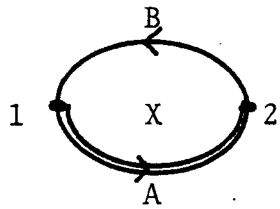


, the face-poset of a 1-polycell





corresponding to the pollycell



§4 The equivalence SPoly → SPos

We now associate an S-polycell to each S-poset.

If R is a subset of the poset Q, $\Delta(R)$ is a subcomplex of $\Delta(Q)$. If $|\Delta(Q)|$ is a geometric realization of $\Delta(Q)$ we denote the subcomplex of $|\Delta(Q)|$ which is a realization of $\Delta(R)$ by $|\Delta(R)|$. For Q an S-poset with least element 0 we denote $Q - \{0\}$ by Q' and, for each element $a \neq 0$ of Q, write $(0,a]$ for $[0,a] - \{0\}$ and $(0,a)$ for $(0,a] - \{a\}$.

4.1 Definition Let Q be an S-poset and let $|\Delta(Q')|$ be a choice of geometric realization of $\Delta(Q')$. We define $G(Q)$ to be the underlying polyhedron of $|\Delta(Q')|$ with the following cell and marked face structures: for $k \geq 1$ and each element $a \in Q'$ with rank k, the underlying polyhedron of $|\Delta(0,a]|$ is a closed $(k-1)$-cell; for rank $a \geq 2$, the marked face of $|\Delta(0,a]|$ is $|\Delta(0,b]|$, where b is the unique element covered by a in the maximal subtree Q_* of Q.

<u>4.2 Proposition</u> *For* Q *an* S*-poset*, $G(Q)$ *is an* S*-polycell with a canonical cone structure on each cell.*

Before giving the proof of Proposition 4.2 we need:

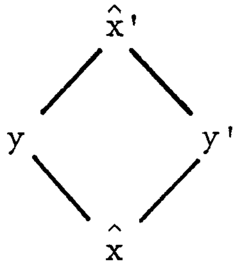
<u>4.3 Lemma</u> *For each element* a *of* Q' *with* $\rho(a) = k \geq 2$, $\Delta(0,a)$ *is a weak* $(k-2)$*-pseudomanifold (without boundary).*

<u>Proof</u> If $x_0 < x_1 < \ldots < x_q$ is a maximal chain in $(0,a)$ then $0 < x_0 < x_1 < \ldots < x_q < a$ is a maximal chain in $[0,a]$. All maximal chains in $[0,a]$ have length $\rho(a) = k$ so $q = k-2$.

Thus $\Delta(0,a)$ is a pure $(k-2)$-dimensional complex.

Suppose $x_0 < x_1 < \ldots < x_{k-3}$ is a chain of length $k-3$ in $(0,a)$ (When $k = 2$ we have the empty chain.) Then $x_0 < x_1 < \ldots < x_{k-3}$ is contained in the chain $c: 0 < x_0 < \ldots < x_{k-3} < a$ of length $k-1$ in $[0,a]$ and $0 = \rho(0) < \rho(x_0) < \rho(x_1) < \ldots < \rho(x_{k-3}) < \rho(a) = k$. It follows that $\rho(x)$, $x \in c$, runs through all the integers $0, 1, \ldots, k$ except one, say q such that $1 \leq q \leq k-1$. Thus there is one pair $\hat{x} < \hat{x}'$ of adjacent elements in c such that $\rho(\hat{x}) = q-1$, $\rho(\hat{x}') = q+1$ and, for any other pair $x < x'$ of adjacent elements, $\rho(x') = \rho(x) + 1$. This means that (since $z < z'$ in Q $\Rightarrow \rho(z) < \rho(z')$) for any chain c' of length k in $[0,a]$ containing c, the single element y of c' not contained in c must satisfy $\hat{x} < y < \hat{x}'$.

Now $[\hat{x}, \hat{x}']$ in Q is of the form



Hence there are two chains of length k in $[0,a]$ which contain c. Therefore $x_0 < x_1 < \ldots < x_{k-3}$ is contained in exactly two chains of length $k-2$ in $(0,a)$. That is, each $(k-3)$-simplex of the complex $\Delta(0,a)$ is a face of exactly two $(k-2)$-simplices.

It follows that $\Delta(0,a)$ is a weak $(k-2)$-pseudomanifold. □

4.4 Lemma *For each element* $a \in Q'$ *with* $\rho(a) \geq 2$, $\Delta(0,a)$ *is shellable.*

Proof By definition $\Delta[0,a]$ is shellable. Therefore $\Delta(0,a)$ is semishellable (Bjorner [4], p. 161) and, since $\Delta(0,a)$ is a weak pseudomanifold, $\Delta(0,a)$ is shellable. □

Proof of 4.2 Simplicial complexes are identified with their underlying polyhedra.

If a is an element of Q' with $\rho(a) = 1$ then $|\Delta(0,a]|$ is a vertex.

For $a \in Q'$ with $\rho(a) = k \geq 2$, $|\Delta(0,a)|$ is a shellable weak $(k-2)$-pseudomanifold by Lemmas 4.3, 4.4. Hence (Proposition 1.4) $|\Delta(0,a)|$ is a $(k-2)$-sphere. $|\Delta(0,a]|$ is a (simplicial) cone on $|\Delta(0,a)|$ and is thus a $(k-1)$-ball with a canonical cone-complex characteristic map.

The interiors of the simplices of $|\Delta(Q')|$ partition $|\Delta(Q')|$. Since $\mathrm{Bd}\,|\Delta(0,a]| = |\Delta(0,a)|$, the open cell $\mathrm{Int}\,|\Delta(0,a]|$ of $G(Q)$ is the union of the interiors of those simplices which belong to $|\Delta(0,a]|$ but not to $|\Delta(0,a)|$. Such simplices correspond to chains in Q which have a as greatest element. Since a chain has a unique greatest element, it follows that the interior of each simplex of $|\Delta(Q')|$ is contained in exactly one open cell of $G(Q)$. The open cells of $G(Q)$ therefore partition $G(Q)$.

Each chain in $(0,a)$ has a greatest element of rank less than $\rho(a) = k$. Hence the interior of each simplex of $|\Delta(0,a)|$ is contained in an open cell of $G(Q)$ of dimension less than $k-1 = \dim|\Delta(0,a]|$. The interiors of the simplices of $|\Delta(0,a)|$ partition $|\Delta(0,a)|$ therefore $\mathrm{Bd}\,|\Delta(0,a]| = |\Delta(0,a)| \subseteq G(Q)^{(k-2)}$.

It has now been shown that $G(Q)$ is a regular complex with a canonical cone structure on each cell. Thus $G(Q)$ has a canonical cone-complex structure. Since Q has a greatest element 1 each cell of $G(Q)$ is a face of the closed cell $|\Delta(0,1]|$. Hence $G(Q)$ is a cone-cell. The marked face structure makes $G(Q)$ a polycell.

Treating the simplicial complex $|\Delta(Q')|$ as a cone-complex we can prove that $|\Delta(Q')| = \mathrm{Sd}\,G(Q)$ by induction on the skeleta of $G(Q)$. Suppose that $|\Delta(Q')| \cap G(Q)^{(k-1)} = \mathrm{Sd}\,G(Q)^{(k-1)}$. Then (see the remarks at the beginning of this proof) each k-cell of $G(Q)^k$ is the underlying space of a (simplicial) cone on its subdivided boundary and the characteristic maps of $G(Q)^k$ are such that $|\Delta(Q')| \cap G(Q)^k = \mathrm{Sd}\,G(Q)^k$. There is no difficulty in starting the process.

For each element $a \in Q'$ with $\rho(a) \geq 2$, $|\Delta(0,a)|$ is shellable by Lemma 4.4. Hence $|\Delta(0,a]| = C\,|\Delta(0,a)|$ is shellable; that is, the barycentric subdivision of the face $|\Delta(0,a]|$ of $G(Q)$ is shellable. It follows that $G(Q)$ is an S - polycell. □

<u>4.5 Proposition</u> G *defines a functor* SPos → SPoly . □

<u>4.6 Proposition</u> *The functors* F *and* G *define an adjoint equivalence*

$$F : \mathsf{SPoly} \rightleftarrows \mathsf{SPos} : G .$$

<u>Proof</u> <u>$F \circ G \simeq 1_{\mathsf{SPos}}$</u> For Q an S - poset, the elements of $F \circ G(Q)$ are faces $|\Delta(0,a]|$, $a \in Q'$, of $G(Q)$ and $|\Delta(0,a]| \leq |\Delta(0,b]|$ in $F \circ G(Q)$ if and only if $|\Delta(0,a]| \subseteq |\Delta(0,b]|$ in $G(Q)$, that is, if and only if $a \leq b$ in Q. The function $\xi: Q \to F \circ G(Q)$ defined by

$$\xi(a) = \begin{cases} |\Delta(0,a]| & a \in Q' \\ \emptyset & a = 0 \end{cases}$$

is therefore a poset isomorphism.

In the maximal subtree $F \circ G(Q)_*$ of $F \circ G(Q)$, $|\Delta(0,a]| \leq_* |\Delta(0,b]|$ if and only if $|\Delta(0,a]|$ belongs to the flag in $|\Delta(0,b]|$. This occurs if and only if $a \leq_* b$ in Q_*. The map ξ therefore induces a poset isomorphism $Q_* \to F \circ G(Q)_*$. It follows that ξ an SPos - isomorphism. Naturality is easily checked so we have $F \circ G \simeq 1_{\mathsf{SPos}}$.

<u>$G \circ F \simeq 1_{\mathsf{SPoly}}$</u> Let A and B be polycells. An S - poset isomorphism $F(A) \to F(B)$ defines an SPoly - isomorphism $A \to B$. By the remarks above we have, for X an S - polycell, $F \circ G \circ F(X) \cong F(X)$. There is thus an SPoly - isomorphism $G \circ F(X) = X$ and we find that $G \circ F \simeq 1_{\mathsf{SPoly}}$. □

An immediate consequence of Proposition 4.6 is :

<u>4.7 Proposition</u> *If* P , P' *are skeleta of* SPoly , SPos *respectively then* P *is isomorphic to* P' . □

By Proposition 2.10 each category in the class EΓ is isomorphic to a full subcategory of P . We have therefore shown that each member of EΓ is isomorphic to a full subcategory of a combinatorial category.

Extra shellability or collapsibility conditions can be imposed on S - polycells to obtain subcategories of P in EΓ which are isomorphic to similarly defined subcategories of P'. (The notion of collapsing is defined in III §3 and an extra condition on S - polycells is discussed in Chapter V.) It may be that the category Cv ∈ EΓ based on convex polytopes can be defined in such a way. It is known [20, p. 37] that there exists a purely combinatorial characterization of convex polytopes and, according to Danaraj and Klee, this characterization could consist of various strong shellability conditions.

CHAPTER III

EQUIVALENCES OF CATEGORIES OF T-COMPLEXES

The aim of this chapter is to define a category of MT-*complexes* for each $M \in E\Gamma$, and to show that, for $M \in E\Gamma$, there is an equivalence of categories between MT-complexes and simplicial T-complexes. This result is obtained by proving:

(i) the categories of simplicial T-complexes and Δ_IT-complexes are isomorphic;

(ii) there is an equivalence of categories MT-complexes $\to$ Δ_IT-complexes for $M \in E\Gamma$.

A feature of our proof of (ii) is that we work as far as possible in the model category. In particular, collapsing of polycells plays a central role. The use of (cubical) collapsing in a similar context was introduced by Brown-Higgins to construct the homotopy ω-groupoid of a filtered space [11] and to construct the ω-groupoid structure on a cubical T-complex [12]. Rourke-Sanderson [33] and Hintze [26] also use collapsing in work on Δ_I-sets and **P**-sets respectively.

The chapter is laid out as follows. The T-complex axioms are introduced and applied to M-sets in §1. An isomorphism between simplicial and Δ_IT-complexes is constructed in §2, using work of Fritsch [24]. In §3, a definition is given of collapsing in cone-complexes and the duality between the notions of collapsing and thin fillers of boxes is noted. This duality allows us to translate work on collapsing into reasoning about elements of MT-complexes. Accordingly, §§4, 5 consider particular collapses of certain $\vec{SC}$ -complexes. These collapses are used in §§6, 7, 8 to prove (ii) above and hence the main equivalence theorem.

§1 T-complexes

The notion of a T-complex was introduced in a simplicial context by Dakin [19] . The cubical version has played an important part in work of Brown and Higgins [10, 11, 12] .

1.1 Definition [10, 12, 19] A *simplicial (cubical)* T-*complex* (K,T) is a simplicial (cubical) set K having in each dimension $n \geq 1$ a set $T_n \subset K_n$ of elements (which are called *thin*) satisfying the axioms:

(T1) Every degenerate element of K is thin;

(T2) Every box in K has a unique thin filler;

(T3) If all faces but one of a thin element of K are thin then so is the remaining face.

Simplicial T-complexes are the objects of a category ΔTC whose morphisms are simplicial maps preserving thin elements. The category $\square$TC has cubical T-complexes as objects and cubical maps preserving thin elements as morphisms.

In order to define a category of MT-complexes for $M \in \Gamma$ we have to equip M-sets with thin elements satisfying a set of axioms similar to that of definition 1.1 . There is a notion of a box and filler in an M-set so axioms T2 and T3 may be used. On the other hand, we have to dispense with axiom T1 since the fact that each $M \in \Gamma$ has only injective morphisms implies that an M-set has no degeneracy maps. It turns out that a definition of MT-complexes using T2 and T3 is satisfactory.

Before defining a box in an M-set we note the following. If $f: A \to B$ is a Poly-morphism then, for any category $M \in \Gamma$ having objects A' and B' Poly-isomorphic to A and B respectively, there is a unique morphism $f': A' \to B'$ in M such that

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \cong\uparrow & & \downarrow\cong \\ A & \xrightarrow{f} & B \end{array}$$

commutes (see Proposition I4.7). Since M is skeletal we can refer to f' , without confusion, as the *M-morphism corresponding to* f .

We call an object of M an *M-cell.*

<u>1.2 Definition</u> For a category $M \in \Gamma$, let X be an n-dimensional M-cell with (n-1)-faces $X_0, X_1, \ldots, X_q$. For an M-set $K: M^{op} \to \mathrm{Set}$, a *box* B in K is a set $\{x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_q\}$ of (n-1)-cells of K such that $x_j \in K(X_j')$, where X_j' is the M-cell corresponding to X_j , and if $A \subset X_k \cap X_r$ is a face of X and $\delta^k: A' \to X'_k$, $\delta^r: A' \to X'_r$ are the M-morphisms corresponding to the inclusions $A \hookrightarrow X_k$, $A \hookrightarrow X_r$ then $K(\delta^k)x_k = K(\delta^r)x_r$.

<u>1.3 Definition</u> A *filler* of the box B is an n-cell $x \in K(X)$ such that $\partial_{X_j} x = x_j$, $j \neq i$.

A more concise definition of a box is given in §3 .

<u>1.4 Definition</u> For $M \in \Gamma$, an MT-*complex* (K,T) is an M-set K having, associated to each M-cell X ($\dim X \geq 1$), a subset $T(X)$ of $K(X)$ whose elements (which are called *thin*) satisfy the following axioms:

(T2) Every box in K has a unique thin filler;

(T3) If all but one of the $(n-1)$-faces of a thin n-cell of K are thin then so is the remaining $(n-1)$-face.

For each $M \in \Gamma$ we can define a category MTC whose objects are MT-complexes and whose morphisms are M-set morphisms which preserve thin elements. Where no confusion arises we will write K for the T-complex (K,T).

A functor $MTC \to \Delta_I TC$ for $M \in E\Gamma$ can be defined here. By Proposition II 2.5, Δ_I is isomorphic to a full subcategory Δ_M of M. Hence we can associate a Δ_I-set $r_M K$ to each M-set $K: M^{op} \to \mathrm{Set}$ merely by restricting K to Δ_M. Each M-set morphism $g: K \to L$ restricts to a Δ_I-set morphism $r_M g: r_M K \to r_M L$ so we have a functor r_M: M-sets $\to$ Δ_I-sets. If K is an MT-complex then $r_M K$ is a Δ_IT-complex and if $g: K \to L$ preserves thin cells so does $r_M g$. We can thus state:

<u>1.5 Proposition</u> *For* $M \in E\Gamma$, *there is a functor* $r_M: MTC \to \Delta_I TC$ *defined by restriction.* □

§2 The isomorphism $\Delta TC \to \Delta_I TC$

This section is concerned with the proof of the following result.

<u>2.1 Theorem</u> *The categories of simplicial* T-*complexes and* Δ_IT-*complexes are isomorphic.*

Since Δ_I is a wide subcategory of the simplicial category Δ there is a forgetful functor ξ:simplicial sets $\to$ Δ_I - sets . A T-complex structure on a simplicial set K is inherited (insofar as it applies) by ξK . We obtain easily:

<u>2.2 Proposition</u> ξ *defines a functor* $\Delta TC \to \Delta_I TC$. □

The simplicial T-complex ηK associated with a Δ_IT-complex K is obtained by equipping K with a set of degeneracy maps. Various authors [24, 33, 39] have shown that a Kan Δ_I-set K admits a (non-canonical) system of degeneracy maps. In particular, Fritsch [24] uses certain sequences of fillers of boxes to construct degenerate elements. By following Fritsch's method, but using only thin fillers, we can construct a canonical system of degeneracy maps for a Δ_IT-complex K such that each degenerate element is thin; that is, we can define a simplicial T-complex.

We write d_i and s_i for the standard face and degeneracy maps of a simplicial set. For L a simplicial set and $0 \le n < m$ let $A(n,m)$ be the set of maps $a:L_n \to L_m$ of the form $a = s_{i_k} s_{i_{(k-1)}} \dots s_{i_o}$, where $k = m - n - 1$ and $i_o < i_1 < \dots < i_k$.

<u>2.3 Definition</u> [24] Let $a = s_{i_k} s_{i_{(k-1)}} \dots s_{i_o}$ and $a' = s_{j_k} s_{j_{(k-1)}} \dots s_{j_o}$ be elements of $A(n,m)$. We say that a' is a *predecessor* of a if $a \ne a'$ and there exists $j \in \{0,1,\dots,m\}$ such that $d_j s_{i_{k+1}} a = a'$.

<u>2.4 Lemma</u> [24] *The relation 'predecessor of' generates a partial order* $\le_p$ *on the set* $A(n,m)$.

Proof Clearly, a reflexive, transitive relation $\leq_p$ is generated.

If a' is a predecessor of a then $i_q \leq j_q$ for $q = 0,1,\ldots,k-1$ and $i_k = j_k$. Hence $\leq_p$ is antisymmetric.□

2.5 Lemma *There exists a (non-canonical) total order* $\leq_t$ *on* $A(n,m)$ *such that* $a' \leq_p a \Rightarrow a' \leq_t a$.

Proof This follows from the properties of a finite poset. □

A system of degeneracy maps for a $\Delta_I T$-complex K can now be defined inductively. It is sufficient to construct all (well-defined) degenerate elements $s_{i_k} s_{i_{(k-1)}} \ldots s_{i_o} z$ for $k \geq 0$, $i_o < i_1 < \ldots < i_k$, and z a non-degenerate element of K. Setting $a = s_{i_k} s_{i_{(k-1)}} \ldots s_{i_o}$, az is defined as a face of a thin element z_a.

Assume the following hold.

(i) For each non-degenerate element y of K of dimension $<n$ and for $r \geq 0$, each element $s_{j_r} s_{j_{(r-1)}} \ldots s_{j_o} y$, $j_o < j_1 < \ldots < j_r$, has been defined.

(ii) For a non-degenerate element $z \in K_n$ and $0 \leq r < k = m - n - 1$, each element $s_{j_r} s_{j_{(r-1)}} \ldots s_{j_o} z$, $j_o < j_1 < \ldots < j_r$, has been defined.

(iii) Each

element bz such that $b \leq_p a$ in $A(n,m)$ has been defined.

The element az is given below. If b is a composite of degeneracy maps, $\not{d}_j b$ represents the map obtained by moving d_j as far as possible to the right, using the simplicial set identities; thus $\not{d}_3 s_4 s_1$ represents $s_3 s_2 d_s$. There are two cases:

(1) $i_k > i_{(k-1)}+1$ (and $k = 0$)

Let z'_a be the thin filler of the box

$(\underset{\rightarrow}{d}_o az, \underset{\rightarrow}{d}_1 az, \ldots, \underset{\rightarrow}{d}_{i_k} az, -, \underset{\rightarrow}{d}_{i_k+2} az, \ldots, \underset{\rightarrow}{d}_m az)$.

Form the box

$(\underset{\rightarrow}{d}_o s_{i_k+1} az, \ldots, \underset{\rightarrow}{d}_{i_k-1} s_{i_k+1} az, -, z'_a, z'_a, \underset{\rightarrow}{d}_{i_k+3} s_{i_k+1} az, \ldots, \underset{\rightarrow}{d}_{m+1} s_{i_k+1} az)$

and denote the thin filler by z_a. Let $az = d_{i_k} z_a$.

(2) $i_k = i_{(k-1)} + 1$

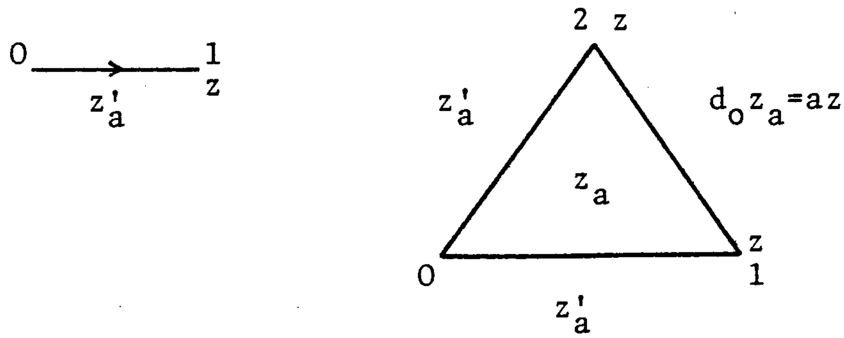
Set $a' = s_{i_{(k-1)}} s_{i_{(k-2)}} \ldots s_o$ and let

$r = \min\{r' \mid d_{r'} a = a'\}$. By the inductive hypothesis $a'z$ is a face of $z_{a'}$. Form the box

$(\underset{\rightarrow}{d}_o s_{i_k+1} az, \ldots, \underset{\rightarrow}{d}_{r-1} s_{i_k+1} az, -, z_{a'}, \ldots, z_{a'}, \underset{\rightarrow}{d}_{i_k+3} s_{i_k+1} az, \ldots, \underset{\rightarrow}{d}_{m+1} s_{i_k+1} az)$

and take z_a to be the thin filler. Let $az = d_r z_a$.

The induction starts as follows. For z an 0-dimensional element of K and $a = s_o$ (case (1)), z'_a is the thin filler of the box $(\underset{\rightarrow}{d}_o az, -) = (z, -)$; z_a is the thin filler of the box $(-, z'_a, z'_a)$; and $az = d_o z_a$.



It is easily checked (making use of 2.4, 2.5) that the elements of the boxes used in cases (1), (2) have been defined earlier in the inductive process.

The degenerate elements satisfy the simplicial set identities, and a simple inductive argument shows that each degenerate element is thin. If we write ηK for the Δ_IT-complex K together with the system of degeneracy maps, we have:

<u>2.7 Proposition</u> ηK *is a simplicial* T-*complex*. □

There is an obvious alternative characterization of ηK .

<u>2.8 Proposition</u> *For* K *a* Δ_IT-*complex, the simplicial* T-*complex* ηK *associated with* K *is the complex* K *together with the system of degeneracy maps defined inductively as follows. For* $n \geq 0$ *and* z *an* n-*dimensional element,* $s_i z$ $(0 \leq i \leq n)$ *is the unique thin filler of the box.*
$b_i z = (s_{i-1}d_0 z, \ldots, s_{i-1}d_{i-1}z, z, -, s_i d_{i+1} z, \ldots, s_i d_n z)$.
(*For* $\dim z = 0$, $b_0 z$ *reduces to* $(z, -)$.) □

Notice that though the definition of ηK given in Proposition 2.8 is neater it is necessary to go through the work leading up to the first definition. When the box $b_i z$ is filled with $s_i z$, the free face $d_{i+1}s_i z$ of $s_i z$ is not known. Reasoning similar to that used in connection with Proposition 2.7 is required to show $d_{i+1}s_i z = z$, so that the simplicial set identities are satisfied.

Since a Δ_ITC - morphism $f: K \to L$ preserves thin elements it is compatible with the degeneracy maps given above. There is thus a canonical ΔTC - morphism $\eta f: \eta K \to \eta L$ associated with f . We obtain:

<u>2.9 Proposition</u> η *defines a functor* $\Delta_I\text{TC} \to \Delta\text{TC}$. □

It is immediate that $\xi \circ \eta = 1_{\Delta_I TC}$.

For K a simplicial T-complex and $z \in K_n$, $s_i z$ is the thin filler of the box
$(s_{i-1}d_o z,\ldots,s_{i-1}d_{i-1}z,z,-,s_i d_{i+1}z,\ldots,s_i d_n z)$. It follows from Proposition 2.8 that $\eta \circ \xi K = K$ and we find that $\eta \circ \xi = 1_{\Delta TC}$. This completes the proof of Theorem 2.1 .

§3 Structures and collapsing

We now start on the geometric preparation for the proof of the equivalence $MTC \to \Delta_I TC$ for $M \in E\Gamma$. The first step is to introduce collapsing in the context of cone-complexes and also the notion of a thin expansion of a structure in an MT-complex.

We will be particularly concerned with Δ_I and the categories in $E\Gamma$. For convenience we write $E\Gamma^+ = E\Gamma \cup \{\Delta_I\}$.

A cone-complex is a CW-complex. The notion of CW-collapse is one of the basic ideas in Whitehead's simple-homotopy theory. We quote the definition given in Cohen [18, p.14] .

3.1 Definition Let (X,Y) be a finite CW pair. Then $X \overset{e}{\searrow} Y$, that is , X *collapses to* Y *by an elementary collapse if*

(1) $X = Y \cup e^{n-1} \cup e^n$ where e^n and e^{n-1} are not in Y ;

(2) there exists a ball pair $(Q^n,Q^{n-1}) \cong (I^n,I^{n-1})$ and a map $\phi\colon Q^n \to X$ such that

(a) ϕ is a characteristic map for $\bar{e}^n$,

(b) $\phi|Q^{n-1}$ is a characteristic map for $\bar{e}^{n-1}$,

(c) $\phi(P^{n-1}) \subset Y^{n-1}$ where $P^{n-1} = \mathrm{cl}(\partial Q^n - Q^{n-1})$.

(Here the term 'characteristic map' has its ordinary meaning and does not refer to our special map $C\partial e^n \to e^n$.)

We take 3.1 as our basic definition of an elementary collapse of a cone-complex. However, we are mainly interested in $\overrightarrow{SC}$ - complexes (definition II 1.10) since the objects of $M \in E\Gamma^+$ are S-polycells (cells of $\overrightarrow{SC}$ - complexes). By Proposition II 1.8 , condition (2) of 3.1 is satisfied by any pair (X,Y) of $\overrightarrow{SC}$ - complexes for which (1) holds and $e^{n-1} \subset \bar{e}^n$. Hence 3.1 reduces in the $\overrightarrow{SC}$ - complex case to a form precisely analogous to the usual simplicial definition of an elementary collapse, namely:

<u>3.2 Definition</u> Let (X,Y) be a pair of $\overrightarrow{SC}$ - complexes. There is an *elementary collapse* from X to Y , written $X \overset{e}{\searrow} Y$, if for some $s \geq 1$ there is an open s-cell a of X and an open (s-1)-cell $b \subset \bar{a}$ such that

$$X = Y \cup \bar{a} \; , \; Y \cap \bar{a} = \partial a - b \; .$$

We say that X *collapses* to Y , $X \searrow Y$, if there is a sequence of elementary collapses

$$X = X_0 \overset{e}{\searrow} X_1 \overset{e}{\searrow} \dots \overset{e}{\searrow} X_q = Y \; .$$

The cell a in definition 3.2 is referred to as the *major cell* of the elementary collapse $X \overset{e}{\searrow} Y$ and b is called the *minor cell*.

<u>3.2 Definition</u> For $M \in E\Gamma^+$, an $\overrightarrow{SC}$ - complex each of whose faces is SPoly - isomorphic to an M-cell is called an $M\text{-}\overrightarrow{SC}$ - *complex*.

It is standard that an ordered simplicial complex defines a simplicial set. Similarly, for $M \in E\Gamma^+$, an M-set can be associated to each M-$\overrightarrow{SC}$-complex.

<u>3.4 Definition</u> For $M \in E\Gamma^+$, let U be an M-$\overrightarrow{SC}$-complex. The M-set $U_M: M^{op} \to Set$ is defined on the objects of M by $U_M(X) = \{k | k: X \to U$ is an $\overrightarrow{SC}$-morphism$\}$ and on morphisms by $U_M(X \xrightarrow{f} Y)(Y \xrightarrow{k} U) = X \xrightarrow{k \circ f} U$.

For M-$\overrightarrow{SC}$-complexes U, V and an $\overrightarrow{SC}$-morphism $g: U \to V$ a morphism $g_M: U_M \to V_M$ of M-sets can be defined by

$$g_M(X \xrightarrow{k} U) = X \xrightarrow{g \circ k} V .$$

We thus have a faithful functor from the full subcategory of $\overrightarrow{SC}$ whose objects are M-$\overrightarrow{SC}$-complexes to the category of M-sets.

Let Z be a face of the M-$\overrightarrow{SC}$-complex U of definition 3.4 and let Z' be the M-cell $\overrightarrow{SC}$-isomorphic to Z. Let the cell $Z_M \in U_M(Z')$ be the $\overrightarrow{SC}$-morphism $Z' \to U$ with range Z. An important special case of 3.4 is $U =$ an n-dimensional M-cell. Then $U_M: U' \to U$ is the single top-dimensional cell of the M-set U_M. That is, U_M is the free M-set on the n-cell $U_M \in U_M(U')$.

From now on we omit the suffix M and write U for the M-set U_M and Z for the cell $Z_M \in U_M(Z')$.

<u>3.5 Definition</u> For $M \in E\Gamma^+$, let U be an M-$\overrightarrow{SC}$-complex. For any M-set K, a morphism $\mathcal{U}: U \to K$ of M-sets is called a (U-) *structure* in K.

The notions of a structure and a collapse can be used to give a definition equivalent to 1.2 of a box in an M-set.

(Although we only deal with $M \in E\Gamma^+$ here, the definition is easily extended to all $M \in \Gamma$) .

<u>3.6 Definition</u> For a category $M \in E\Gamma^+$, let X by an M-cell and let $X \overset{e}{\searrow} H$ be an elementary collapse of X . A *box* in an M-set K is a structure $\mathcal{H}: H \to K$.

A *filler* of the box $\mathcal{H}$ may be defined, equivalently to 1.3 , to be a structure $\mathcal{X}: X \to K$ extending $\mathcal{H}$. (That is, $\mathcal{X} \circ i = \mathcal{H}$, where $i: H \to X$ is the inclusion).

If K is an MT-complex there is a unique thin filler of $\mathcal{H}$. Intuitively we have, corresponding to the elementary collapse $X \overset{e}{\searrow} H$, a dual operation of filling the box $\mathcal{H}$ thinly. This idea can be generalized as follows.

<u>3.7 Definition</u> For $M \in E\Gamma^+$, let $V \subset U$ be a pair of M-$\overrightarrow{SC}$-complexes and suppose there is a collapse $C: U \searrow V$. For K an MT-complex and $\mathcal{V}: V \to K$ a structure in K , we say that a structure $\mathcal{U}: U \to K$ is a *thin expansion of* $\mathcal{V}$ *corresponding to* C if $\mathcal{U}$ extends $\mathcal{V}$ and $\mathcal{U}(\bar{a})$ is a thin cell of K for each major cell a of C .

<u>3.8 Proposition</u> *There exists a unique thin expansion of* $\mathcal{V}$ *corresponding to* C .

<u>Proof</u> Suppose C is the collapse $U = U_0 \overset{e}{\searrow} U_1 \overset{e}{\searrow} \dots \overset{e}{\searrow} U_q = V$. Assume that there is a unique structure $\mathcal{U}_j: U_j \to K$ extending $\mathcal{V}$ such that $\mathcal{U}_j(\bar{a})$ is thin if a is a major cell of C .

Let a_j, b_j be the major and minor cells respectively of the elementary collapse $U_{j-1} \overset{e}{\searrow} U_j$. The restriction of $\mathcal{U}_j$ to $\partial a_j - b_j$ defines a box $\mathcal{H}$ in K . We obtain a structure $\mathcal{U}_{j-1}: U_{j-1} \to K$ extending $\mathcal{U}_j$ by setting $\mathcal{U}_{j-1}(\bar{a})$ = thin filler of the box $\mathcal{H}$.

The result follows by induction, taking $\mathcal{U}_q = \mathcal{V}$. □

It follows from Proposition 3.8 that we can define certain structures in MT-complexes by means of collapses of $\overrightarrow{\text{M-SC}}$ - complexes. The next two sections are concerned with collapses required for the proof of equivalence $\text{MTC} \to \Delta\text{TC}$. The idea behind the proof is to 'barycentrically subdivide' the cells of an MT-complex K , that is, to think of $x \in K(X)$ as the 'sum' of a collection of simplices of K defined by a structure $SdX \to K$ in K . In order to work with such structures we need to describe certain collapses of SdX and VX where X is an S-polycell (see definitions II 2.1, 2.2). Note that, for Z a $\overrightarrow{\text{CC}}$ - or $\overrightarrow{\text{SC}}$ - complex, CZ and SdZ denote the $\overrightarrow{\text{CC}}$ - cone and subdivision unless otherwise stated.

§4 Particular collapses in SdX and VX

For X an S-polycell, we specify collapses in SdX using a total order $\zeta_s(X)$ on the set of cells in the interior of SdX . Recall that there is a total order $\zeta(X)$ on the set of faces of X (I 5.3) .

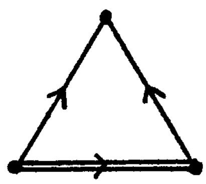
__4.1 Definition__ The order $\zeta_s(X)$ is defined by induction on the dimension of X .

Assume that $\zeta_s(Y)$ is defined for $\dim Y < n$ and let $\dim X = n$. To each open cell e_λ in $SdBdX$ associate an ordered pair (i,j) where i is the position in $\zeta(X)$ of the unique face Z of X with $e_\lambda \subset \text{Int}\, Z$, and j is the position of e_λ in $\zeta_s(Z)$. Let $\zeta_s(BdX)$ be the lexicographic ordering of the cells of SdBdX. Treating SdX as CSdBdX with cone point v, $\zeta_s(\mathrm{BdX})$ induces an ordering of the cells of Int Sd X - v. The ordering $\zeta_s(X)$ is obtained from this by taking v to be the greatest cell.

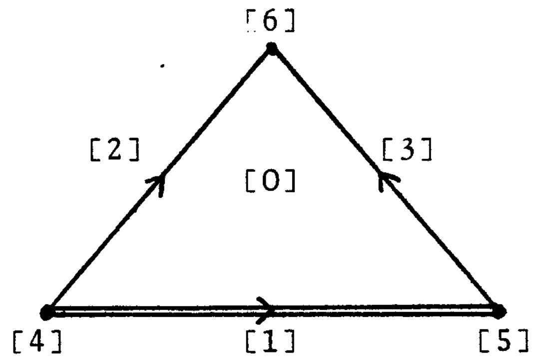
There is no difficulty in starting the induction since, for X an 0-polycell, SdX = X = an 0-cell.

<u>4.2 Example</u> Consider the (Poly-) 2-simplex

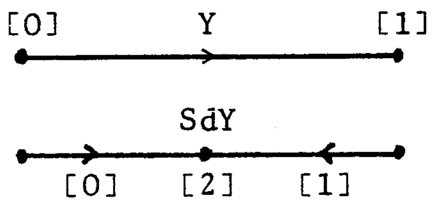
Δ^2 =



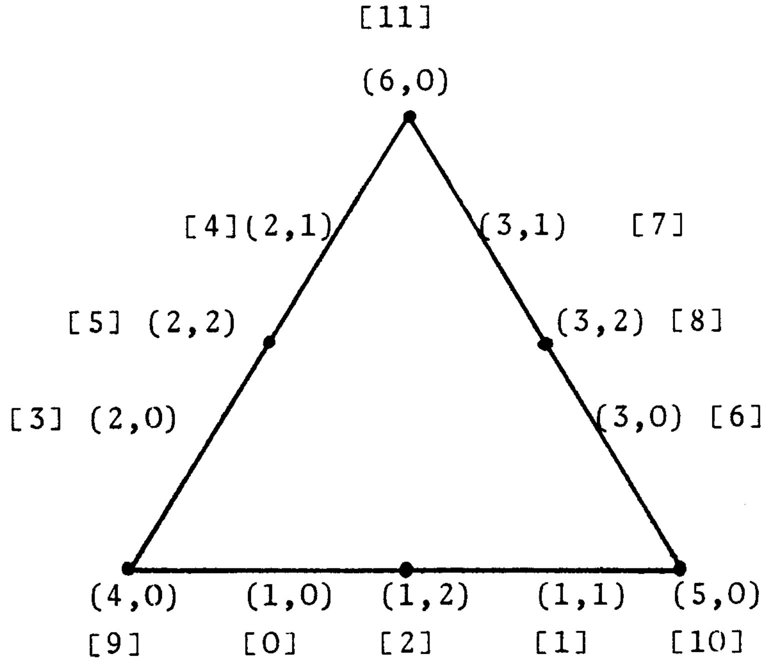
The order $\zeta(\Delta^2)$ is



For any 1-polycell Y, $\zeta_s(Y)$ is defined by

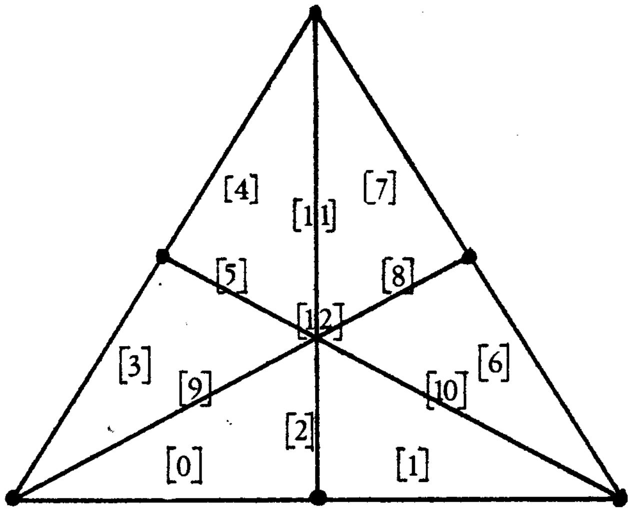


Therefore we can assign ordered pairs to the open cells of $\mathrm{SdBd}\Delta^2$ and obtain $\zeta_s(\mathrm{Bd}\Delta^2)$ as follows



(Numbers in square brackets refer to the order $\zeta_s(\text{Bd}\Delta^2)$.)

This gives $\zeta_s(\Delta^2)$ =



A crucial feature of $\zeta_s(X)$ is that it follows the order $\zeta(X)$ because of the lexicographic ordering used in defining $\zeta_s(\text{Bd}X)$.

A method of associating collapses with orderings of cells is required. Let U, V be $\overrightarrow{SC}$ - complexes such that there is a collapse $U \searrow V$. For any subcomplex U' of U containing V , a *good* elementary collapse $U' \overset{e}{\searrow} U''$ is one which is part of a collapse $U' \searrow V$.

<u>4.3 Definition</u> For any total order ω on the set of open cells of $U - V$ the *collapse* $U \searrow V$ *associated with* ω is defined as follows.

Suppose U has been collapsed to U'. The next elementary collapse has major and minor cells a and b, where a is the least cell of U' in the order ω which can act as a major cell in a good collapse of U', and b is the least cell of U' which can be paired (as minor cell), with a in a good collapse of U'.

We now show that certain collapses are possible in SdX, where X is an S-polycell.

<u>4.4 Proposition</u> *For* $n \geq 1$ *and* X *an* n-*dimensional* S-*polycell there is an* $(n-1)$-*simplex* F_t *of* $SdBdX$ *such that* $SdBdX - \mathrm{Int}\, F_t$ *is* $\overrightarrow{SC}$-*collapsible (to a point).*

<u>Proof</u> By Proposition II 1.9, $SdBdX$ is shellable. Let $F_1, F_2, \ldots, F_t$ be a shelling. Then $SdBdX - \mathrm{Int}\, F_t$ has a shelling $F_1, F_2, \ldots, F_{t-1}$ satisfying condition (ii) of Definition II 1.1 at each stage.

It is standard (Rushing [34, p.17]) that a simplicial complex with such a shelling is (simplicially) collapsible. For the complex $SdBdX - \mathrm{Int}\, F_t$ the notions of $\overrightarrow{SC}$ and simplicial collapsing coincide. □

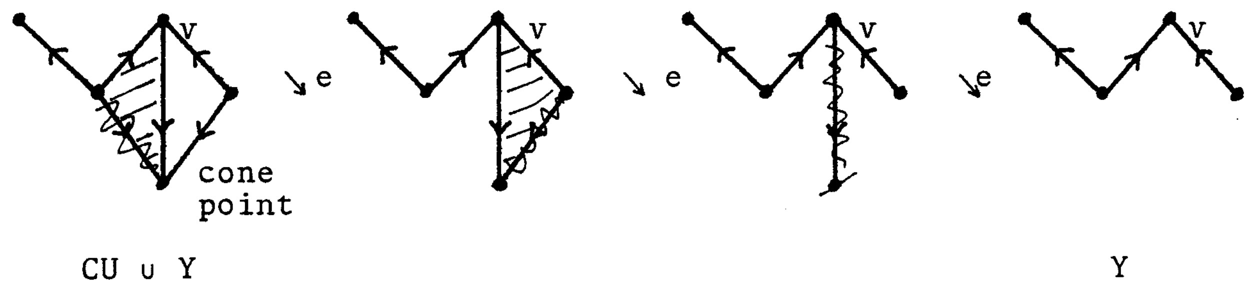
Let $U \subset Y$ be a pair of $\overrightarrow{SC}$-complexes and consider the $\overrightarrow{CC}$-cone CU. Here, and in the sequel, we identify U with $U \times \{0\} \subset CU$ using the canonical isomorphism i and write $CU \cup Y$ for the adjunction space $CU \cup_i Y$. Clearly, $CU \cup Y$

is an $\vec{SC}$ - complex. For any collapse $C: U \searrow V$ there is an $\vec{SC}$ collapse $CU \cup Y \searrow CV \cup Y$ induced by C defined thus: follow the sequence of elementary collapses of C but, at each stage, instead of deleting the major and minor cells a, b of C delete $\text{Int } C\bar{a} \cup \text{Int } C\bar{b}$.

For a collapse $U \searrow v$, v = a vertex, there is an induced collapse $CU \cup Y \searrow Y$ consisting of the induced collapse $CU \cup Y \searrow Cv \cup Y$ followed by the elementary collapse deleting $(\text{Int } Cv) \cup$ cone point. For example, take $C: U \searrow v$ to be:

(cells to be deleted are indicated at each stage).

The collapse $CU \cup Y \searrow Y$ induced by C is:



For X an S-polycell with $\dim X \geq 1$, SdX can be identified with $CSdBdX$. Therefore a collapse $SdBdX - \text{Int } F_t \searrow$ vertex as in Proposition 4.4 induces a collapse $SdX - \text{Int } CF_t = C(SdBdX - \text{Int } F_t) \cup SdBdX$

$$\searrow SdBdX .$$

We thus have:

<u>4.5 Proposition</u> *For* $n \geq 1$ *and* X *an* n-*dimensional* S-*polycell there is a (simplicial) open* n-*cell* e^n *of* SdX *such that there is a collapse* $SdX - e^n \searrow SdBdX$. □

<u>4.6 Definition</u> The open n-cell pX of SdX is defined to be the least cell in the order $\zeta_s(X)$ such that there is a collapse $SdX - pX \searrow SdBdX$.

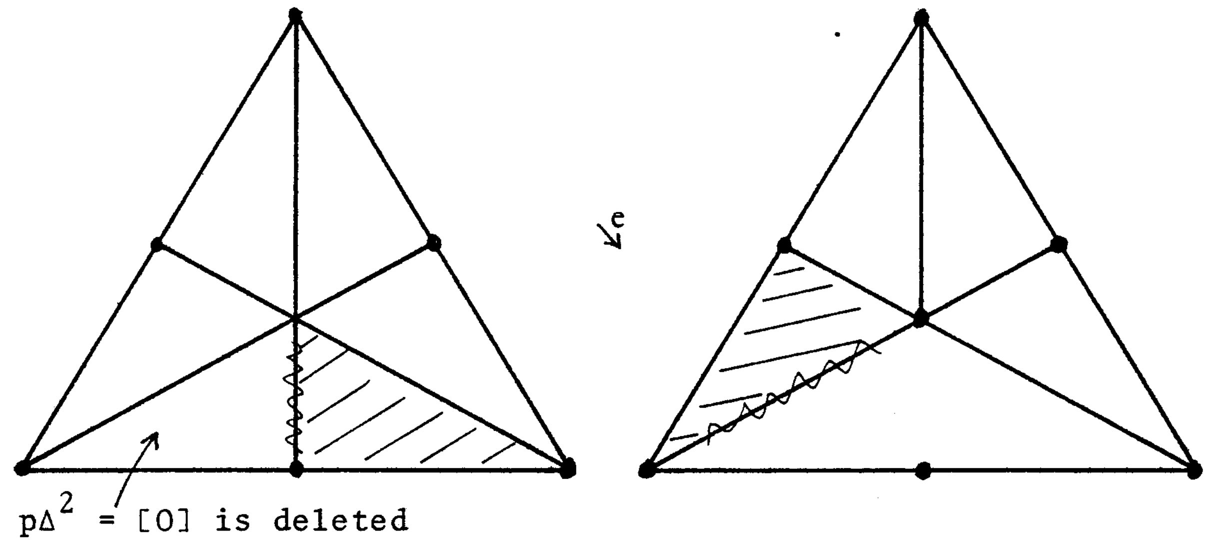
For Y an 0-dimensional polycell we set $pY = SdY = Y$.

<u>4.7 Definition</u> For $n \geq 1$ and X an n-dimensional S-polycell, the collapse

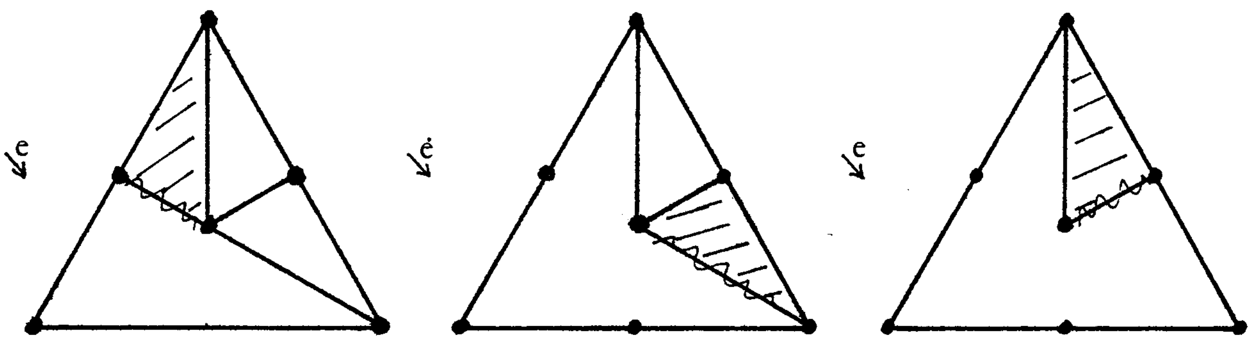
$$A(X):\ SdX - pX \searrow SdBdX$$

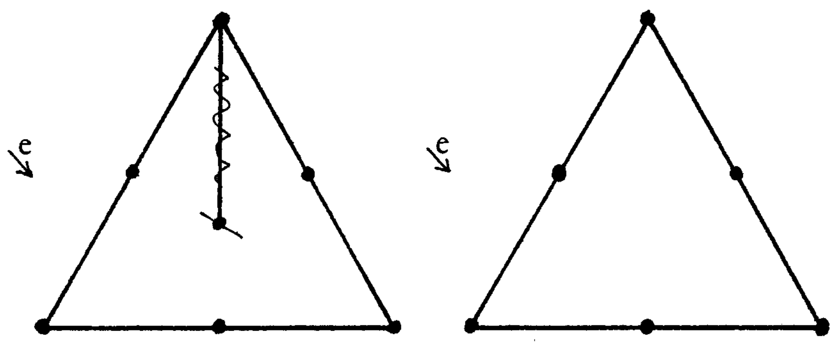
is the collapse associated with the order $\zeta_s(X)$ on the set of cells of $\mathrm{Int}\,SdX - pX$.

<u>4.8 Example</u> In 4.4 the order $\zeta_s(\Delta^2)$ was given. We continue with this example and describe the collapse $A(\Delta^2)$ in $Sd\Delta^2 - p\Delta^2$.



$Sd\Delta^2 - p\Delta^2$





$SdBd\Delta^2$

<u>4.9 Proposition</u> *Let* X *be an* n*-dimensional* S*-polycell* $(n \geq 1)$. *For each* $(n-1)$*-face* t *of* SdBdX *there is a unique sequence*

$$t = t_o, T_1, t_1, T_2, t_2, \ldots, T_q, t_q, T_{q+1} = \overline{pX}, \quad q \geq 0 ,$$

of (distinct) faces of SdX *satisfying, for* $1 \leq i \leq q$:

(i) $\dim T_i = n$, $\dim t_i = n-1$;

(ii) $\operatorname{Int} T_i$ *and* $\operatorname{Int} t_i$ *are a pair of major and minor cells in the collapse* A(X) *in* SdX - pX ;

(iii) $t_i \subset T_i \cap T_{i+1}$ *and* $t \subset T_1$.

<u>Proof</u> Now SdX is a weak n-pseudomanifold with boundary. Hence the $(n-1)$-face $t \subset SdBdX = BdSdX$ is contained in exactly one n-face T_1 . Either $T_1 = \overline{pX}$ or $\operatorname{Int} T_1$ is a major cell of the collapse A(X) . (Each n-cell of SdX - pX is deleted in A(X) and, being of maximum dimension, must be a major cell.) If $T_1 = \overline{pX}$ we are done. If $\operatorname{Int} T_1$ is a major cell there is an $(n-1)$-face t_1 of T_1 such that $\operatorname{Int} t_1$ is the minor cell paired with $\operatorname{Int} T_1$ in A(X) .

Assume that there is a unique sequence $t_o, T_1, t_1, \ldots, T_r, t_r$ satisfying conditions (i)-(iii) . By the definition of A(X) , $t_r \not\subset BdSdX$ and so there is exactly one n-face of SdX other than T_r containing t_r. Denote this n-face by T_{r+1}. In $A(X)$, the elementary collapse deleting $\text{Int}\, T_r \cup \text{Int}\, t_r$ must precede the elementary collapses deleting $\text{Int}\, T_i \cup \text{Int}\, t_i$, $1 \leq i < r$. This can not occur if $t_r \subset T_i$ for $1 \leq i < r$. Hence $T_{r+1} \neq T_i$ for $1 \leq i \leq r$. Either $T_{r+1} = p\overline{X}$ or $\text{Int}\, T_{r+1}$ is a major cell of $A(X)$ and there is an $(n-1)$-face $t_{r+1} \subset T_{r+1}$ such that $\text{Int}\, t_{r+1}$ is the minor cell paired with $\text{Int}\, T_{r+1}$.

The sequence $t = t_o, T_1, t_1, \ldots, T_{q+1} = p\overline{X}$ is thus defined by an inductive process, which (since $Sd\overline{X}$ has a finite number of n-faces) must halt at $p\overline{X}$. □

<u>4.10 Definition</u> For $n \geq 1$, X an n-dimensional S-polycell and t an (n-1)-face of SdBdX the *tube* T(t) on t in SdX is defined to be the subsequence

$$t = t_o, T_1, t_1, \ldots, T_q, t_q$$

of the sequence given in 4.9.

If $q = 0$ then $T(t) = \{t\}$ is a *trivial tube.*

<u>4.11 Definition</u> Let X be an n-dimensional S-polycell $(n \geq 1)$. For an (n-1)-face t of SdBdX the collapse

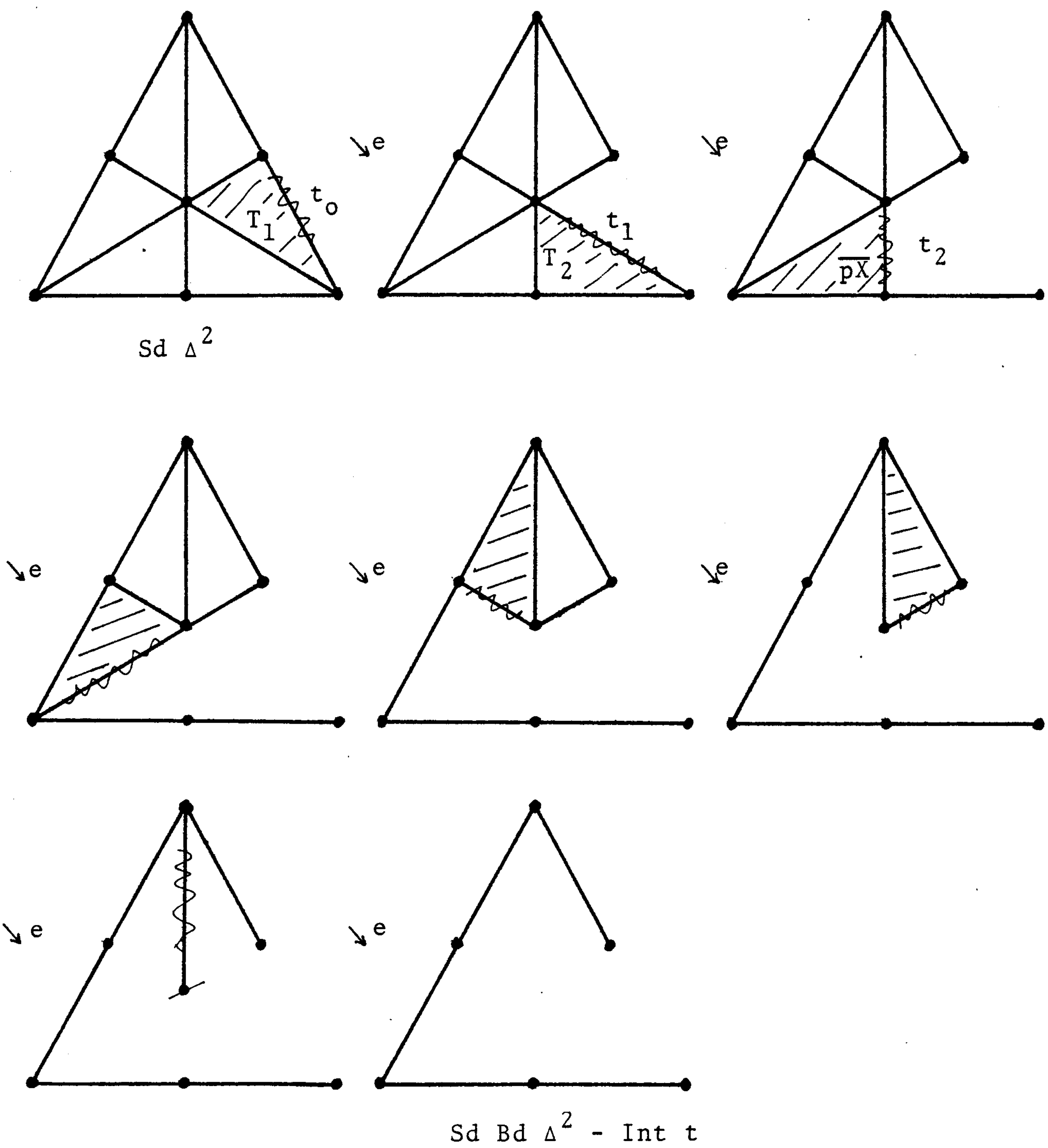
$$B(X,t) : SdX \searrow SdBdX - \text{Int}\, t$$

proceeds as follows. Take the tube T(t) to be $t = t_o, T_1, \ldots, t_q$ and set $T_{q+1} = p\overline{X}$. For $i = 1, 2, \ldots, q+1$, perform the elementary collapse deleting $\text{Int}\, T_i \cup \text{Int}\, t_{i-1}$. Then follow the collapse $A(X)$, ignoring an elementary collapse if its major and minor cells have already been deleted.

We have immediately :

4.12 Proposition *For any* (n-1)*-face* t *of* SdBdX *the major cells of the collapses* B(X,t) *and* A(X) *coincide except that* pX *is a major cell of* B(X,t) *but not of* A(X) . □

4.13 Example Compare the following collapse $B(\Delta^2,t)$ in $Sd\Delta^2$ with the collapse $A(\Delta^2)$ given in 4.8 .

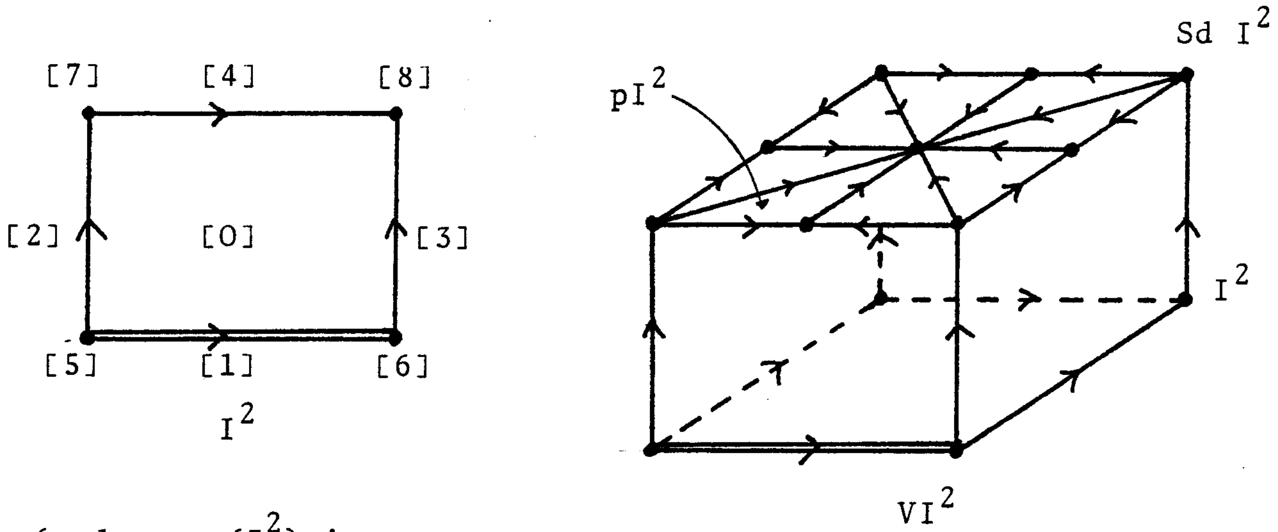


Two collapses in VX , for X an S-polycell, will be needed later on (see Definition II 2.2 and Remark II 2.3(ii)). For any face Y of X , we refer to the subcomplexes $Y\times\{0\}$, $Sd(Y\times\{1\})$ of VY as Y , SdY respectively.

The fact that, with the standard order $\zeta(X)$ on the set of faces of X , $\dim Y > \dim Z \Rightarrow Y <_{\zeta(X)} Z$ ensures that the following definition is meaningful.

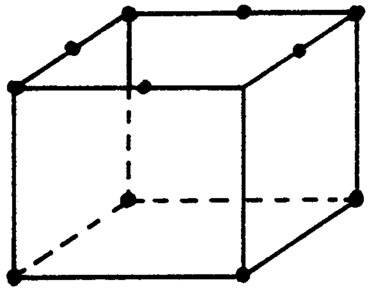
<u>4.14 Definition</u> For X an S-polycell, the collapse $A_0(VX): VX \searrow X$ proceeds as follows. In the order $\zeta(X)$, for each face Y of X , perform the elementary collapse deleting (Int VY) ∪ pY then carry out the collapse A(Y) in SdY - pY . (IF dim Y = 0 pY = SdY so that there is no collapse A(Y) .

<u>4.15 Example</u>

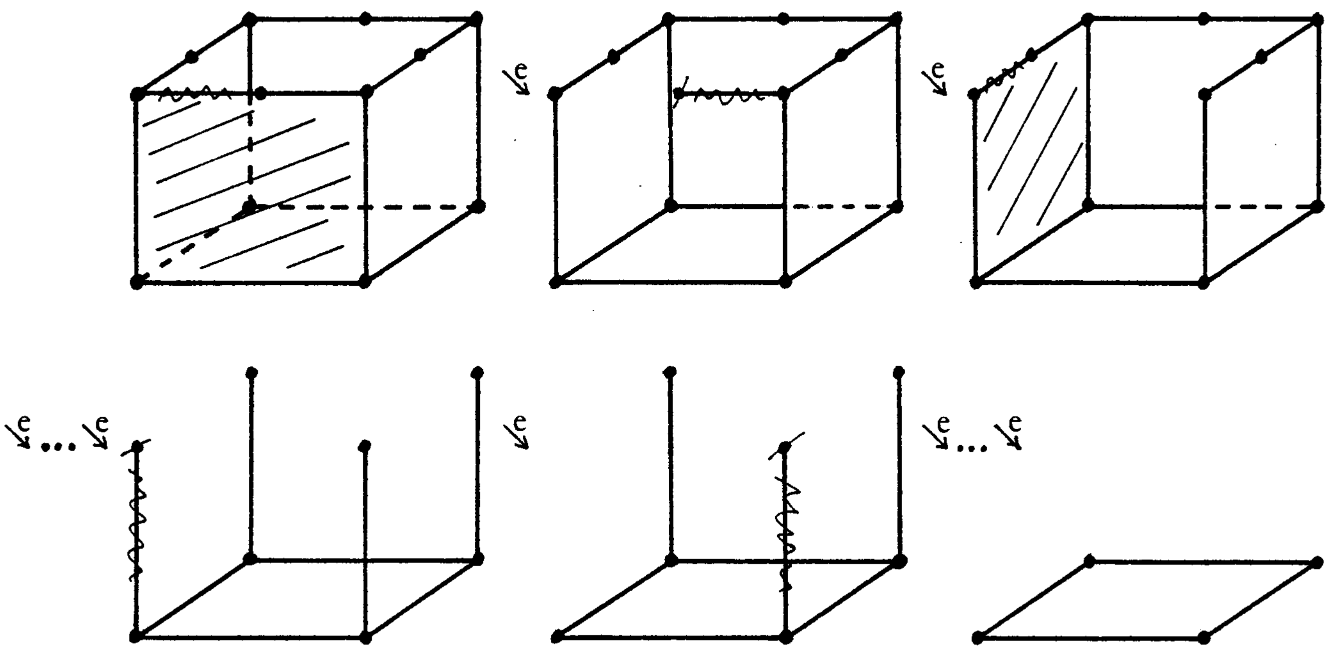


(order $\zeta(I^2)$ in square brackets)

The first elementary collapse of $A_0(VI^2)$ deletes (Int VI^2) ∪ pI^2 . Next comes the collapse $A(I^2)$ in $SdI^2 - pI^2$, leaving



The process is repeated for each face of I^2 in the order $\zeta(I^2)$.



The second of our two collapses in VX is roughly inverse to A_0 .

<u>4.16 Definition</u> .For X an S-polycell, the collapse $A_1(VX): VX \searrow SdX$ proceeds thus: for each face Y of X , in the order $\zeta(X)$, perform the elementary collapse deleting $\mathrm{Int}\, VY \cup \mathrm{Int}\, Y'$.

<u>4.17 Proposition</u> *Each major cell of the collapse* $A_1(VX)$ *is major in* $A_0(VX)$. *The only cells which are major in* $A_0(VX)$ *but not in* $A_1(VX)$ *are the major cells of the collapses* $A(Y)$, Y *a face of* X . □

<u>4.18 Definition</u> Let Z be a subcomplex of the $\vec{\bar{S}C}$ - complex U . A collapse C_Z: $Z \searrow Z_1$ is said to be a *restriction* of a collapse C_U: $U \searrow U_1$ if each elementary collapse in C_Z is also an elementary collapse in C_U .‡

Note that the elementary collapses which belong to both C_Z and C_U need not occur in the same order in C_Z and C_U. If Y is a face of an S-polycell X the order $\zeta(X)Y$ induced by $\zeta(X)$ need not agree with $\zeta(Y)$. Thus the order in which cells of VY are collapsed out in $A_o\ (VY)$: $VY \searrow Y$ (or $A_1(VY)$: $VY \searrow SdY$) may differ from the order in which they are collapsed out in $A_o\ (VX)$ (respectively $A_1(VX)$). However, we obviously have:

<u>4.19 Proposition</u> *For any face* Y *of an* S-*polycell* X , *the collapse* $A_i(VY)$ $(i = 0,1)$ *is a restriction of the collapse* $A_i(VX)$. □

<u>§5 The collapse $A(\Delta^n)$ in $Sd\Delta^n - p\Delta^n$</u>

The collapse $A(\Delta^n)$: $Sd\Delta^n - p\Delta^n \searrow SdBd\Delta^n$, where Δ^n is a (Poly) n-simplex, is particularly important in the proof of equivalence $MTC \to \Delta_I TC$ for $M \in E\Gamma$. Here we derive the crucial properties of the collapse.

There is no difficulty in proving that, for $n \geq 1$ and any n-simplex F of $Sd\ \Delta^n$, there is a collapse $Sd\Delta^n$ - Int $F \searrow SdBd\Delta^n$. We therefore have (see Definition 4.6)::

<u>5.1 Proposition</u> *The cell* $p\Delta^n$ *of* $Sd\Delta^n$ *is the least cell in the order* $\zeta_s(\Delta^n)$. □

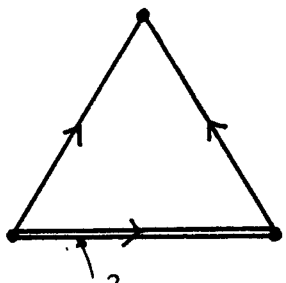
‡ There is an abuse of language here. Precisely, we require that each pair of major and minor cells of C_Z is also a pair of major and minor cells in C_U .

There follows immediately from the definition of $\zeta_S(\Delta^n)$:

<u>5.2 Corollary</u> *For* $n \geq 1$, $(\overline{p\Delta^n})_* = \overline{p(\Delta^n_*)}$ □

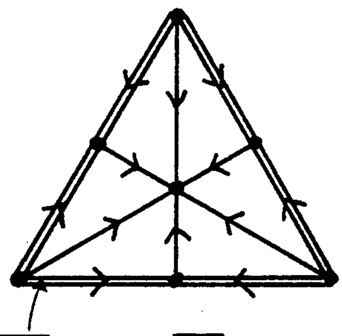
<u>Example</u> (See 4.2 and 4.8)

Δ^2



Δ^2_*

$Sd\Delta^2$



$\overline{p\Delta^2_*} = (\overline{p\Delta^2})_* = \overline{p(\Delta^2{}_*)}$

In order to define a particular collapse $CSd\Delta^n \searrow Sd\Delta^n$ we note:

(1) The collapse $A(\Delta^n)$: $Sd\Delta^n - p\Delta^n \searrow SdBd\Delta^n$ obviously defines a collapse
$Sd\Delta^n - (p\Delta^n \cup p\Delta^n_*) \searrow SdBd\Delta^n - p\Delta^n_*$ which will also be denoted by $A(\Delta^n)$.

(2) Since there is a canonical $\overrightarrow{SC}$ - isomorphism $Sd\Delta^n \to CSdBd\Delta^n$ which maps $\overline{p\Delta^n}$ onto $\overline{Cp\Delta^n_*}$ there is a collapse $A(\Delta^n)$: $CSdBd\Delta^n - \text{Int}\, C\overline{p\Delta^n_*} \searrow SdBd\Delta^n$.

<u>5.3 Definition</u> The collapse $\mathcal{D}: CSd\Delta^n \searrow Sd\Delta^n$, $n \geq 1$, proceeds as follows. First $\text{Int}\, C\overline{p\Delta^n} \cup \text{Int}\, C\overline{p\Delta^n_*}$ is deleted. Then the collapse
$C(Sd\Delta^n - (p\Delta^n \cup p\Delta^n_*)) \cup Sd\Delta^n \searrow C(SdBd\Delta^n - p\Delta^n_*) \cup Sd\Delta^n$
induced by the collapse $A(\Delta^n)$ in $Sd\Delta^n - (p\Delta^n \cup p\Delta^n_*)$ is performed. Finally the collapse $A(\Delta^n)$ in $CSdBd\Delta^n - \text{Int}\, C\overline{p\Delta^n_*}$ is carried out, so that $Sd\Delta^n$ remains.

A central result of this section can now be given. For an example of the process involved see 4.8 . We identify $Sd\Delta^n$ with $CSdBd\Delta^n$.

<u>5.4 Proposition</u> *For* $n \geq 2$ *, the collapse* $A(\Delta^n)$ *in* $Sd\Delta^n - p\Delta^n = C(SdBd\Delta^n - p\Delta^n_*) \cup SdBd\Delta^n$ *satisfies :*

(i) $A(\Delta^n)$ *is the collapse* $C(SdBd\Delta^n - p\Delta^n_*) \cup SdBd\Delta^n \searrow SdBd\Delta^n$ *induced by a collapse* $C_n: SdBd\Delta^n - p\Delta^n_* \searrow$ vertex *which restricts to* $A(Y): SdY - pY \searrow SdBdY$ *for each face* $Y(\cong \Delta^k, k < n)$ *of dimension* ≥ 1 *in* $Bd\Delta^n$.

(ii) For each face X *of* $Bd\Delta^n$ *such that* $\dim X \geq 1$ *and* $X \not\subset \Delta^n_*$, $A(\Delta^n)$ *restricts to the collapse* $\mathcal{D}: CSdX \searrow SdX$.

<u>Proof</u> The proof starts with the definition of a collapse $C_n: SdBd\Delta^n - p\Delta^n_* \searrow$ vertex such that the induced collapse $C_n': C(SdBd\Delta^n - p\Delta^n_*) \cup SdBd\Delta^n \searrow SdBd\Delta^n$ satisifes (i) and (ii) above. It is then shown that $C_n' = A(\Delta^n)$.

Now $SdBd\Delta^n - p\Delta^n_*$ can be identified with $SdCBd\Delta^n_* \cup (Sd\Delta^n_* - p\Delta^n_*)$, where the cone point is the unique vertex v_n of $\Delta^n - \Delta^n_*$. We start C_n by carrying out the collapse $A(\Delta^n_*): Sd\Delta^n_* - p\Delta^n_* \searrow SdBd\Delta^n_*$. This leaves $SdCBd\Delta^n_*$, which is collapsed to the vertex v_n by downward induction on the skeleta of $Bd\Delta^n_* = (\Delta^n_*)^{(n-2)}$.

Suppose that $SdCBd\Delta^n_*$ has been collapsed to $SdC(\Delta^n_*)^k$, where $0 \leq k \leq n-2$. Denote the faces of Δ^n in the order $\zeta(\Delta^n)$ by $\Delta^n = \Delta^n(0)$, $\Delta^n_* = \Delta^n(1)$, $\Delta^n(2)$, ..., $\Delta^n(q)$. By the definition of $\zeta(\Delta^n)$ there exist $r, s \geq 1$ such that $K = \{\Delta^n(r+1), \Delta^n(r+2), \ldots, \Delta^n(r+s)\}$ is the set of $(k+1)$-faces of $C(\Delta^n_*)^k$ and $K' = \{\Delta^n(r+s+1) , \Delta^n(r+s+2) , \ldots, \Delta^n(r+2s)\}$ is the set of k-faces of Δ^n_* . Further, the $\overrightarrow{SC}$ structure of $C(\Delta^n_*)^k$ is such that K' is identical to the set of distinguished faces of elements of K .

$(\chi)_k$
- (a) For $i = (r+1),(r+2),\ldots,(r+s)$, carry out the elementary collapse deleting $p\Delta^n(i) \cup p\Delta^n(i)_*$ then the collapse $A(\Delta^n(i))$ in $Sd\Delta^n(i) - (p\Delta^n(i) \cup p\Delta^n(i)_*)$. (It is clear that the k-face $p\Delta^n(i)_* \subset Sd(\Delta^n_*)^k$ is a free face of $p\Delta^n(i)$.)
- (b) Then perform the collapse $A(\Delta^n i)$ in $Sd\Delta^n(i) - p\Delta^n(i)$ for $i = (r+s+1), \ldots, (r+2s)$.

At this stage, $SdC(\Delta^n_*)^k$ has been collapsed down to $SdC(\Delta^n_*)^{(k-1)}$. Thus (taking $p\Delta^n(i)_* = \Delta^n(i)_*$ for $\Delta^n(i)_*$ an 0-cell at $k = 0$) the collapse C_n has been defined inductively.

It is obvious that the collapse C'_n induced by C_n satisfies conditions (i) of the Proposition . Consider (ii) .

Let X be a face of $Bd\Delta^n$ such that $X \not\subset \Delta^n_*$ and $\dim X = k \geq 1$ $(X \cong \Delta^k)$. We have to show that each elementary collapse of $\mathcal{D}$: $CSdX \searrow SdX$ also belongs to C'_n . That the elementary collapse deleting $\mathrm{Int}\, C\overline{pX} \cup \mathrm{Int}\, C\overline{pX}_*$ and the collapse $C(SdX - (pX \cup pX_*)) \cup SdX \searrow C(SdBdX - pX_*) \cup SdX$ occur in C'_n follows from part (a) of $(\chi)_{k-1}$. There remains the collapse $A(X)$ in $CSdBdX - \mathrm{Int}\, C\overline{pX}_*$.

It is proved below that $A(X)$ in $CSdBdX - \mathrm{Int}\, C\overline{pX}_*$ is identical to C'_k: $C(SdX - pX_*) \cup SdBdX \searrow SdBdX$, induced by C_k: $SdX - pX_* \searrow$ unique vertex v_x of $X-X_*$. Since v_x is the vertex v_n of Δ^n , the elementary collapse of C'_k with major cell $\mathrm{Int}\, Cv_x$ belongs to C'_n . Thus (ii) follows if the elementary collapses of C_k belong to C_n . The collapse

$A(X_*)$ in $SdX_* - pX_*$ occurs in the step $(\chi)_{(k-1)}$ (b) of C_n . For $j < k-1$, the stage $(\chi)_j$ of C_k is part of the stage $(\chi)_j$ of C_n . Therefore C_n' is as required.

We now prove that C_n' is the collapse $A(\Delta^n)$.

For $U \subset V$ a pair of $\vec{SC}$ - complexes , let ω be a total order on the open cells of $U - V$. We say that a collapse $\tilde{C}: U \searrow V$ follows ω if , when U has been collapsed to U' and the next elementary collapse of $\tilde{C}$ has major and minor cells a and b , then a is the least cell of $U' - V$ in the order ω and b is least cell of $U' - V$ which can be paired with a in an elementary collapse of U' . It is clear (see definitions 4.1 , 4.3 , 4.7) that if C_n' follows the order $\zeta_s(\Delta^n)$ on the set of open cells of $\mathrm{Int}\, Sd\,\Delta^n - p\Delta^n$ then $C_n' = A(\Delta^n)$. Furthermore, (by the use of $\zeta_s(Bd\Delta^n)$ in the definition of $\zeta_s(\Delta^n)$ and the fact that C_n' is the collapse induced by C_n) if $C_n: SdBd\Delta^n - p\Delta^n_* \searrow v_n$ follows the order $\zeta_s(Bd\Delta^n)$ on the set of open cells of $SdBd\Delta^n - p\Delta^n_* - v_n$ then C_n' follows $\zeta_s(\Delta^n)$. We therefore have to show that C_n follows $\zeta_s(Bd\Delta^n)$.

Assume that, for $k < n$, C_k follows $\zeta_s(Bd\Delta^k)$. Suppose that C_n has collapsed $SdBd\Delta^n - p\Delta^n_*$ to U and that the next pair of major and minor cells of C_n is (a,b) . There are two cases.

1. a and b belong to the collapse $A(Y)$ for $Y \subset Bd\Delta^n$, $\dim Y = r \geq 1$;
2. $a = pX$, $b = pX_*$ where $X \subset Bd\Delta^n$, $X \neq \Delta^n_*$ and $\dim X \geq 1$.

Case 1 The definition of C_n is such that if Z is a face of $\mathrm{Bd}\Delta^n$ with $Z <_{\zeta(\Delta^n)} Y$ then $\mathrm{Int\,Sd}\,Z$ has already been collapsed out. From the inductive hypothesis C_r follows $\zeta_s(\mathrm{Bd}\Delta^r)$ so that $A(Y) = C_r'$ follows $\zeta_s(Y)$. Hence, by the lexicographic nature of $\zeta_s(\mathrm{Bd}\Delta^n)$, a is the cell of $U - v_n$ which is least in $\zeta_s(\mathrm{Bd}\Delta^n)$ and b is the least cell of $U - v_n$ which can partner a in an elementary collapse of U.

Case 2 As in case 1, $\mathrm{Int\,Sd}\,Z$ has already been collapsed out for any face $Z \subset \mathrm{Bd}^n$ such that $Z <_{\zeta(\Delta^n)} X$. Thus the cell of U which is least in $\zeta_s(\mathrm{Bd}\Delta^n)$ is $a = pX$.

The choice of b must come from the top-dimensional cells of $\mathrm{Bd}\,\overline{pX}$. So far no cell of SdX has been deleted therefore no cell in $\mathrm{Int\,Sd}\,X$ may be used as b . This leaves only pX_* .

It has now been shown that C_n follows $\zeta_s(\mathrm{Bd}\Delta^n)$. The inductive process is started by taking $C_1' = A(\Delta 1)$. □

The results which follow are, in varying degrees, consequences of 5.4 . Again, $\mathrm{Sd}\Delta^n$ is identified with $\mathrm{CSdBd}\Delta^n$.

5.5 Definition For $n \geq 0$, let μ_n be the unique Poly - isomorphism $\Delta^n \to \overline{p\Delta^n}$.

5.6 Proposition *For* $n \geq 2$ *and each* $(n-1)$ *- face* $X \neq \Delta^n_*$ *of* $\mathrm{Bd}\Delta^n$ *, the tube* $T(\overline{pX}) = \{t_o, T_1, t_1, \ldots, T_q, t_q\}$ *is as follows:* $t_o = \overline{pX}$, $T_1 = C\overline{pX}$, $t_1 = C\overline{pX_*}$ *and, for* $j > 1$, $T_j = CT'_{j-1}$, $t_j = Ct'_{j-1}$ *where* T'_{j-1} , t'_{j-1} *belong to the tube* $T'(\overline{pX_*}) = \{\overline{pX_*} = t'_o, T'_1, \ldots, t'_{q-1}\}$ *in* $\mathrm{Sd}\Delta^n_*$. *Furthermore* $t_q = \mu_n(X) \subset \overline{p\Delta^n}$.

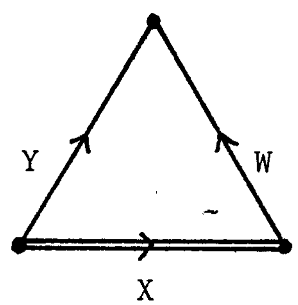
The tube on $\overline{p\Delta^n_*}$ *is trivial, with* $T(\overline{p\Delta^n_*}) = \{\overline{p\Delta^n_*}\}$.

<u>Proof</u> Obviously, $T_1 = C\,\overline{pX}$ and, by the definition of $\mathcal{D}$: CSdX $\searrow$ SdX , $t_1 = C\,\overline{pX}_*$. Since pX_* is a face of $SdBd\Delta^n_*$, there is a tube $T'(\overline{pX}_*) = \{\overline{pX}_* = t'_o, T'_1, \ldots, t'_{q-1}\}$ in $Sd\Delta^n_*$. The collapse $A(\Delta^n)$ in $Sd\Delta^n - p\Delta^n$ restricts to the collapse $C(Sd\Delta^n_* - p\Delta^n_*) \searrow CSdBd\Delta^n_*$ induced by the collapse $A(\Delta^n_*)$ in $Sd\Delta^n_* - p\Delta^n_*$. Hence $T_j = CT'_{j-1}$, $t_j = Ct'_{j-1}$ for $j = 2, \ldots, q$.

Induction on the dimension n is used to show that $t_q = \mu_n(X)$. This is clearly true for Δ^1 . Assume true for Δ^{n-1} . Then $t'_{q-1} = \mu_{n-1}(X_*) \subset \overline{p\Delta^n_*}$ and $t_q = Ct'_{q-1}$ is the unique face of $\overline{p\Delta^n}$ containing $\mu_{n-1}(X_*)$ and the single vertex in $\overline{p\Delta^n} - \overline{p\Delta^n_*}$. Now μ_n agrees with μ_{n-1} on Δ^n_* and maps the single vertex v_n in $\Delta^n - \Delta^n_*$ to the vertex in $\overline{p\Delta^n} - \overline{p\Delta_*}$. X is the unique face of Δ^n containing X_* and v_n . Hence $t_q = \mu_n(X)$. □

<u>5.7 Definition</u> For $n \geq 2$, let X and Y be (n-1)-faces of Δ^n and denote the (n-2)-face in which X and Y intersect by Z . We say Δ^n is a *pseudocylinder* $\Delta^n: X \Rightarrow Y$ if the unique Poly-isomorphism $X \to Y$ maps $Z \subset X$ onto $Z \subset Y$.

For $n = 1$, we say there are pseudocylinders $\Delta^1: \Delta^1_o \Rightarrow \Delta^1_1$ and $\Delta^1: \Delta^1_1 \Rightarrow \Delta^1_o$, where Δ^1_o , Δ^1_1 are the vertices of Δ^1 .



For example, Δ^2 is a pseudocylinder $X \Rightarrow Y$ and $Y \Rightarrow W$ but not $X \Rightarrow W$.

Note that if Δ^n: $X \Rightarrow Y$ is a pseudocylinder then so is Δ^n: $Y \Rightarrow X$. Definition 5.7 is a special case of a concept developed in §2 of the next chapter (where the use of the term 'pseudocylinder' is explained).

<u>5.8 Proposition</u> *If* Δ^n *is a pseudocylinder* $X \Rightarrow Y$ *then* $C\Delta^n$ *is a pseudocylinder* $CX \Rightarrow CY$.

<u>Proof</u> In the $\overrightarrow{CC}$ cone $C\Delta^n$ (Definition I 6.4) X and Y retain their marked face structures and, for each face A of Δ^n, $(CA)_* = A$. Hence, if $Z = X \cap Y$, the unique Poly-isomorphism $CX \to CY$ maps $CZ = CX \cap CY$ onto CZ. □

<u>5.9 Proposition</u> *For* $n \geq 1$, *let* $X \neq \Delta^n_*$ *be an* $(n-1)$*-face of* Δ^n *with tube* $T(\overline{pX}) = \{t_o, T_1, t_1, \ldots, T_q, t_q\}$ *in* $Sd\Delta^n$. *Then* T_j: $t_{j-1} \Rightarrow t_j$ *is a pseudocylinder for* $j = 1,2,\ldots,q$.

<u>Proof</u> Induction on dimension is used.

The result is obvious for Δ^1. Assume it is true for Δ^{n-1} and consider the tube $T(\overline{pX})$ in $Sd\Delta^n$.

We have $t_o = \overline{pX}$, $T_1 = C\overline{pX}$ and $t_1 = C\overline{pX_*}$. The marked face structure of $\overline{pX}$ is that of a $\overrightarrow{CC}$ cone on the face $\overline{pX_*} \subset \overline{pX}$. Hence T_1 is a pseudocylinder $t_o \Rightarrow t_1$. For $j = 2,3,\ldots,q$, $T_j = CT'_{j-1}$ and $t_j = Ct'_{j-1}$ where T'_{j-1} and t'_{j-1} belong to the tube $T'(\overline{pX_*})$ in $Sd\Delta^n_*$. By the inductive hypothesis, T'_{j-1} is a pseudocylinder $t'_{j-2} \Rightarrow t'_{j-1}$. Thus, from 5.8, T_j is a pseudocylinder $t_{j-1} \Rightarrow t_j$. □

In preparation for §7 we have to consider certain cells of $Sd\Delta^n - p\Delta^n$ which are related to major cells of the collapses A in subdivided faces of Δ^n.

<u>5.10 Definition</u> (Recursive) An open cell e^k $(k \geq 1)$ of $SdΔ^n - pΔ^n$ is said to be an $A_{Δn}$ - *cell* if either

(i) e^k is a major cell of the collapse $A(Y)$ in $SdY - pY$ for some face Y of $Δ^n$; or

(ii) there is a $(k + 1)$ - cell a with $e^k \subset \partial a$ such that a and each k-cell of $\partial a - e^k$ is an $A_{Δn}$ - cell .

<u>5.11 Proposition</u> *For* $n \geq 1$, *let* $X \neq Δ^n_*$ *be an* $(n-1)$-*face of* $Δ^n$ *with tube* $T(\overline{pX}) = \{t_0, T_1, t_1, \ldots, T_q, t_q\}$ *in* $SdΔ^n$. *Then for* $j = 1,2,\ldots,q$ *each (open) cell in* $T_j - (t_{j-1} \cup t_j)$ *is an* $A_{Δn}$ - *cell* .

<u>5.12 Lemma</u> *Identify* $SdΔ^n$ *with* $CSdBdΔ^n$ *and let* Y $(\cong Δ^{n-1})$ *be an* $(n-1)$-*face of* $Δ^n$. *If a cell* e^k *of* $SdY - pY$ *is an* A_Y -*cell then* $\text{Int}\, C\bar{e}^k$ *is an* $A_{Δn}$ -*cell* .

<u>Proof</u> First we associate an integer with each $A_{Δr}$-cell in $SdΔ^r - pΔ^r$ $(r \geq 1)$. Let the major cells of each collapse A in $SdΔ^r - pΔ^r$ be $(A_{Δr}, 0)$-cells . For $i > 0$, e^k is an $(A_{Δr}, i)$-cell if i is the least integer such that there is a $(k+1)$-cell a with $e^k \subset \partial a$ and a and the k-cells of $\partial a - e^k$ are $(A_{Δr}, j)$-cells for $j \leq i - 1$.

The Lemma is proved by induction.

By 5.4(i) , if e^k is an $(A_Y, 0)$-cell (that is, a major cell of a collapse A in $SdY - pY$) then $\text{Int}\, C\bar{e}^k$ is an $A_{Δn}$-cell . Assume the result holds for (A_Y, j)-cells $j \leq i - 1$. For e^k an (A_Y, i)-cell , there is a $(k+1)$-cell a with $e^k \subset \partial a$ such that a and k-cells of $\partial a - e^k$ are (A_Y, j)-cells for $j \leq i - 1$. Hence (note that the A_Y-cell a is an $A_{Δn}$-cell) each cell of dimension $\geq k + 1$ of $C\bar{a}$

other than $\operatorname{Int} C\bar{e}^k$ is an A_{Δ^n}-cell. Thus $\operatorname{Int} C\bar{e}^k$ is an A_{Δ^n}-cell . □

Proof of Proposition 5.11 We use induction on dimension.

Assume the result holds for Δ^{n-1} . Consider $T(\overline{pX}) = \{\overline{pX} = t_o, T_1, t_1, \ldots, T_q, t_q = \mu_n(X)\}$. By 5.6 , for $2 \le j \le q$, $T_j = C\,T'_{j-1}$ and $t_j = C\,t'_{j-1}$ where T'_{j-1} , t'_{j-1} belong to the tube $T'(\overline{pX_*})$ in $Sd\Delta^n_*$. By the inductive assumption, each cell in $T'_{j-1} - (t'_{j-2} \cup t'_{j-1})$ is an $A_{\Delta^n_*}$-cell . Hence, using 5.12 , we find that each cell in $T_j - (t_{j-1} \cup t_j)$ for j = 2, 3, ..., q is an A_{Δ^n}-cell.

The case of $T_1 = C\overline{pX}$ remains. Consider the cone $CSdX \subset CSdBd\Delta^n = Sd\Delta^n$. Since X is an (n-1) - simplex there is a tube $T^X(\overline{pY}) = \{\overline{pY} = t^X_o , T^X_1 , \ldots , t^X_r = \mu_X(Y) \subset \overline{pX}\}$ in SdX for each (n-2) - face $Y \neq X_*$. Hence there is a sequence $C\overline{pY} = Ct^X_o, CT^X_1, \ldots, Ct^X_r = C\,\mu_X(Y) \subset C\overline{pX}$ of faces of CSdX such that $CT^X_j : Ct^X_{j-1} \Rightarrow Ct^X_j$ is a pseudocylinder (5.8 , 5.9) . We have immediately that if $\nu : Ct^X_{j-1} \to Ct^X_j$ is the unique Poly - isomorphism then, for $k \ge 0$ and each k-face Z of Ct^X_{j-1} , either $\nu(Z) = Z$ or Z and $\nu(Z)$ are both faces of a (k+1)-face $L_Z \subset CT^X_j$ such that $L_Z \cap Ct^X_{j-1} = Z$, $L_Z \cap Ct^X_j = \nu(Z)$.

By the inductive assumption, each cell of $T^X_j - (t^X_{j-1} \cup t^X_j)$ is an A_X-cell so that (5.12) each (open) cell of $CT^X_j - (Ct^X_{j-1} \cup Ct^X_j)$ is an $A_\Delta n$-cell. Therefore, when Z, $\nu(Z)$ are faces of L_Z , the cell $\operatorname{Int} L_Z$ and the k-cells of $L_Z - (Z \cup \nu(Z))$ are $A_\Delta n$-cells. Thus if Z is an $A_\Delta n$-cell then so is $\nu(Z)$. It follows, denoting the unique Poly -

isomorphism $C\overline{pY} \to C\mu_X(Y)$ by ν_Y, that $\nu_Y(Z)$ is an A_{Δ^n}-cell for each A_{Δ^n}-cell Z of $C\overline{pY}$.

Now the collapse $A(\Delta^n)$ restricts to the collapse $A(X)$: $CSdBdX - \operatorname{Int} C\overline{pX}_* \searrow SdBdX$ (5.4 (ii)). For each $(n-2)$-face $Y \neq X_*$ of X, the tube $T^{CSdBdX}(\overline{pY})$ in $CSdBdX$ has $t_o^{CSdBdX} = \overline{pY}$, $T_1^{CSdBdX} = C\overline{pY}$, $t_1^{CSdBdX} = C\overline{pY}_*$. Hence each cell of $C\overline{pY} - (\overline{pY} \cup C\overline{pY}_*)$ is an A_{Δ^n}-cell. Clearly, $\nu_Y(\overline{pY}) = \mu_X(Y)$ and $\nu_Y(C\overline{pY}_*) = C\,\mu_X(Y)_*$ so that, from the previous paragraph, each (open) cell of $C\mu_X(Y) - (\mu_X(Y) \cup C\,\mu_X(Y)_*)$ is an A_{Δ^n}-cell. Each cell of $C\overline{pX} - (\overline{pX} \cup C\overline{pX}_*)$ apart from $\operatorname{Int} C\overline{pX}$ is a cell of $C\mu_X(Y) - (\mu_X(Y) \cup C\,\mu_X(Y)_*)$ for some $(n-2)$-face Y. Thus, since $\operatorname{Int} C\overline{pX}$ is a major cell of $A(\Delta^n)$, each cell in $C\overline{pX} - (\overline{pX} \cup C\overline{pX}_*) = T_1 - (t_o \cup t_1)$ is an A_{Δ^n}-cell.

To start the inductive process we note that the proposition is obviously true in the case of Δ^1. □

§6 The functor e_M from $\Delta_I T$-complexes to MT-complexes

We now construct a functor e_M: $\Delta_I TC \to MTC$ for each category M in our class $E\Gamma$ (definition II 2.4) of special model categories.

Let $E\Gamma^+ = E\Gamma \cup \{\Delta_I\}$. Recall that, for $M \in E\Gamma^+$, we can define M-sets (that is, functors $K: M^{op} \to \mathrm{Set}$) and also M-$\overrightarrow{SC}$-complexes, namely $\overrightarrow{SC}$-complexes built from the models. Further, each M-$\overrightarrow{SC}$-complex U defines an M-set (definition 3.4) which we also denote by U. An U-structure in K (3.5) is then a morphism of M-sets $\mathcal{U}: U \to K$.

Now let K be a $\Delta_I T$-complex. Our aim is to define an extension $e_M K: M^{op} \to \mathrm{Set}$ of K such that $e_M K$ is an

MT-complex. It might be thought, since Δ_I is a subcategory of M, that $e_M K$ could be defined directly as a Kan extension. Certainly we can extend $K: \Delta_I^{op} \to \mathrm{Set}$ to a functor $K': M^{op} \to \mathrm{Set}$ in this way; but we require that if K is a Δ_IT-complex then the extension is an MT-complex, and this property seems unlikely to be given by the Kan extension process.

Instead we use a subdivision process. Note that, for K a Δ_IT-complex and X an M-polycell, we could define an X-cell in the M-set $e_M K$ to be a SdX-structure in K (map of Δ_I-sets $SdX \to K$). In fact we take the X-cells of $e_M K$ to be SdX-structures of a particular kind which depend on the T-complex structure of K and the collapse $A(X): SdX - pX \searrow SdBdX$ described in Definition 4.7 . It is this tight control which allows us to obtain the isomorphism $L \cong e_M r_M(L)$ in §8 .

<u>6.1 Definition</u> For K a Δ_IT-complex and Z an $\overrightarrow{SC}$-complex, let U be a subcomplex of SdZ containing $SdY - pY$ for each face Y of Z of dimension ≥ 1 and let $\mathcal{U}: U \to K$ be a structure in K . We say that $\mathcal{U}$ is *special* if, for each Y and each major cell a of the collapse $A(Y)$ in $SdY - pY$, $\mathcal{U}(\bar{a})$ is a thin cell of K .

In other words $\mathcal{U}$ is special if, for each face Y of Z , the restriction of $\mathcal{U}$ to $SdY - pY$ is the thin expansion, corresponding to $A(Y): (SdY - pY) \searrow SdBdY$, of the restriction of $\mathcal{U}$ to $SdBdY$ (see Definition 3.7) .

<u>6.2 Definition</u> Let K be a $\Delta_I T$-complex. For $M \in E\Gamma$, the M-set $e_M K$ is defined as follows.

For an M-cell X ,

$e_M K(X)$ = the set of special SdX-structures in K ;

and, for an M-morphism $f: X \to Y$,

$e_M K(f)(U_Y) = U_Y \circ sf$, where sf is the map of Δ_I-sets $SdX \to SdY$ induced by f .

For $n \geq 1$, let the n-dimensional M-cell X have (n-1)-faces $X_o, X_1, \ldots, X_q, Y$ and denote the M-cell corresponding to X_i by $X_i^!$.

<u>6.3 Proposition</u> *If B is a box $\{ V_o, V_1, \ldots, V_q \}$ in $e_M K$ with $V_i \in e_M K(X_i^!)$ then there is a unique filler $U \in e_M K(X)$ such that $U(\overline{pX})$ is thin in K .*

<u>Proof</u>

<u>Existence</u> Let $H = X - (\operatorname{Int} X \cup \operatorname{Int} Y)$. The box B defines a special structure $V_H: SdH \to K$ with $V_H | SdX_i = V_i \circ j_i$ (j_i = the canonical Δ_I-set morphism $SdX_i \to SdX_i^!$) . The collapse $A(Y): SdY - pY \searrow SdBdY$ defines a collapse $SdBdX - pY \searrow SdH$. Let $V_J: (SdBdX - pY) \to K$ be the thin expansion of V_H corresponding to the latter collapse. The SdX-structure U is defined to be the thin expansion of V_J corresponding to the collapse $B(X,\overline{pY}): SdX \searrow SdBdX - pY$ (see Definition 4.11).

<u>Uniqueness</u> By Proposition 3.8 , V_J is the unique special (SdBdX - pY) - structure in K extending V_H . By Proposition 4.12 the only major cell of $B(X, \overline{pY})$ not major in $A(X)$ is pX . Hence any filler $U' \in e_M K(X)$ satisfying $U'(\overline{pX})$ is thin must be an SdX - structure extending V_J such that $U'(\bar{a})$ is

thin in K for each major cell of $\mathcal{B}(X, \overline{pY})$. That is , (Proposition 3.8) $\mathcal{U}'$ must be identical to the unique thin expansion $\mathcal{U}$ of $\mathcal{V}_J$. □

<u>6.4 Proposition</u> *For* $n \geq 2$, *let* X *be an* n-*dimensional* M-*cell with* (n-1)-*faces* $X_0, X_1, \ldots, X_q$, Y *and consider* $\mathcal{U} \in e_M K(X)$ *(that is, a special* SdX - *structure* $\mathcal{U}$ *in* K *)* . *If* $\mathcal{U}(\overline{pX})$ *and* $\mathcal{U}(\overline{pX}_i)$ *for* $i = 0, 1, \ldots, q$ *are thin elements of* K *then so is* $\mathcal{U}(\overline{pY})$.

<u>Proof</u> If Z is an (n-1)-dimensional M-cell each (n-1)-cell of SdZ - pZ is a major cell of the collapse $A(Z)$. (Each (n-1)-cell is deleted in $A(Z)$ and, being of maximum dimension, must be a major cell.) Thus $\mathcal{U}(\bar{e}^{n-1})$ is thin in K for each (n-1)-cell e^{n-1} of SdBdX - pY .

Suppose the collapse $\mathcal{B}(X,\overline{pY})$: SdX $\searrow$ SdBdX - pY proceeds: $SdX = V_0 \overset{e}{\searrow} V_1 \overset{e}{\searrow} \ldots \overset{e}{\searrow} V_r = SdBdX - pY$. Assume $\mathcal{U}(\bar{e}^{n-1})$ is thin for each (n-1)-cell e^{n-1} of V_i and let a and b be the major and minor cells of the elementary collapse $V_{i-1} \overset{e}{\searrow} V_i$. Since $\mathcal{U}$ is a special structure with $\mathcal{U}(\overline{pX})$ thin, $\mathcal{U}$ is the thin expansion corresponding to $\mathcal{B}(X,\overline{pY})$ of the restriction of $\mathcal{U}$ to SdBdX - pY . Thus $\mathcal{U}(\bar{a})$ is thin, which deals with a = an (n-1)-cell . Further, if b is an (n-1)-cell then $\mathcal{U}(\bar{b})$ is an (n-1)-face of a thin n-element of K whose other (n-1)-faces are thin. Hence $\mathcal{U}(\bar{b})$ is thin by axiom (T3) for the $\Delta_I T$ - complex K .

We therefore have, by induction, that $\mathcal{U}(\bar{e}^{n-1})$ is thin in K for each (n-1)-cell e^{n-1} of SdX . It follows that $\mathcal{U}(\overline{pY})$ is thin . □

In view of Propositions 6.3 , 6.4 there is no difficulty in proving:

6.5 Proposition *For* $M \in E\Gamma$ *, there is a functor* $e_M\colon \Delta_I TC \to MTC$ *defined on objects by* $K \to e_M K$ *, where a cell* $U\colon SdX \to K$ *of* $e_M K(X)$ *is thin if* $U(\overline{pX})$ *is thin in* K *. For a* $\Delta_I TC$*-morphism* $f\colon K \to L$ *, the* MTC*-morphism* $e_M f\colon e_M K \to e_M L$ *is given by* $e_M f(U\colon SdX \to K) = f \circ U\colon SdX \to L$. □

§7 The natural equivalence $r_M \circ e_M \simeq 1$

Recall that a functor $r_M\colon MTC \to \Delta_I TC$ for each $M \in E\Gamma$ was defined in §1 . We now show that there is a natural equivalence $r_M \circ e_M \simeq 1$. Our proof makes use of work of §5 ; the lemmas below translate the geometric results of that section into T-complex language.

Take Δ^n to be the Poly n-simplex in $Ob(\Delta_I)$.

7.1 Lemma *For* K *a* Δ_I*T-complex, let* $x \in K(\Delta^n)$ *. If* $\Delta^n\colon Y \Rightarrow Z$ *is a pseudocylinder and* $\partial_F x$ *is thin for each face* $F \not\subset Y \cup Z$ *of* Δ^n *then* $\partial_Y x = \partial_Z x$ *. (See* I 6.2 *for the notation* $\partial_F x$ *.)*

Proof The standard order $\zeta(\Delta^n)$ on the set of faces of Δ^n induces a total order $\zeta_0(\Delta^n)$ on the vertex set (see I §5). Denote by Δ^n_i the unique (n-1)-face of Δ^n not containing the vertex v_i in $\zeta_0(\Delta^n)$ and let $\delta_i\colon \Delta^{n-1} \to \Delta^n$ be the unique Poly-morphism with image Δ^n_i . The canonical isomorphism between our category Δ_I and the usual combinatorial version (I §6) is such that, writing d_i for the usual face maps of the Δ_I T-complex K , $d_i\colon K_n \to K_{n-1}$ is the image under K of $\delta_i\colon \Delta^{n-1} \to \Delta^n$.

Without loss of generality, we can assume $Y = \Delta^n_k$, $Z = \Delta^n_j$ and $j < k$. Then, since the vertex - orderings $\zeta_o(Y)$ and $\zeta_o(Z)$ are compatible with the vertex - orderings induced by $\zeta_o(\Delta^n)$ on Y and Z respectively, v_j is the j'th vertex in $\zeta_o(Y)$ and v_k is the vertex in position $(k-1)$ of $\zeta_o(Z)$. The unique Poly - isomorphism $f: Y \to Z$ preserves the order $\zeta_o(Y)$. From the definition of a pseudocylinder, $f(v_j) = v_k$. Hence v_k is in the position j of $\zeta_o(Z)$; that is, $j = k - 1$ and $Y = \Delta^n_{j+1}$. If we take a 'face' of $x \in K(\Delta^n)$ to mean x or any element $d_{i_k} d_{i_{(k-1)}} \dots d_{i_o} x$, $\partial_F x$ is thin for each face $F \not\subset Y \cup Z$ of Δ^n implies $(K)_j$: each face of x apart from faces of $d_j x$ and $d_{j+1} x$ is thin.

Using Proposition 2.8 and induction on dimension, we can show that if $(K)_j$ holds then x is a degenerate element $s_j y$ in the simplicial T-complex πK associated with K. Hence $d_j x = d_{j+1} x$; that is, $\partial_Y x = \partial_Z x$. □

For any Poly n-simplex Δ^n let μ_n be the unique Poly - isomorphism $\Delta^n \to \overline{p\Delta^n}$ (5.5).

<u>7.2 Lemma</u> *For* $n \geq 1$, *if* $U: (Sd\Delta^n - p\Delta^n) \to K$ *is a special structure in the* $\Delta_I T$*-complex* K *then* $U(\overline{pX}) = U(\mu_n(X))$ *for each* $(n-1)$*-face* X *of* Δ^n.

<u>Proof</u> In the case $X = \Delta^n_*$ we have $\overline{pX} = \mu_n(X)$ so that the result follows trivially.

Take $X \neq \Delta^n_*$ and consider the tube $T(\overline{pX}) = \{t_o, T_1, t_1, \dots, T_q, t_q\}$ in $Sd\Delta^n$. For $j = 1,2,\dots,q$, $T_j: t_{j-1} \Rightarrow t_j$ is a pseudocylinder (see 5.9) and each cell in

$T_j - (t_{j-1} \cup t_j)$ is an $A_{\Delta}n$-cell (5.11). It is clear, since $\mathcal{U}$ is special, that $\mathcal{U}(\bar{e}^k)$ is thin in K for each $A_{\Delta}n$-cell e^k. Hence, by 7.1, $\mathcal{U}(t_{j-1}) = \mathcal{U}(t_j)$ for $j = 1,2,\ldots,q$. From 5.6, $t_o = \overline{p}X$ and $t_q = \mu_n(X)$ so that we have $\mathcal{U}(\overline{p}X) = \mathcal{U}(\mu_n(X))$. □

<u>7.3 Lemma</u> *For* $n \geq 0$, *a special structure* $\mathcal{U}: Sd\Delta^n \to K$ *in the* $\Delta_I T$-*complex* K *is uniquely determined by the simplex* $\mathcal{U}(\overline{p\Delta}^n) \in K_n$.

<u>Proof</u> We use induction on n. Since $\overline{p\Delta}^o = \Delta^o$, $\mathcal{U}: Sd\Delta^o \to K$ is obviously determined by $\mathcal{U}(\overline{p\Delta}^o)$.

Assume the result holds for dimension n-1 and let $\mathcal{U}: Sd\Delta^n \to K$ be special. Then, by 7.2 $\mathcal{U}(\overline{p}X) = \mathcal{U}(\mu_n(X))$ for each (n-1)-face X of Δ^n. By the inductive hypothesis, $\mathcal{U}(pX)$ determines the restriction $\mathcal{U}|SdX$ of $\mathcal{U}$ and so $\mathcal{U}|SdBd\Delta^n$ is fixed. We have $\mathcal{U}|(Sd\Delta^n - p\Delta^n)$ is the unique thin expansion of $\mathcal{U}|SdBd\Delta^n$ corresponding to the collapse $A(\Delta^n)$. Therefore $\mathcal{U}$ is uniquely determined by $\mathcal{U}(\overline{p\Delta}^n)$. □

<u>7.4 Proposition</u> *For each category* M *in* EΓ, *the functors* $r_M \circ e_M$, $1: \Delta_I TC \to \Delta_I TC$ *are naturally equivalent.*

<u>Proof</u> Let Δ^n denote the n-simplex in Ob(M). For a $\Delta_I T$-complex K and each $n \geq 0$, $(r_M \circ e_M K)_n$ is the set of special $Sd\Delta^n$ - structures in K. Define $\theta: r_M \circ e_M K \to K$ by $\theta_n\ (\mathcal{U}: Sd\Delta^n \to K) = \mathcal{U}(\overline{p\Delta}^n)$. We prove that θ is a $\Delta_I TC$-isomorphism using induction on dimension.

Clearly, θ_o is a bijection $(r_M \circ e_M K)_o \to K_o$. Assume that $\theta_o, \theta_1, \ldots, \theta_{n-1}$ satisfy the conditions for a $\Delta_I TC$ - isomorphism insofar as they apply.

Let $x \in K_n$. By the inductive assumption, the set $\{\theta_{n-1}^{-1}\, \partial_Y x \mid Y$ an $(n-1)$-face of $\Delta^n\}$ of special $Sd\Delta^{n-1}$-structures in K forms a shell in $r_M \circ e_M K$. Hence there is a special $SdBd\Delta^n$-structure $\mathcal{U}''_x$ such that $\mathcal{U}''_x | SdY = (\theta_{n-1}^{-1}\, \partial_Y x) \circ j_Y$ (j_Y = the canonical Δ_I-set isomorphism $SdY \to Sd\Delta^{n-1}$). The thin expansion $\mathcal{U}'_x$ of $\mathcal{U}''_x$ corresponding to $A(\Delta^n)$: $Sd\Delta^n - p\Delta^n \searrow SdBd\Delta^n$ is a special $(Sd\Delta^n - p\Delta^n)$ - structure in K so that (7.2) $\mathcal{U}'_x\, (\mu_n(Y)) = \mathcal{U}'_x\, (\overline{pY}) = \partial_Y x$. Hence we can form a special $Sd\Delta^n$-structure $\mathcal{U}_x$ extending $\mathcal{U}'_x$ in K such that $\mathcal{U}_x\, (\overline{p\Delta^n}) = x$. It follows from 7.3 that θ_n has an inverse defined by $\theta_n^{-1}\,(x) = \mathcal{U}_x$. Clearly, θ_n is compatible with face maps and θ_n, θ_n^{-1} preserve thin elements. Hence $\theta_o, \theta_1, \ldots, \theta_n$ satisfy the conditions for a $\Delta_I TC$ - isomorphism insofar as they apply. There is thus an isomorphism $\theta\colon r_M \circ e_M K \to K$.

It is easily checked that θ is a natural equivalence $r_M \circ e_M \simeq 1$. □

§8 The equivalence of categories

We now construct a natural equivalence $1 \simeq e_M \circ r_M$ for $M \in E\Gamma$ using the construction VX (see II 2.2 and III §4).

Recall that for each M-cell X there is an M-cell $V'X$ Poly - isomorphic to VX. We can thus define VX - structures in an MT - complex. As before, we identify X with $X \times \{0\} \subset VX$ and SdX with $Sd(X \times \{1\}) \subset VX$.

For K an MT - complex, the cells of $e_M \circ r_M K(X)$ are special SdX - structures in $r_M K$. Since $r_M K$ is defined as a restriction of K we can obviously identify each cell of

$e_M \circ r_M K(X)$ with a special SdX-structure in K.

<u>8.1 Definition</u> For $M \in E\Gamma$, let K be an MT-complex. For an M-cell X and a cell $x \in K(X)$, the structure $\Phi x: VX \to K$ is defined as follows. First specify the structure $\Phi^o x: X \to K$ by $\Phi^o x(X) = x$ then let Φx be the thin expansion of $\Phi^o x$ corresponding to the collapse $A_o(VX): VX \searrow X$.

Let $\phi x: SdX \to K$ be the restriction of Φx to SdX.

<u>8.2 Lemma</u>

(i) For Y *a face of* X, *the restriction of* Φx *to* $VY \subset VX$ *is the structure* $(\Phi \partial_Y x) \circ i_{VY} : VY \to K$ *(where, if* Y' *is the* M-*cell* Poly-*isomorphic to* Y, i_{VY} *is the* Poly-*isomorphism* $VY \to VY'$ *).*

(ii) There is a map $\phi_X: K(X) \to e_M \circ r_M K(X)$ *defined by* $x \mapsto \phi x$.

<u>Proof</u>

(i) This follows from Proposition 4.19.

(ii) The collapse $A_o(VX)$ restricts to the collapse $A(Y)$ in SdY for each face Y of X. Hence the restriction ϕx is a special SdX-structure in $r_M K(X)$. □

<u>8.3 Definition</u> For X an M-cell, let U be a cell of $e_M \circ r_M K(X)$, that is, a special SdX-structure in K. The VX-structure $\Psi U: VX \to K$ is the thin expansion of U corresponding to the collapse $A_1(VX): VX \searrow SdX$. The cell $\psi U \in K(X)$ is $\Psi U(X)$.

We have immediately.

<u>8.4 Lemma</u> *There is a map* $\psi_X\colon e_M \circ r_M K(X) \to K(X)$ *given by* $U \mapsto \psi U$. □

<u>8.5 Proposition</u> *For each category* M *of* EΓ *there is a natural equivalence of functors* $\phi\colon 1 \to e_M \circ r_M$.

<u>Proof</u> Let K be an MT-complex , $M \in E\Gamma$. We show that there is an MTC - isomorphism $\phi\colon K \to e_M \circ r_M K$ with inverse ψ .

Let Y be a face of the M-cell X and take $x \in K(X)$. By 8.2 (i) the structure $\phi x\colon SdX \to K$ restricts to the structure $(\phi\partial_Y x) \circ si_Y : SdY \to K$ (where, if Y' is the M-cell Poly - isomorphic to Y , $si_Y\colon SdY \to SdY'$ is induced by the Poly - isomorphism $i_Y\colon Y \to Y'$) . Hence the maps $\phi_X\colon K(X) \to e_M \circ r_M K(X)$, X an M-cell, are compatible with face maps.

The maps ϕ_X and ψ_X are inverse if :

(i) for each $x \in K(X)$, $\Phi x = \Psi\phi x$;

(ii) for each $U \in e_M \circ r_M K(X)$, $\Psi U = \Phi\psi U$.

Consider (i) . The restriction to SdX of each of the structures Φx , $\Psi\phi x : VX \to K$ is ϕx . By 4.17 , $\Phi x(\bar{a})$ is thin in K for every major cell a of the collapse $A_1(VX)$. But $\Psi\phi x$ is the unique thin expansion of ϕx corresponding to $A_1(VX)$. Therefore $\Phi x = \Psi\phi x$.

In case (ii) we have that ΨU and $\Phi\psi U$ restrict to the structure $\Phi^o\psi U\colon X \to K$. Since U is a special SdX - structure $\Psi U(\bar{a})$ is thin in K for every major cell a of $A_o(VX)$. Thus, as $\Phi\psi U$ is the thin expansion of $\Phi^o\psi U$ corresponding to $A_o(VX)$, $\Psi U = \Phi\psi U$.

We have now shown that $\phi : K \to e_M \circ r_M K$ is an isomorphism of M-sets with inverse $\psi : e_M \circ r_M K \to K$.

Let X be an n-dimensional M-cell. By the definition of $A_0(VX)$, the cells $\Phi x(VX)$ and $\Phi x(Y)$, for Y an n-face of VX other than X or pX ($\subset SdX$) , are thin in K . Hence (axiom T3) x is thin in K if and only if ϕx is thin in $e_M \circ r_M K$. Thus ϕ and ψ preserve thin elements and ϕ is an MTC - isomorphism.

There is no difficulty in checking naturality. □

Combined with 7.4 , Proposition 8.5 gives the main results of this chapter. First :

__8.6 Theorem__ *For* $M \in E\Gamma$ *, the functors* $r_M : MTC \to \Delta_I TC$ *and* $e_M : \Delta_I TC \to MTC$ *are inverse equivalences of categories.* □

Secondly, using the isomorphism $\Delta_I TC \to \Delta TC$, where ΔTC is the category of simplicial T-complexes (see §2) :

__8.7 Theorem__ *For* $M \in E\Gamma$ *, there is an equivalence of categories* $MTC \to \Delta TC$. □

It was shown in Proposition II 2.7 that $E\Gamma$ contains an infinite number of non-isomorphic categories. We have therefore constructed an infinite set of non-trivially equivalent algebraic categories.

CHAPTER IV

DEGENERACY STRUCTURES IN MT-COMPLEXES

The model categories $M \in \Gamma$ which we have defined have only injective morphisms. Thus there is no structure of degenerate elements in M-sets. It would be of interest to have generalized model categories C with non-injective morphisms and hence C-sets equipped with degeneracies.

For $M \in \Gamma$, we might define a category M_D by taking M and adding non-injective morphisms. Checks on the suitability of M_D as a model category could be to obtain (i) a Kan M-set K admits a set of degeneracies which give K an M_D-set structure; or (ii) the categories of MT-complexes and M_DT-complexes are equivalent (compare with III §2). The definition of M_D is not obvious, however. There is a wide variety of non-injective morphisms preserving the cell and marked face structures of M-cells and it is not clear which should be included in M_D.

We do not take this approach here. Instead we note that, for K a cubical or simplicial T-complex and $x \in K_n$, a degenerate element $\varepsilon_i x$ or $s_i x \in K_{n+1}$ may be characterized as the unique thin element having two faces equal to x and certain degenerate elements as other faces, the arrangement of the faces being governed by a *cylinder* structure on I^{n+1} or a *pseudocylinder* structure on Δ^{n+1}. We define pseudocylinder structures on certain S-polycells. Then, for $M \in E\Gamma$ and K an MT-complex, a degenerate element $\varepsilon_J x \in K(U)$ is defined to be the unique thin element with two faces equal to x and other faces degenerate according to a pseudocylinder structure J on U. It can be shown that, for $M \in E\Gamma$, an MT-complex has a canonical degeneracy structure (Theorem 4.2).

A skeleton P of the category SPoly of S-polycells is an important member of the class $E\Gamma$. As a consequence of the degeneracy structure in a PT-complex there are functors

$$\rho_\Delta : PTC \to \Delta TC \quad , \qquad \rho_\square : PTC \to \square TC$$

defined essentially by restriction. The degeneracy structure in a PT-complex allows us also to define a pair of functors

$$\sigma : \square TC \to \Delta TC \quad , \qquad \tau : \Delta TC \to \square TC$$

which we claim are inverse equivalences of categories.

Throughout this chapter, the face and degeneracy maps of a simplicial set are denoted by d_i, s_i respectively and the face and degeneracy maps of a cubical set are denoted by ∂_i^α, ε_i.

§1 An approach to degeneracy structures in MT-complexes.

We now look at degeneracy maps in cubical $(\square)$ and simplicial (Δ) T-complexes. It is clear that the following holds.

1.1 Proposition

(i) *For* K *a* $\square$T-*complex and* $x \in K_n$, $\varepsilon_j x$ $(1 \le j \le n+1)$ *is the unique thin element of* K_{n+1} *with*

$$\partial_i^\alpha \varepsilon_j x = \begin{cases} \varepsilon_{j-1}\, \partial_i^\alpha x & , \quad i < j \\ x & , \quad i = j \\ \varepsilon_j\, \partial_{i-1}^\alpha x & , \quad i > j . \end{cases}$$

(ii) *For* L *a* ΔT-*complex and* $z \in L_n$, $s_j z$ $(0 \le j \le n)$ *is the unique thin element of* Z_{n+1} *with*

$$d_i s_j z = \begin{cases} s_{j-1} d_i z , & i < j \\ z , & i = j, j+1 \\ s_j d_{i-1} z , & i > j+1 . \end{cases} \quad \square$$

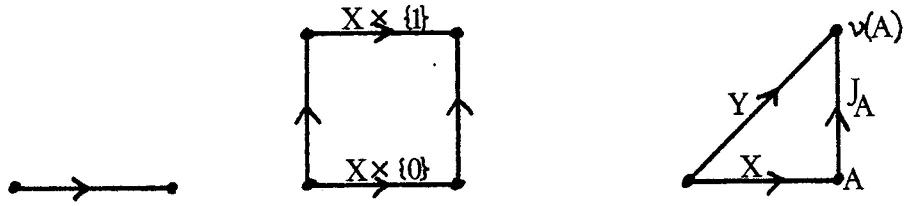
Recall (I 3.1) that, for a cone-complex X , the *cylinder* on X is the complex $X \times I$. For $n \ge 0$ the complex I^{n+1} has n+1 cylinder structures defined by the isomorphisms

$$\gamma_j : (I^n \times I) \to I^{n+1} , \quad 1 \le j \le n+1$$
$$((t_1,\dots,t_n),t) \mapsto (t_1,\dots,t_{j-1},t,t_j,\dots,t_n)$$

Each face Y of I^{n+1} not contained in $\gamma_j(I^n \times \{0\}) \cup \gamma_j(I^n \times \{1\})$ has a cylinderstructure induced by γ_i . We say Y has the structure of a *sub-cylinder* of $\gamma_j(I^n \times I)$.

For K a □T-complex and $x \in K_n$, the degenerate element $\varepsilon_j x$ can be associated with $\gamma_j(I^n \times I)$. From 1.1 we have $\varepsilon_j x$ is the unique thin element in K_{n+1} such that $\partial_j^\alpha \varepsilon_j x = x$ and every other (n-)face of $\varepsilon_j x$ is a degenerate element associated with a sub-cylinder of $\gamma_j(I^n \times I)$.

Degenerate elements in simplicial sets are related to *pseudocylinder* structures $\Delta^n : X \Rightarrow Y$ on Δ^n . (See Definition III 5.7 and the proof of Lemma III 7.1 . We can treat the object Δ^n of the simplicial model category Δ as a polycell because a vertex-ordering on a simplex is equivalent to a marked face structure.) The term *pseudocylinder* is used to emphasize the analogy between $\Delta^n: X \Rightarrow Y$ and the $\overrightarrow{CC}$-cylinder $X \times I$ (I 6.3) . If $\nu: X \to Y$ is the $\overrightarrow{CC}$-isomorphism and A is a k-face of X then either $\nu(A) = A$ or there is a (k+1)-face J_A such that $J_A \cap X = A$, $J_A \cap Y = \nu(A)$.



X   X × I $\Delta^2: X \Rightarrow Y$

Moreover, $\Delta^n: X \Rightarrow Y$ induces a pseudocylinder structure $J_A: A \Rightarrow \nu(A)$ on J_A . We say $J_A: A \Rightarrow \nu(A)$ is a *sub-pseudocylinder* of $\Delta^n: X \Rightarrow Y$.

For $j = 0, 1, \ldots, n+1$, let Δ_j^{n+1} be the n-face of Δ^{n+1} not containing the vertex j . For L a ΔT-complex and $z \in L_n$, the degenerate element $s_j z$ can be associated with $\Delta^{n+1}: \Delta_j^{n+1} \Rightarrow \Delta_{j+1}^{n+1}$. From 1.1 we have that $s_j z$ is the unique thin element in L_{n+1} such that $d_j s_j z = d_{j+1} s_j z = z$ and every other (n-) face of $s_j z$ is a degenerate element associated with a sub-pseudocylinder of $\Delta^{n+1}: \Delta_j^{n+1} \Rightarrow \Delta_{j+1}^{n+1}$.

In view of the remarks above it is reasonable to attempt to define degeneracy structures on MT-complexes by means of pseudocylinder structures on M-cells; that is, to define a degenerate element $\varepsilon_J x$ in an MT-complex as a thin element with shell determined by a pseudocylinder structure J .

One point must be borne in mind, however. For each shell of the form 1.1 (i) , (ii) in a □T (respectively ΔT)-complex there is, by definition, a thin element whose faces agree with those of the shell. Given an analogous shell in an MT-complex K, we have to show that there exists a thin element in K with the required faces. This can not be done for all model categories $M \in \Gamma$. Consider Δ_I and $\Box_I$, the wide subcategories with injective morphisms of the simplicial and cubical model categories Δ and $\Box$ respectively. While a Δ_IT-complex admits a canonical degeneracy structure to become a ΔT-complex (III §2) the situation is different for $\Box_I$.

<u>1.2 Example</u> (R. Brown) We define a $\Box_I$T-complex $\Box_I\mathbb{Z}$ as follows:

$$(\Box_I\mathbb{Z})_n \quad = \quad \text{the set of sequences } \{m,m+1,\dots,m+n\} \;,\quad m \in \mathbb{Z} \;;$$

for $i = 1,2,\dots,n$,

$$\partial_i^0 \{m,m+1,\dots,m+n\} = \{m,m+1,\dots,m+n-1\}$$

$$\partial_i^1 \{m,m+1,\dots,m+n\} = \{m+1,m+2,\dots,m+n\} \;;$$

each element of $\Box_I\mathbb{Z}$ of dimension ≥ 1 is thin.

Since there is no element of $(\Box_I\mathbb{Z})_1$ with identical 0-faces, $\Box_I\mathbb{Z}$ does not admit a degeneracy structure.

We will show that if M is one of the 'nice'model categories in $E\Gamma$ then a degeneracy structure may be defined in an MT-complex.

The next two sections consider pseudocylinder structures on $\overrightarrow{SC}$-complexes (S-shellable marked cone-complexes) and give the geometric preparation for § 4 , where degenerate elements in MT-complexes are discussed.

§2 Pseudocylinder structures on $\vec{SC}$-complexes

For X an $\vec{SC}$-complex, a pseudocylinder $J(X)$ should resemble the cylinder $X \times I$. On the other hand, a fairly general structure would be of interest. In our notion of a pseudocylinder, $A \times I$ (for each face A of X) is replaced by a 'stack' of faces satisfying certain conditions.

<u>2.1 Definition</u> Let X by an n-dimensional $\vec{SC}$-complex. A *pseudocylinder* $J(X)$ consists of:

(i) an $(n+1)$-dimensional $\vec{SC}$-complex UJ ;

(ii) two subcomplexes X^o, X^1 of UJ with $\vec{SC}$-isomorphisms

$$i^o\colon X \to X^o \ , \quad i^1\colon X \to X^1 \quad ;$$

(iii) for $k \geq 0$ and each k-face A of X, a *stack* on A, namely a sequence

$$J_A = \{i^o(A) = A_o, \tilde{A}_1; A_1, \ldots, \tilde{A}_q, A_q = i^1(A)\} \ , \ q \geq 0$$

of (distinct) faces of UJ such that the sets J_A partition the set of faces of UJ and the following hold for $j = 1, 2, \ldots, q$:

(a) $\tilde{A}_j$ is a $(k+1)$-face, A_j is a k-face;
$A_o \subset \tilde{A}_1$, $A_q \subset \tilde{A}_q$; and, for $j = 1, 2, \ldots, q-1$,
$(\tilde{A}_1 \cup \ldots \cup \tilde{A}_j) \cap \tilde{A}_{j+1} = \tilde{A}_j \cap \tilde{A}_{j+1} = A_j$.

(b) There is an SPoly-isomorphism $\nu_j\colon A \to A_j$.
For B a $(k-1)$-face of A, the $(k-1)$-face $\nu_j(B)$ of A_j belongs to J_B and $j < \ell \Rightarrow \nu_j(B) \leq \nu_\ell(B)$ in J_B .

(c) The k-faces of $\tilde{A}_j$ are A_{j-1}, A_j and, for each $(k-1)$-face B of A, the k-faces which lie between $\nu_{j-1}(B)$ and $\nu_j(B)$ in J_B . (Set $\nu_0 = i^o|A, A_o$.)

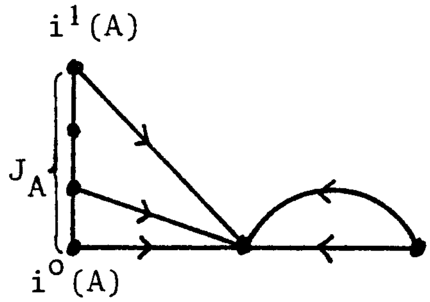
(Where there is a possibility of confusion we write X^α_J , i^α_J for X^α , i^α ($\alpha = 0\ 1$) .)

If $J_A = \{i^o(A) = i^1(A)\}$ then J_A is said to be a *trivial stack* on A .

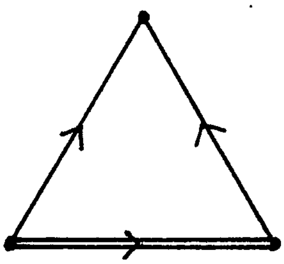
We define the *trivial pseudocylinder* $\emptyset(X)$ to be the complex X with the trivial stack $\emptyset_A = \{A\}$ on each face A of X .

Examples

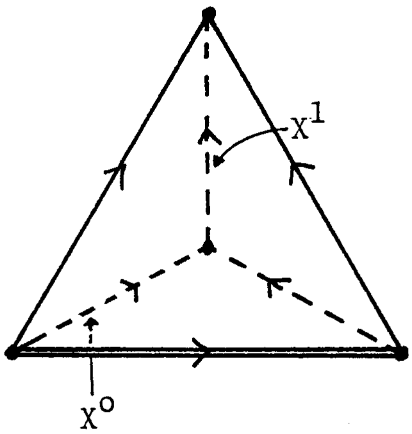
X



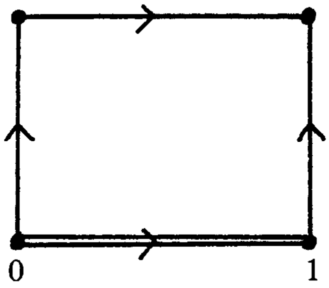
J(X) .



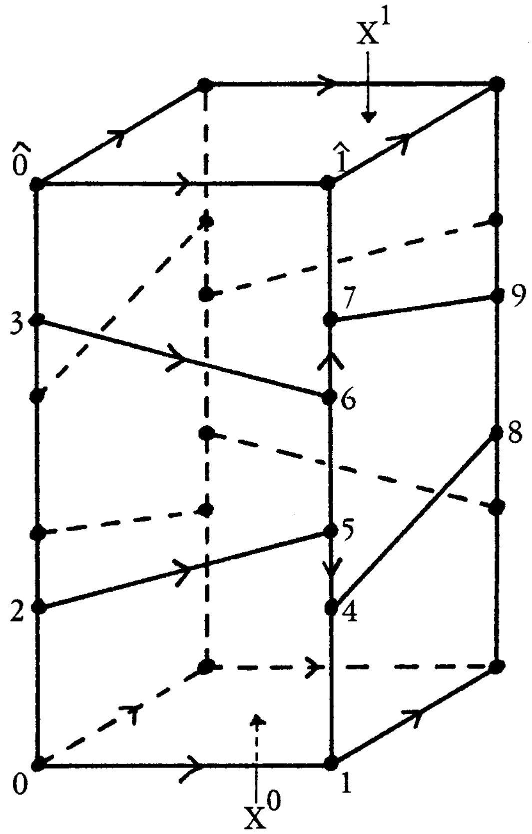
X



J(X)



X



J(X)

2.2 Remarks

(i) A complex UJ may have a multiplicity of pseudocylinder structures J(X). That is, there may exist distinct pseudocylinders J(X), J'(X) with UJ = UJ'. In the third example above, J(X) and J'(X) can be defined with UJ = UJ', $X_J^\alpha = X_{J'}^\alpha$, and $i_J^\alpha = i_{J'}^\alpha$ (α = 0,1). The structures of J(X), J'(X) are indicated by the stacks J_{01}, J'_{01} on the 1-face 01 of X.

$$J_{01} = \{01, 02541, 25, 2365, 36, 3\hat{0}\hat{1}76, \hat{0}\hat{1}\}$$
$$J'_{01} = \{01, 02541, 45, 456798, 67, 3\hat{0}\hat{1}76, \hat{0}\hat{1}\}.$$

Examples of distinct pseudocylinders J(X), J'(X) with UJ = UJ' (but with $X_J^\alpha \neq X_{J'}^\alpha$) occur in (iv) and (v) below.

(ii) We will be particularly concerned with pseudocylinders $J(X)$ such that X and UJ are S-polycells. In this case, the choice of faces $X^o, X^1 \subset UJ$ determines i^o, i^1 (the unique SPoly-isomorphims $X \to X^o$, $X \to X^1$) and the stack J_X (= $\{X^o, UJ, X^1\}$). The example considered in (i) above shows, though, that even here $J(X)$ is not determined by UJ, X^o, X^1.

(iii) For any $\vec{SC}$-complex X, the cylinder $X \times I$ has a canonical pseudocylinder structure $\Pi(X)$ where $U\Pi = X \times I$; $X^o = X \times \{0\}$, $X^1 = X \times \{1\}$; i^o, i^1 are the canonical isomorphisms $X \to X \times \{0\}$, $X \to X \times \{1\}$; and, for each face A of X, $\Pi_A = \{A \times \{0\}, A \times I, A \times \{1\}\}$.

(iv) The definition (III 5.7) of a simplicial pseudocylinder is a special case of 2.1. For $n \geq 1$, each simplicial pseudocylinder Δ^n: $X \Rightarrow Y$ defines a unique $J(\Delta^{n-1})$ with $UJ = \Delta^n$, $i^o(\Delta^{n-1}) = X$ and $i^1(\Delta^{n-1}) = Y$. Also, for each pseudocylinder structure $J(\Delta^{n-1})$ on Δ^n we have Δ^n: $i^o_J(\Delta^{n-1}) \Rightarrow i^1_J(\Delta^{n-1})$.

(v) For each pseudocylinder $J(X)$ there is an 'inverse' $J^{-1}(X)$ where $UJ^{-1} = UJ$, $i^o_{(J^{-1})} = i^1_J$, $i^1_{(J^{-1})} = i^o_J$ and if $J_A = \{A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q\}$ for A a face of X then $J^{-1}_A = \{A_q, \tilde{A}_q, A_{q-1}, \ldots, \tilde{A}_1, A_o\}$.

Propositions 2.4, 2.5 below bring out the analogy between cylinders and pseudocylinders. Conditions (a), (b), (c) of 2.1 are tailored to give these results and to allow us to define sub-pseudocylinder structures on certain faces of a pseudocylinder (2.8).

<u>2.3 Definition</u> Let $J(X)$ be a pseudocylinder. For A a face of X, let $\underset{\sim}{J}_A$ be the union of the faces in the stack J_A.

We define $\underset{\sim}{J}$ to be the space $\cup J$ with the following $\overrightarrow{CC}$ structure. The closed cells of $\underset{\sim}{J}$ are the closed cells of X^o, X^1, and $\underset{\sim}{J}_A$ for each face A of X with non-trivial stack J_A. An arbitrary choice of characteristic maps is made. We take $X^o \cup X^1$ to have the inherited marked face structure and set $(\underset{\sim}{J}_A)_* = i^o(A) \subset X^o$.

<u>2.4 Proposition</u> *For* $J(X)$ *a pseudocylinder,* $\underset{\sim}{J}$ *is a* $\overrightarrow{CC}$*-complex.*

<u>Proof</u> The result follows if we show that $\underset{\sim}{J}$ is a regular cell complex.

Consider the non-trivial stack $J_A = \{A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q\}$ on the k-face A. For $j = 1,2,\ldots,q$, $\tilde{A}_j$ is a PL $(k+1)$-ball and A_j is a PL k-ball (Proposition II 1.7). Assume $\tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_t$, $t \geq 1$, is a PL $(k+1)$-ball. By 2.1(a), $(\tilde{A}_1 \cup \ldots \cup \tilde{A}_t) \cap \tilde{A}_{t+1} = A_t$ and so $\tilde{A}_1 \cup \ldots \cup \tilde{A}_{t+1}$ is a PL $(k+1)$-ball with boundary $\mathrm{Bd}(\tilde{A}_1 \cup \ldots \cup \tilde{A}_t) \cup \mathrm{Bd}\,\tilde{A}_{t+1} - \mathrm{Int}\,A_t$. We thus have $\underset{\sim}{J}_A = (\tilde{A}_1 \cup \ldots \cup \tilde{A}_q)$ is a $(k+1)$-ball with (see 2.1(c))

$$\mathrm{Bd}\,\underset{\sim}{J}_A = i^o(A) \cup i^1(A) \cup \bigcup_{B \subset A} \underset{\sim}{J}_B$$

and $\mathrm{Int}\,\underset{\sim}{J}_A = \mathrm{Int}\,\tilde{A}_1 \cup \mathrm{Int}\,A_1 \cup \ldots \cup \mathrm{Int}\,\tilde{A}_q$.

Since $X^o \cup X^1$ is a regular complex each closed cell of $\underset{\sim}{J}$ is a ball.

Since the sets J_A partition the set of faces of $\cup J$, either an open cell e_λ of $\cup J$ is a cell of $X^o \cup X^1$ or there is a unique cell $\operatorname{Int} \underset{\sim}{J}_A$ of $\underset{\sim}{J}$ containing $\bar{e}_\lambda$. Hence the open cells of $\underset{\sim}{J}$ partition the space UJ .

It follows from the definition of $\operatorname{Bd} \underset{\sim}{J}_A$ that the boundary of a closed n-cell of $\underset{\sim}{J}$ $(n \geq 0)$ is contained in $\underset{\sim}{J}^{(n-1)}$.

We now have that $\underset{\sim}{J}$ is a regular complex. □

For X an $\overrightarrow{SC}$-complex, let $i_o: X \to X \times \{0\}$, $i_1: X \to X \times \{1\}$ be the canonical isomorphisms.

<u>2.5 Proposition</u> *If* $J(X)$ *is a pseudocylinder with no trivial stacks then there is a* $\overrightarrow{CC}$*-isomorphism* $X \times I \to \underset{\sim}{J}$ *which restricts to the isomorphisms*

$$i_J^o \circ (i_o)^{-1} : X \times \{0\} \to X^o , \quad i_J^1 \circ (i_1)^{-1} : X \times \{1\} \to X^1$$

and which maps $A \times I$ *onto* $\underset{\sim}{J}_A$ *for each face* A *of* X .

<u>Proof</u> Since $X^o \cap X^1 = \emptyset$ there is an isomorphism $f_{-1}: (X \times \{0\}) \cup (X \times \{1\}) \to X^o \cup X^1$ which restricts to $i_J^o \circ (i_o)^{-1}$, $i_J^1 \circ (i_1)^{-1}$.

Assume, for $k \geq 0$, there is an isomorphism

$$f_{k-1}: (X\times\{0\}) \cup (X\times\{1\}) \cup (X^{(k-1)}\times I) \to X^o \cup X^1 \cup \bigcup_{B\subset X^{(k-1)}} \underset{\sim}{J}_B$$

which restricts to $i_J^o \circ (i_o)^{-1}$, $i_J^1 \circ (i_1)^{-1}$ and satisfies $f_{k-1}(B \times I) = \underset{\sim}{J}_B$ for B a face of $X^{(k-1)}$. Consider a k-face A of X . Since $\operatorname{Bd} \underset{\sim}{J}_A = i_J^o(A) \cup i_J^1(A) \cup \bigcup_{B\subset A} \underset{\sim}{J}_B$

we have $f_{k-1}(\operatorname{Bd}(A \times I)) = \operatorname{Bd} \underset{\sim}{J}_A$. As $(A \times I)_* = A \times \{0\}$ and $(\underset{\sim}{J}_A)_* = i_J^o(A)$, the cone structures of $A \times I$ and $\underset{\sim}{J}_A$ may be used to extend $f_{k-1}| \operatorname{Bd}(A \times I)$ to a $\overrightarrow{CC}$-isomorphism $A \times I \to \underset{\sim}{J}_A$. We thus have a $\overrightarrow{CC}$-isomorphism

$$f_k: (X\times\{0\}) \cup (X\times\{1\}) \cup (X^k\times I) \to X^o \cup X^1 \cup \bigcup_{A\subset X^k} \underset{\sim}{J}_A$$

of the required form, and the result follows by induction. □

<u>2.6 Definition</u> For X an $\overrightarrow{SC}$-complex, let $J(X)$ be a pseudocylinder. For Y a subcomplex of X, a *sub-pseudocylinder* of $J(X)$ is a pseudocylinder $L(Y)$ such that:

(i) UL is a subcomplex of UJ ;

(ii) for each face A of Y, $i_L^o(A)$ and $i_L^1(A)$ belong to the stack J_A ;

(iii) each stack of $L(Y)$ is a subsequence of a stack of $J(X)$.

We obtain:

<u>2.7 Proposition</u>

(i) *A sub-pseudocylinder* $L(Y)$ *of* $J(X)$ *is uniquely characterized by* UL . *The subcomplexes* Y^o , Y^1 *are the unions of the faces of* UL *which are earliest, respectively latest, in stacks of* $J(X)$.

(ii) *Let* Y *and* V *be subcomplexes of* X *and* $\underset{A\subseteq Y}{\cup} J_A$ *respectively. There is a sub-pseudocylinder* $L(Y)$ *of* $J(X)$ *with* $UL = V$ *if and only if, for each face* A *of* Y :

(a) *The set of elements of* J_A *contained in* V *is non-empty and is of the form* $\{A_{s-1}, \tilde{A}_s, A_s, \ldots, \tilde{A}_t, A_t\}$ *were* $1 \le s \le t$.

(b) *If* $\dim A = k \ge 1$ *and* B *is a* $(k-1)$*-face of* A *then the unique face of* $A_s(A_t)$ *in* J_B *is the first (last) element of* J_B *contained in* V . □

<u>2.8 Proposition</u> *Let* $J(X)$ *be a pseudocylinder and let* A *be a face of* X *with stack* $J_A = \{A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q\}$.

We have:

(i) *There is a unique sub-pseudocylinder* $J_\sigma(A)$ *of* $J(X)$ *with* $UJ_\sigma = \tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_q$. *Here* $i^0_{J_\sigma}(A) = i^0_J(A) = A_0$ *and* $i^1_{J_\sigma}(A) = i^1_J(A) = A_q$.

(ii) *For each face* $\tilde{A}_j$ $(1 \le j \le q)$ *in* J_A *there is a unique sub-pseudocylinder* $J_j(A)$ *of* $J(X)$ *with* $UJ_j = \tilde{A}_j$. *Here* $i^0_{J_j}(A) = A_{j-1}$ *and* $i^1_{J_j}(A) = A_j$. □

In preparation for a T-complex construction given in §4 we specify a collapse in certain pseudocylinders. Recall (I 5.3) that there is a total order $\zeta(X)$ on the faces of a polycell X .

<u>2.9 Definition</u> Let X be an S-polycell and let $J(X)$ be a pseudocylinder such that the stack J_A on a face A of X is $\{A_0, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q\}$. The collapse $C_J: UJ \searrow X^0$ proceeds as follows. For each face A of X , in the order $\zeta(X)$, carry out the sequence of elementary collapses deleting

$$\operatorname{Int} \tilde{A}_q \cup \operatorname{Int} A_q \; , \; \operatorname{Int} \tilde{A}_{q-1} \cup \operatorname{Int} A_{q-1} \; , \; \ldots \; , \; \operatorname{Int} \tilde{A}_1 \cup \operatorname{Int} A_1 \; .$$

We have immediately from the definition (III 4.18) of a restriction of a collapse:

<u>2.10 Proposition</u> *With the notation of* 2.9 , *for any face* $\tilde{A}_j$ *in* J_A *the collapse* $C_{J_j} : \tilde{A}_j \searrow A_{j-1}$ *is a restriction of the collapse* C_J . □

<u>§3 Rectifiers on pseudocylinders</u>

The notion of a *rectifier* on a pseudocylinder is central to the proof given in the next section of the existence of degeneracy structures in MT-complexes.

We first define 'sums' of pseudocylinders. Note that if (Y,Z) is an $\overrightarrow{SC}$-pair and $f\colon Z \to W$ is an $\overrightarrow{SC}$-morphism then the adjunction space $W \cup_f Y$ has a canonical $\overrightarrow{SC}$-complex structure.

<u>3.1 Definition</u> For X an $\overrightarrow{SC}$-complex, let $J(X)$ and $L(X)$ be pseudocylinders.‡ We define $(J+L)(X)$ to be the pseudocylinder such that:

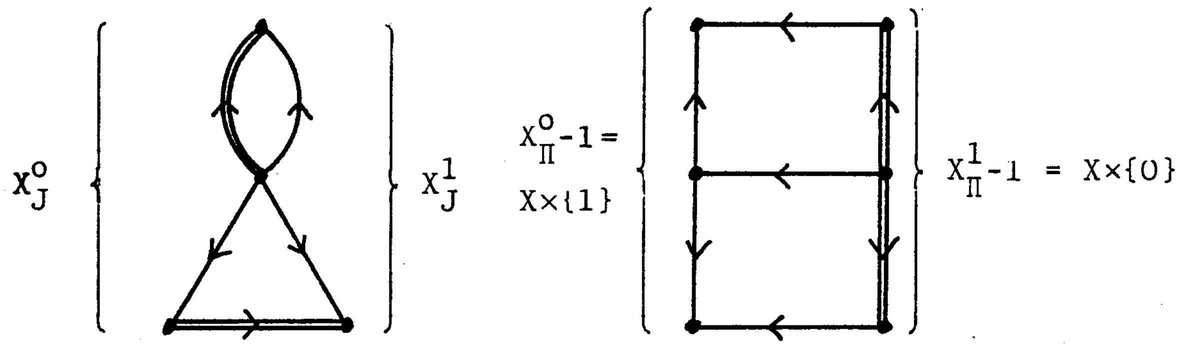
$$U(J + L) = UJ \cup_f UL$$

where f is the $\overrightarrow{SC}$-morphism $X_L^o \to UJ$ induced by $i_J^1 \circ (i_L^o)^{-1} : X_L^o \to X_J^1$; $i^o_{(J+L)} = i_J^o$, $i^1_{(J+L)} = i_L^1$, and if $J_A = \{i_J^o(A) = A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q = i_J^1(A)\}$, $L_A = \{i_L^o(A) = A'_o, \tilde{A}'_1, A'_1, \ldots, \tilde{A}'_r, A'_r = i_L^1(A)\}$ for A a face of X then $(J+L)_A = \{A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q = A'_o, \tilde{A}'_1, A_1, \ldots, \tilde{A}'_r, A_r\}$.

We are particularly concerned with the following special case. Recall that the pseudocylinders $\Pi(X)$ and $J^{-1}(X)$ were discussed in 2.2 (iii), (v) .

<u>3.2 Definition</u> For $J(X)$ a pseudocylinder, the *extension* $EJ(X)$ is defined to be the pseudocylinder $(J + \Pi^{-1})(X)$.

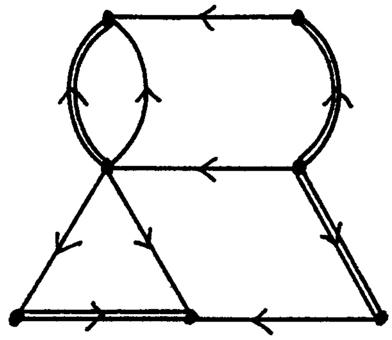
<u>Example</u>



$J(X)$ $\Pi^{-1}(X)$

‡ UJ and UL are required to be disjoint. Later on, we use the pseudocylinder $E\Pi(X) = (\Pi + \Pi^{-1})(X)$, where $U\Pi = U\Pi^{-1} = X \times I$. In this case we take $U\Pi^{-1}$ to be a distinct copy of $X \times I$.

$X^{o}_{EJ} = X^{o}_{J}$



$X^{1}_{EJ} = X \times \{0\}$

$EJ(X) = (J + \Pi^{-1})(X)$

For each face A of X, the stack Π^{-1}_{A} is non-trivial so that the stack $(J + \Pi^{-1})_{A}$ is non-trivial. Hence, from Proposition 2.5, we have

3.3 Proposition *For any pseudocylinder* $J(X)$ *there is a* $\overrightarrow{CC}$*-isomorphism* $X \times I \to \underset{\sim\sim}{EJ}$ *which restricts to the isomorphisms* $i^{o}_{EJ} \circ (i_{o})^{-1} : X \times \{0\} \to X^{o}_{J}$, $i_{o} \circ (i_{1})^{-1} : X \times \{1\} \to X \times \{0\}$ *and maps* $A \times I$ *onto* $\underset{\sim\sim}{EJ}_{A}$ *for each face* A *of* X. □

This result ensures that the definition below is meaningful.

3.4 Definition For a pseudocylinder $J(X)$, a *rectifier* RJ on $J(X)$ is the space $UEJ \times I$ with a $\overrightarrow{CC}$ structure as follows. Identify $UEJ \times \{0\}$ with UEJ. For each k-face A of X let

$$RJ_{A} = \underset{\sim\sim}{EJ}_{A} \times I$$

and

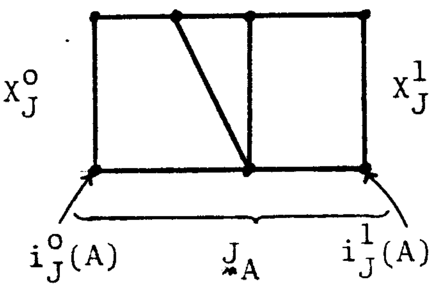
$$rJ_{A} = (i^{o}_{EJ}(A) \times I) \cup (\underset{\sim\sim}{EJ}_{A} \times \{1\}) \cup (i^{1}_{EJ}(A) \times I) .$$

Let RJ_{A} be a closed $(k+2)$-cell of RJ and rJ_{A} be a closed $(k+1)$-cell. Set $(RJ_{A})_{*} = rJ_{A}$ and $(rJ_{A})_{*} = i^{o}_{EJ}(A)$.

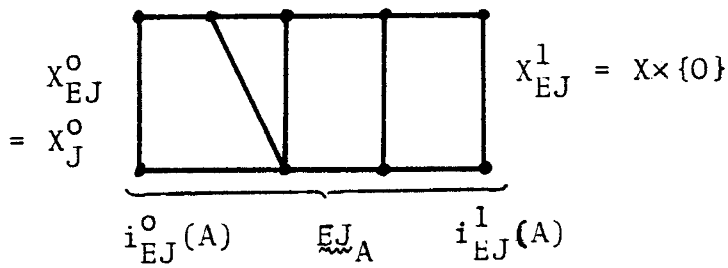
The characteristic maps of RJ_{A} and rJ_{A} are not specified (so that there is a multiplicity of rectifiers on $J(X)$).

The subcomplex rJ of RJ is defined by $rJ = \bigcup_{A \subset X} rJ_A$.

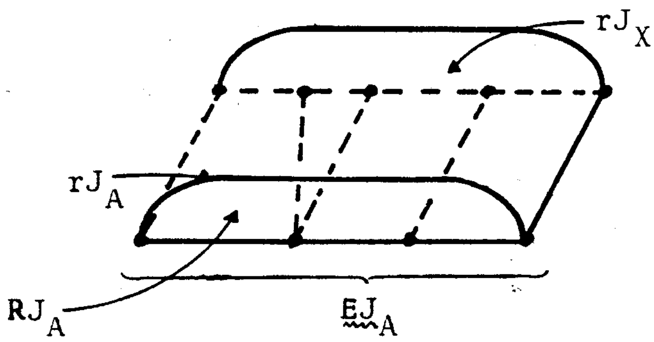
Example



J(X)



EJ(X)



RJ

We have easily (see the Appendix for S-shellability):

3.5 Proposition *Let* J(X) *be a pseudocylinder . Then:*

(i) *a rectifier* RJ *is an* $\vec{SC}$*-complex;*

(ii) *for each face* A *of* X *,* RJ_A *is a rectifier on the sub-pseudocylinder* $J_\sigma(A)$ *of* J(X) (see 2.8)

(iii) *there is an* $\vec{SC}$*-isomorphism* $\kappa : X \times I \to rJ$ *which restricts to the isomorphisms*

$i^o_{EJ} \circ (i_o)^{-1} : X \times \{0\} \to X^o_J$; $i_o \circ (i_1)^{-1} : X \times \{1\} \to X \times \{0\}$

and maps $A \times I$ *onto* rJ_A *for each face* A *of* X .

We are mainly interested in rectifiers on pseudocylinders $J(X)$ with X an S-polycell. Here RJ is an S-polycell, $rJ = rJ_X$, and $\kappa : X \times I \to rJ$ is the unique SPoly isomorphism.

For such pseudocylinders, a particular collapse in RJ is required to specify a construction in MT-complexes given in the next section. The total order $\zeta(X)$ on the faces of a polycell X (I 5.3) is used.

<u>3.6 Definition</u> Let X be an S-polycell and let the faces of X be $X = X(0), X(1), \ldots, X(r)$ in the order $\zeta(X)$. For $J(X)$ a pseudocylinder, the collapse $C_R : RJ \searrow UEJ$ proceeds

$$
\begin{aligned}
RJ &\overset{e}{\searrow} RJ - (\mathrm{Int}\, RJ_{X(0)} \cup \mathrm{Int}\, rJ_{X(0)}) \\
&\overset{e}{\searrow} RJ - (\mathrm{Int}\, RJ_{X(0)} \cup \mathrm{Int}\, rJ_{X(0)}) \\
&\qquad - (\mathrm{Int}\, RJ_{X(1)} \cup \mathrm{Int}\, rJ_{X(1)}) \\
&\overset{e}{\searrow} \\
&\quad \vdots \\
&\overset{e}{\searrow} UEJ
\end{aligned}
$$

<u>§4 Degenerate elements in an MT-complex</u>

Throughout this section, M is a model category in the class $E\Gamma$. A pseudocylinder $J(X)$ with UJ = an M-cell is called an M-pseudocylinder.

We define a structure of degenerate elements in an MT-complex K by assigning to an element $x \in K(X)$ and an M-pseudocylinder $J(X)$ another element $\varepsilon_J x \in K(UJ)$ with correct properties as regards faces.

Before giving a precise definition we recall some points concerning $J(X)$. For A a k-face of X let the stack J_A be $\{A_o,\tilde{A}_1,A_1,\ldots,\tilde{A}_q,A_q\}$. For $j = 1,2,\ldots,q$ there is a sub-pseudocylinder $J_j(A)$ of $J(X)$ with $UJ_j = \tilde{A}_j$ (2.8) . By the definition of (the skeletal) M there are unique M-cells A', $\tilde{A}_j'$ SPoly-isomorphic to A and $\tilde{A}_j$ respectively. We write $J_j(A')$ for the obvious pseudocylinder structure on $\tilde{A}_j'$.

The notation $\partial_A x$ (Definition I 6.2) is used.

<u>4.1 Definition</u> Let K be an MT-complex. A *degeneracy structure* in K assigns to each M-cell X and M-pseudocylinder $J(X)$ a function

$$\varepsilon_J : K(X) \to K(UJ)$$

such that the following hold for each element $x \in K(X)$:

(i) $\varepsilon_J x \in K(UJ)$ is thin;

(ii) $\partial_{(X^o)} \varepsilon_J x = \partial_{(X^1)} \varepsilon_J x = x$;

(iii) taking $\dim X = n$, for each $(n-1)$-face A of X and each face $\tilde{A}_j \in J_A$

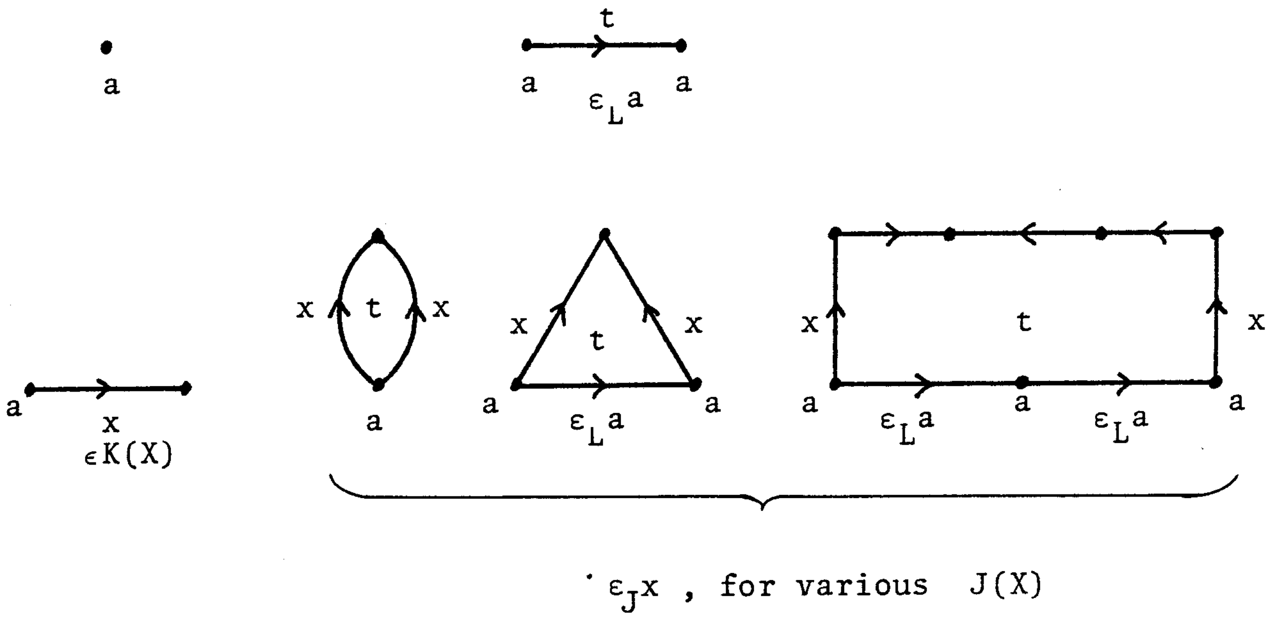
$$\partial_{(\tilde{A}_j)} \varepsilon_J x = \varepsilon_{J_j} \partial_A x$$

We say ε_J is a *degeneracy map* and $\varepsilon_J x$ is a *degenerate element* in K .

Note that the faces $\tilde{A}_j$ in condition (iii) above are the n-faces of UJ other than X^o and X^1 .

Thus an n-face of $\varepsilon_J x$ is either x or a degenerate element associated with a sub-pseudocylinder of $J(X)$.

Examples



(t denotes a thin element)

The remainder of this section is devoted to the proof of:

4.2 Theorem *For each* $M \in E\Gamma$ *, an* MT-*complex has a unique degeneracy structure.*

First, we have:

4.3 Proposition *If there is a degeneracy structure in an* MT-*complex* K *then the structure is unique.*

Proof Suppose that there is a degeneracy structure in K and use induction on dimension.

Assume that, for each M-cell Y of dimension $<n$ and M-pseudocylinder $L(Y)$, there is a unique degeneracy map $\varepsilon_L : K(Y) \to K(UL)$. Let X = an n-dimensional M-cell and J(X) = an M-pseudocylinder. By the existence of a degeneracy structure in K and axiom T2 (III 1.4) we have that, for

$x \in K(X)$, there is a unique thin element $\varepsilon_J x$ satisfying 4.1 (i) - (iii) . That is, there is a unique degeneracy map $\varepsilon_J\colon K(X) \to K(UJ)$. This gives the result. □

Secondly, recall that a skeleton P of the category SPoly is a member of $E\Gamma$ and each category $M \in E\Gamma$ is isomorphic to a full subcategory of P (II 2.10) . Clearly, a PT-complex restricts to an MT-complex $r^M K$ and we have:

<u>4.4 Proposition</u> *For* $M \in E\Gamma$, r^M *defines a functor* $PTC \to MTC$. □

<u>4.5 Proposition</u> *If there is a degeneracy structure in every* PT-*complex then every* MT-*complex,* $M \in E\Gamma$, *has a degeneracy structure.*

<u>Proof</u> We have obtained inverse equivalences of categories $r_M\colon MTC \to \Delta_I TC$, $e_M\colon \Delta_I TC \to MTC$ (III 1.5, 6.5, 8.6) . There is thus a natural equivalence $1_{MTC} \simeq e_M \circ r_M$. It is immediate from the definition of r^M that $e_M \simeq r^M \circ e_P$. Hence $1 \simeq r^M \circ e_P \circ r_M$ and, for any MT-complex K , there is a PT-complex $L = e_P \circ r_M K$ such that K is (naturally) isomorphic to $r^M L$.

A structure of degenerate elements in L induces a degeneracy structure in $r^M L$ and hence in K . □

We now consider degeneracies in PT-complexes. The notions of a *structure* in a PT-complex (III 3.5) and a *thin expansion of a structure corresponding to a collapse* (III 3.7) will be used.

<u>4.6 Definition</u> Let K be a PT-complex. For X a P-cell, $J(X)$ a pseudocylinder, and $x \in K(X)$, the structure $s_J x : UJ \to K$ is defined as follows. Let $(s_J x)^o : X^o_J \to K$ be the structure specified by $(s_J x)^o(X^o_J) = x$ and let $s_J x$ be the thin expansion of $(s_J x)^o$ corresponding to the collapse $c_J : UJ \searrow X^o_J$ given in 2.9.

For any $x \in K(X)$ we have the structure $s_\Pi x : X \times I \to K$, where $\Pi(X)$ is the canonical pseudocylinder structure on $X \times I$ with $X^o_\Pi = X \times \{0\}$, $X^1_\Pi = X \times \{1\}$ (2.2).

<u>4.7 Proposition</u> *For* K *a* PT-*complex,* X *a* P-*cell and* $x \in K(X)$, *we have* $s_\Pi x\ (X \times \{1\}) = s_\Pi x\ (X \times \{0\}) = x$.

<u>Proof</u> (See example which follows.) The proof makes use of a rectifier $R\Pi$ on $\Pi(X)$ (see 3.4).

Consider the subcomplex $UE\Pi$ of $R\Pi$. Since $U\Pi = U\Pi^{-1} = X \times I$, a structure $E : UE\Pi \to K$ is defined by setting $E|\,U\Pi = E|U\Pi^{-1} = s_\Pi x$. Let $Q : R\Pi \to K$ be the thin expansion of E corresponding to the collapse $c_R : R\Pi \searrow UE\Pi$. (3.6).

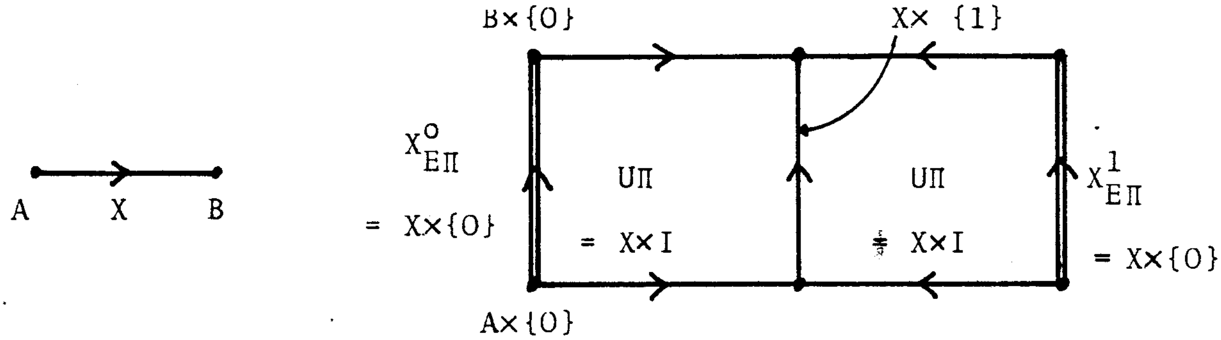
For $k \geq 0$ and each k-face A of X, the $(k+2)$-face $R\Pi_A$ of $R\Pi$ has as $(k+1)$-faces $A \times I \subset U\Pi$, $A \times I \subset U\Pi^{-1}$, $r\Pi_A$ and $R\Pi_B$ for each $(k-1)$-face B of A. By definition, $Q(R\Pi_A)$, $Q(A \times I)$ and $Q(R\Pi_B)$ for each B are thin in K. Hence $Q(r\,\Pi_A)$ is thin by axiom (T3).

By 3.5 there is an SPoly-isomorphism $\kappa : X \times I \to r\Pi$ such that $\kappa(X \times \{0\}) = X^o_{EJ}$, $\kappa(X \times \{1\}) = X^1_{EJ}$ and $\kappa(A \times I) = r\Pi_A$ for each face A of X. Let $\kappa' : X \times I \to R\Pi$ be the SPoly-morphism induced by κ. Then there is a structure

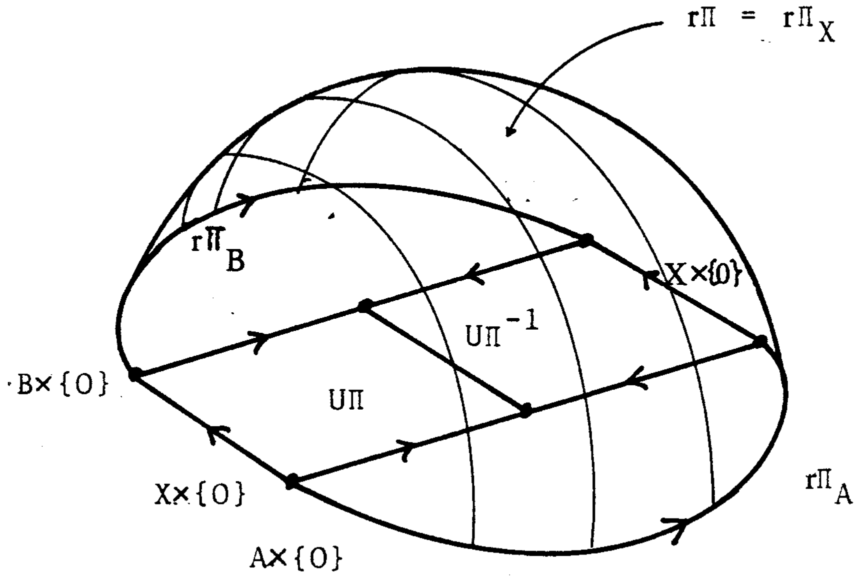
$\mathcal{Q} \circ \kappa' : X \times I \to K$ such that $\mathcal{Q} \circ \kappa' | (X \times \{0\}) = s_\Pi x | (X \times \{0\})$ and $\mathcal{Q} \circ \kappa'(A \times I)$ is thin for each face A of X. Using Proposition III 3.8 we have $\mathcal{Q} \circ \kappa' = s_\Pi x$, the thin expansion of $s_\Pi x | (X \times \{0\})$ corresponding to C_Π.

It follows that $\mathcal{Q}(X^1_{EJ}) = \mathcal{Q} \circ \kappa'(X \times \{1\}) = s_\Pi x(X \times \{1\})$. But $\mathcal{Q}(X^1_{EJ})$ was defined to be $s_\Pi x(X \times \{0\})$ so $s_\Pi x(X \times \{1\}) = s_\Pi x(X \times \{0\}) = x$. □

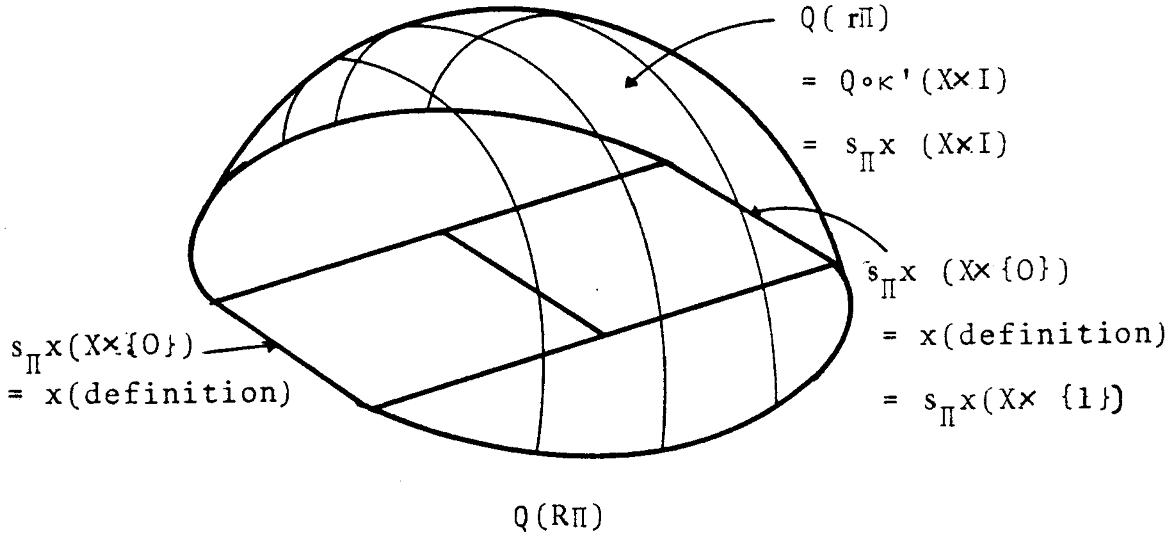
Example



UE Π



RΠ



<u>4.8 Proposition</u> *Let* K *be a* PT-*complex. For* X *a* P-*cell,* J(X) *a pseudocylinder and* $x \in K(X)$ *we have*

$$s_J x(X_J^1) = s_J x(X_J^o) = x .$$

<u>Proof</u> (See the example which follows.) We use a rectifier RJ on J(X) . Let $s_J x(X_J^1) = x'$. From 4.7 , $s_\Pi x'(X \times \{1\}) = s_\Pi x'(X \times \{0\}) = x'$ so we can define a structure E: UEJ $\to$ K by setting $E|UJ = s_J x$ and $E|U\Pi^{-1} = E|X \times I = s_\Pi x'$. Then Q : RJ $\to$ K is defined to be the thin expansion of E corresponding to the collapse C_R : RJ $\searrow$ UEJ .

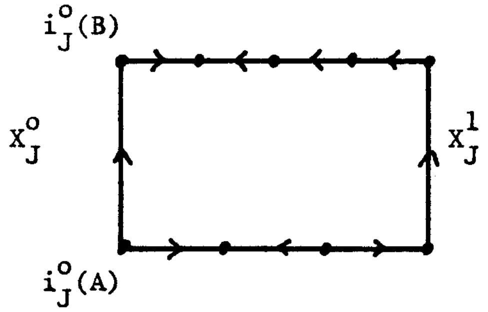
For $k \geq 0$ and A a k-face of X with stack $J_A = \{A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q\}$, $q \geq 0$, the (k+1)-faces of the (k+2)-face RJ_A are the faces $\tilde{A}_j$ for $1 \leq j \leq q$, $A \times I$, rJ_A , and RJ_B for each (k-1)-face B of A . By definition, $Q(RJ_A)$ and all its (k+1)-faces except $Q(rJ_A)$ are thin. Hence $Q(rJ_A)$ is thin by axiom T3 .

Let κ': X × I → RJ be the SPoly-morphism induced by the isomorphism κ: X × I → rJ of 3.5 . The structure

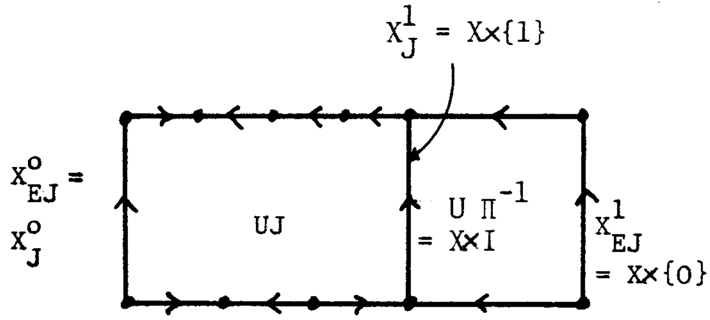
$\mathcal{Q} \circ \kappa' : X \times I \to K$ satisfies $\mathcal{Q} \circ \kappa'|(X \times \{0\}) = s_{\Pi}x|(X \times \{0\})$ and (for each face A of X) $\mathcal{Q} \circ \kappa'(A \times I) = \mathcal{Q}(rJ_A)$ = a thin element. Hence (using III 3.8) $\mathcal{Q} \circ \kappa' = s_{\Pi}x$, the thin expansion of $s_{\Pi}x|(X \times \{0\})$ corresponding to the collapse C_{Π} .

It follows (4.7) that $\mathcal{Q} \circ \kappa'(X \times \{1\}) = \mathcal{Q} \circ \kappa'(x \times \{0\}) = x$; that is $\mathcal{Q}(X^1_{EJ}) = x$. But $\mathcal{Q}(X^1_{EJ}) = s_{\Pi}x'(X \times \{0\}) = x'$. Hence $x' = x$. □

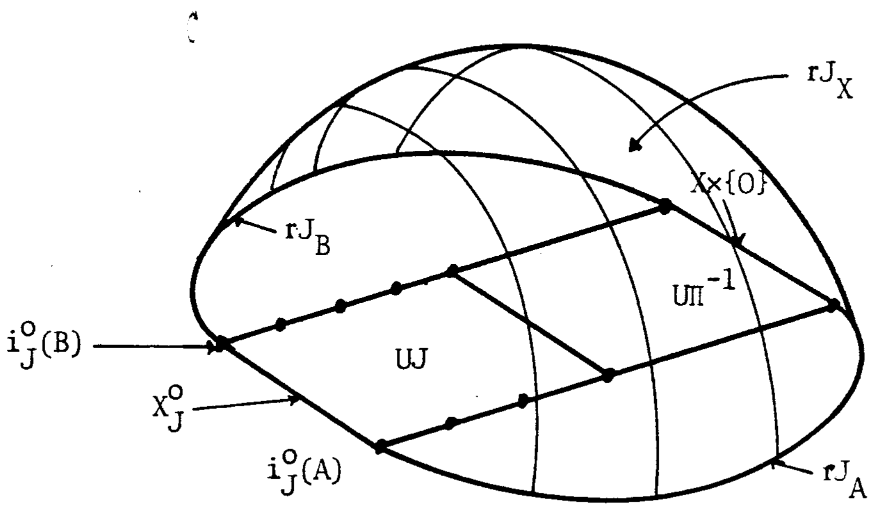
<u>Example</u>



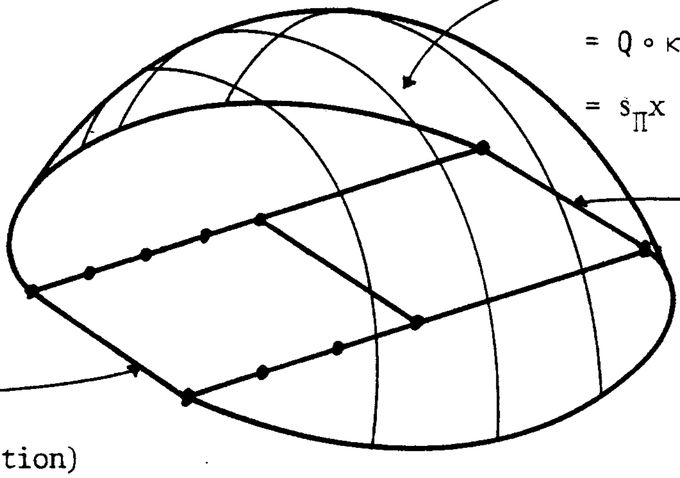
UJ



UEJ



RJ



<u>4.9 Proposition</u>. *Every* PT-*complex* K *has a degeneracy structure.*

<u>Proof</u> For each P-cell X and P-pseudocylinder J(X) we define a function $\varepsilon_J: K(X) \to K(UJ)$ by $\varepsilon_J x = s_J x(UJ)$ for $x \in K(X)$.

The structure $s_J x : UJ \to K$ is the thin expansion of $s_J x|X^o$ corresponding to the collapse $C_J: UJ \searrow X^o$. Thus $\varepsilon_J x$ is thin. From 4.8 , $\partial_{(X^o)}\varepsilon_J x = \partial_{(X^1)}\varepsilon_J x = x$.

Let $\dim X = n$ and consider an (n-1)-face A of X with stack $J_A = \{i_J^o(A) = A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q = i_J^1(A)\}$. For $j = 0,1,\ldots,q$, let $a_j = s_J x(A_j)$. By 2.8 , for $j = 1,2,\ldots,q$, there is a sub-pseudocylinder $J_j(A)$ of J(X) with $UJ_j = \tilde{A}_j$, $i_{J_j}^o(A) = A_{j-1}$ and $i_{J_j}^1(A) = A_j$. Denoting the unique P-cell isomorphic to A by A' , $J_j(A)$ defines a pseudocylinder $J_j(A')$ on $\tilde{A}_j$. There is thus a structure $s_{J_j} a_{j-1}: \tilde{A}_j \to K$. Since the collapse $C_{J_j}: \tilde{A}_j \searrow A_{j-1}$ is a restriction of C_J we have $s_J x|\tilde{A}_j = s_{J_j} a_{j-1}$ and, by 4.8 , $a_j = a_{j-1}$. Hence $a_j = a_o$ for each j and $s_J x \mid \tilde{A}_j = s_{J_j} a_o$. It follows that for each face $\tilde{A}_j \in J_A$, $\partial_{(\tilde{A}_j)}\varepsilon_J x = \varepsilon_{J_j}\partial_A x$ in the notation of 4.1 .

We have now shown that the maps ε_J form a degeneracy structure in K . □

This completes the proof of 4.2 .

<u>4.10 Remark</u> It was noted in 2.2 (v) that a pseudocylinder $J(X)$ has an 'inverse' $J^{-1}(X)$ with $UJ^{-1} = UJ$. It is clear that if $\varepsilon_J\colon K(X) \to K(UJ)$ is a degeneracy map in an MT-complex K then there is a degeneracy map $\varepsilon_{(J^{-1})}\colon K(X) \to K(UJ^{-1})$ such that $\varepsilon_{(J^{-1})} = \varepsilon_J$. That is, our notion of a degeneracy structure in an MT-complex has two copies of every degeneracy map. This does not cause any problems. If necessary, the total order on the faces of UJ (I 5.3) can be used to specify one of the pseudocylinders $J(X)$, $J^{-1}(X)$ so that a choice of ε_J , $\varepsilon_{J^{-1}}$ is fixed.

§5 Functors between categories of T-complexes

The degeneracy structure in an MT-complex ($M \in E\Gamma$) allows us to define certain functors involving M- , simplicial (Δ) - , and cubical (□) T - complexes. For convenience, we restrict ourselves here to $M = P$ = a skeleton of SPoly , and consider functors between PTC , ΔTC and □TC .

Note that P has full subcategories $\Delta_I^!$, $\Box_I^!$ canonically isomorphic to the categories Δ_I , $\Box_I$ defined in I §6 . Throughout this section, $\Delta_I^!$ is identified with Δ_I and $\Box_I^!$ and $\Box_I$. The marked face structure of $\Delta^n \in Ob(\Delta_I)$ is equivalent to a vertex-ordering (I §6). We denote the $(n-1)$-face of Δ^n not containing the vertex i by Δ^n_i. For $\alpha = 0, 1$ the face

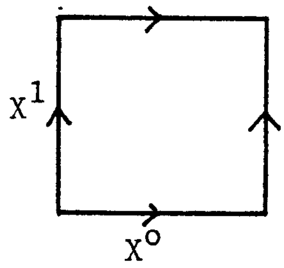
$$\{(t_1,\ldots,t_{i-1},\alpha,t_{i+1},\ldots,t_n) \mid 0 \le t_j \le 1\} \text{ of } I^n \in \mathrm{Ob}\,(\square_I)$$

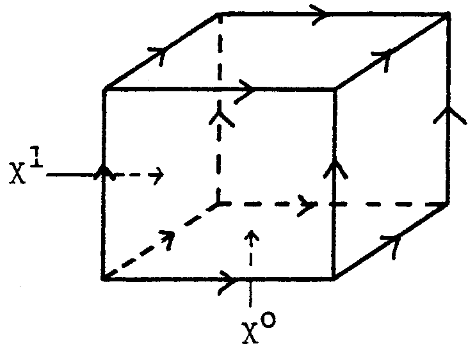
is denoted by $(I^n)^\alpha_i$.

A simplicial T-complex $\rho_\Delta K$ and a cubical T-complex $\rho_\square K$ can be associated with a PT-complex K thus : restrict K to obtain a Δ_IT-complex and a $\square_I$T-complex and define the following degeneracy structures. (The notation of 4.1 is used.) For $0 \le j \le n$, $s_j\colon (\rho_\Delta K)_n \to (\rho_\Delta K)_{n+1}$ is the degeneracy map $s_J\colon K(\Delta^n) \to K(\Delta^{n+1})$ in K, where $J(\Delta^n)$ is the pseudocylinder with $UJ = \Delta^{n+1}$, $i^0_J(\Delta^n) = \Delta^{n+1}_j$ and $i^1_J(\Delta^n) = \Delta^{n+1}_{j+1}$. For $1 \le j \le n+1$, $\varepsilon_j\colon (\rho_\square K)_n \to (\rho_\square K)_{n+1}$ is the degeneracy map $s_L\colon K(I^n) \to K(I^{n+1})$, where $L(I^n)$ is the pseudocylinder with $UL = I^{n+1}$, $i^0_L(I^n) = (I^{n+1})^0_j$ and $i^1_L(I^n) = (I^{n+1})^1_j$. We have:

<u>5.1 Proposition</u> *ρ_Δ defines a functor* PTC $\to$ ΔTC *and $\rho_\square$ defines a functor* PTC $\to$ $\square$TC. $\square$

Not all the degeneracy maps $K(\Delta^n) \to K(\Delta^{n+1})$, $K(I^n) \to K(I^{n+1})$ are used in the definition of $\rho_\Delta K$, $\rho_\square K$. As was noted in 4.10, there are degeneracy maps in K associated with the 'inverses' J^{-1}, L^{-1} of the pseudo-cylinders J, L. More significantly, K has maps $K(I^n) \to K(I^{n+1})$ corresponding to the 'connections' introduced by Brown-Higgins in [10] as extra (cubical) degeneracies. These maps are defined by pseudocylinders of the form:





Recall (III 8.6) that there are inverse equivalences of categories

$$r_P : PTC \to \Delta_I TC \quad , \quad e_P : \Delta_I TC \to PTC$$

and isomorphisms of categories (III §2)

$$\xi : \Delta TC \to \Delta_I TC \quad , \quad \eta = \xi^{-1} : \Delta_I TC \to \Delta TC \ .$$

We have immediately:

<u>5.2 Proposition</u> $\rho_\Delta = \eta \circ r_P$. □

Setting $e'_P = e_P \circ \xi$, this gives:

<u>5.3 Theorem</u> *The functors* $\rho_\Delta : PTC \to \Delta TC$ *and* $e'_P : \Delta TC \to PTC$ *are inverse equivalences of categories.* □

We also believe the following to be true:

<u>5.4 Claim</u> *The functor* $\rho_\Box : PTC \to \Box TC$ *is an equivalence of categories.* □

A sketch proof of this Claim is given in the next section. The construction of the inverse equivalence is not direct.

We use $\rho_\Box$ to obtain a functor $\tau : \Delta TC \to \Box TC$. Our definition of τ is similar to that of the functors $e_M : \Delta_I TC \to MTC$ in III §6 and terminology from III §6 is used.

<u>5.5 Definition</u> For K a simplicial T-complex, the $\square_I$T-complex $\tau_I K$ is defined as follows. For $n \geq 0$, $(\tau_I K)_n$ = the set of special $\mathrm{Sd}\, I^n$-structures in ξK and, for a $\square_I$-morphism $f: I^m \to I^n$, $\tau_I K(f)(V) = V \circ sf$, where sf is the map of Δ_I-sets induced by f. A structure $V \in (\tau_I K)_n$ is thin if $V(pI^n)$ is thin in K.

The cubical T-complex τK is obtained by defining the following degeneracy structure. For $n \geq 1$, $j = 1,2,\ldots,n+1$ and $x \in (\tau_I K)_n$, we let $\varepsilon_j x$ be the unique thin element of $(\tau_I K)_{n+1}$ with

$$\partial_i^\alpha \varepsilon_j x = \begin{cases} \varepsilon_{j-1}\partial_i^\alpha x, & i < j \\ x, & i = j \\ \varepsilon_j \partial_{i-1}^\alpha x, & i > j \,. \end{cases}$$

Reasoning similar to that of III §6 shows that $\tau_I K$ is a $\square_I$T-complex. In view of Example 1.2 there might seem to be a problem in showing that τK is a cubical T-complex. However, we find that τK is actually the complex $\rho_\square \circ e_P^! K$.

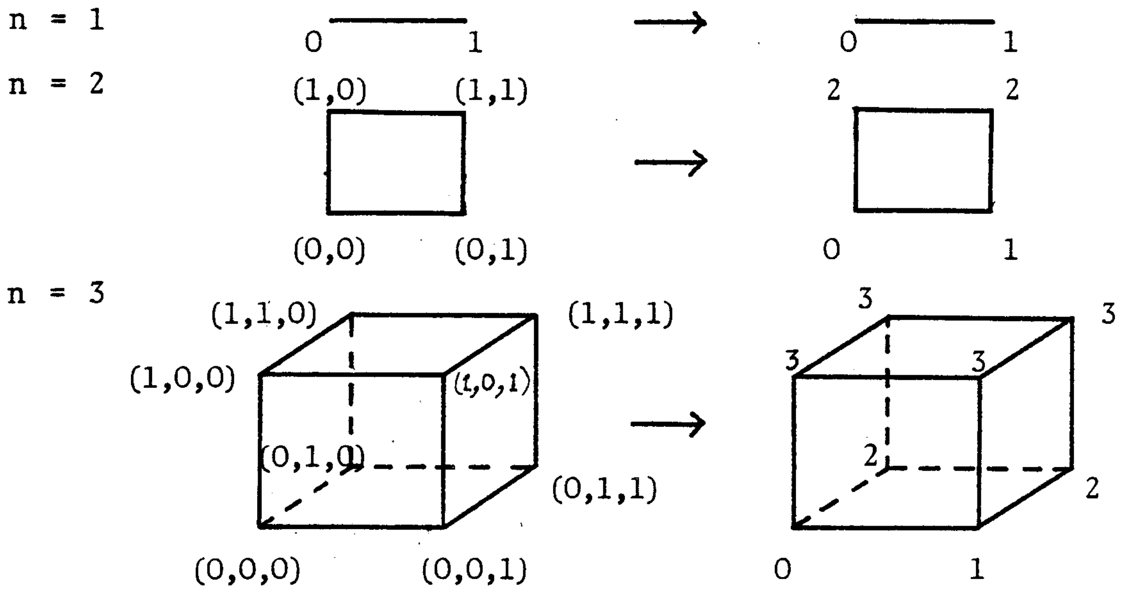
To each ΔTC-morphism $f: K \to L$ we can associate a $\square$TC-morphism $\tau f: \tau K \to \tau L$ given by

$$\tau f(V: \mathrm{Sd}\, I^n \to K) = \xi f \circ V: \mathrm{Sd}\, I^n \to L \,.$$

Thus we have:

<u>5.6 Proposition</u> τ *defines a functor* ΔTC $\to$ $\square$TC *such that* $\tau = \rho_\square \circ e_P^!$. $\square$

A functor $\square$TC $\to$ ΔTC may be defined in a fairly direct way. Consider the function $f_\sigma: \{0,1\}^n \to \{0,1,\ldots,n\}$ given by $f_\sigma(t_1, t_2, \ldots, t_n) = (n - i + 1)$, where i is the least integer such that $t_i = 1$ (set $f_\sigma(0,0,\ldots,0) = 0$).



Intuitively, we obtain a 'simplicial' element x in a cubical set K by making certain faces of x degenerate in accordance with vertex-numbering f_σ .

<u>5.7 Definition</u> For K a cubical set, the Δ_I-set $\sigma_I K$ is defined as follows:

$$(\sigma_I K)_0 = K_0 \quad , \quad (\sigma_I K)_1 = K_1 \quad ;$$

for $n \geq 2$, $(\sigma_I K)_n$ = the set of n-cubes x such that

$$\partial_i^1 x \in \varepsilon_i^{n-i} K_{n-1} \quad , \quad i = 1,2,\ldots,n-1 \quad ;$$

for $x \in K_n$, $d_0 x = \partial_n^1 x$ and $d_i x = \partial^0_{(n-i+1)} x$, $i = 1,2,\ldots,n$.

There is no difficulty in checking that $\sigma_I K$ is a Δ_I-set. It is not obvious that a simplicial set can be associated to K in a similar way: the definition of simplicial degeneracies seems to require the Brown-Higgins 'connections' mentioned earlier. However, a simplicial T-complex may be associated to each cubical T-complex.

<u>5.8 Definition</u> For K a cubical T-complex, the ΔT-complex σK is defined as follows. Make $\sigma_I K$ a Δ_IT-complex by taking $x \in \sigma_I K$ to be thin if x is a thin cube in K. Then let $\sigma K = \eta(\sigma_I K)$. (That is, add the canonical thin degeneracy structure to $\sigma_I K$.)

<u>5.9 Proposition</u> *σ defines a functor* $\square$TC $\to$ ΔTC. $\square$

We believe the following is true.

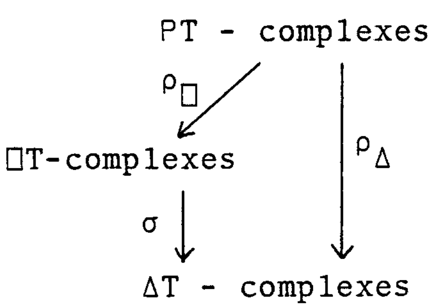
<u>5.10 Claim</u> *The functors* σ: $\square$TC $\to$ ΔTC *and* τ: ΔTC $\to$ $\square$TC *are inverse equivalences of categories.*

We do not give a complete proof of this Claim. In the following section an outline of a possible proof (dealing also with 5.4) is given. A detailed version of this would be lengthy and might not be the best approach to the question.

<u>§6 Suggested proof of 5.4, 5.10</u>

The sketch below is feasible but the statements made have not been verified in detail.

<u>Step 1</u> *The natural equivalence* $\rho_\Delta \simeq \sigma \circ \rho_\square$.

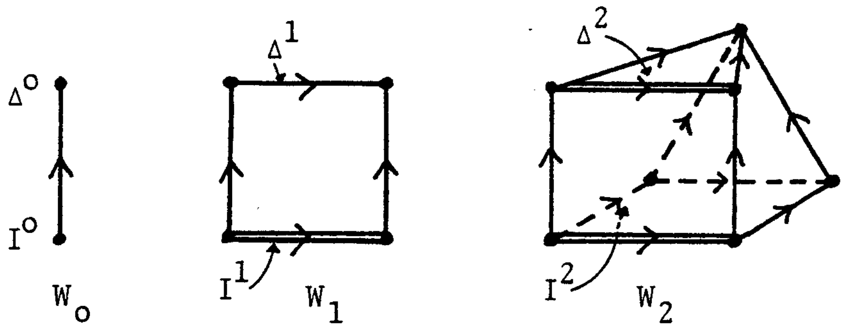


Define a map $f_n: I^n \to \Delta^n$ by induction on dimension, starting with $f_0(I^0) = \Delta^0$. f_n is given by

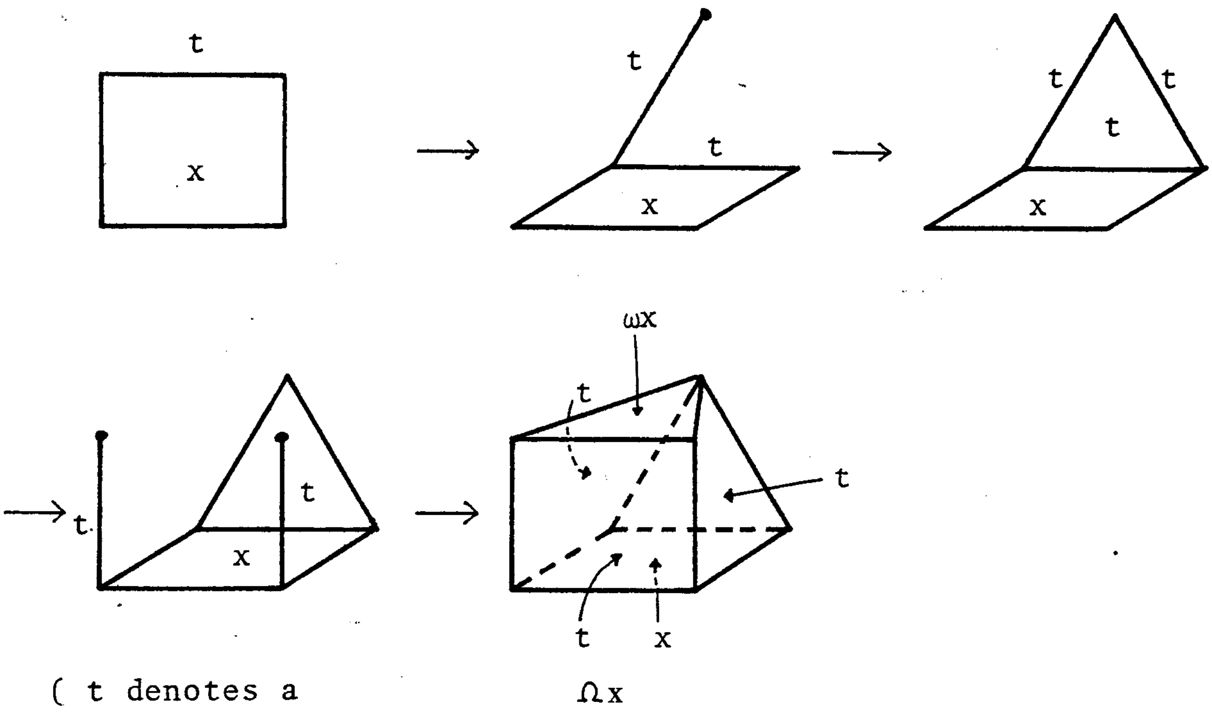
$$I^n = I^{n-1}\times I \xrightarrow{f_{n-1}\times I} \Delta^{n-1}\times I \longrightarrow \left(\begin{array}{l} C\Delta^{n-1} \xrightarrow{\;\cong\;} \Delta^n . \\ = (\Delta^{n-1}\times I) \big/ (\Delta^{n-1}\times\{1\}) \end{array} \right.$$

Let W_n = the mapping cylinder, $M(f_n)$
$= \Delta^n \cup_{(f')} (I^n \times I)$, where $f' : I^n \times \{1\} \to \Delta^n$
is the map $(x,1) \to f(x)$.

We can equip W_n with an $\vec{SC}$ structure to obtain an S-polycell with an (SPoly) n-cube and n-simplex as faces.



For K a PT-complex and $x \in (\sigma \circ \rho_\square K)_n$ an element $\Omega x \in K(W_n)$ can be built using thin fillers and from this an element $\omega x \in (\rho_\Delta K)_n$ is read off .



(t denotes a thin element)

Conversely, we can start with $y \in (\rho_\Delta K)_n$, build an element $\Omega' y \in K(W_n)$ using thin fillers and read off an element $\omega' y \in (\sigma \circ \rho_\square K)_n$. A bijection $(\sigma \circ \rho_\square K)_n \to (\rho_\Delta K)_n$ is obtained and this gives a natural equivalence $\rho_\Delta \simeq \sigma \circ \rho_\square$.

<u>Remark</u> R. Brown has pointed out that the cells W_n have also the following use. A classifying space $B^W G$ on a group G may be defined. Then, through $B^W G$, an equivalence between the simplicial and cubical classifying spaces $B^\Delta G$, $B^\square G$ (with degeneracies factored out) can be obtained.

$$\begin{array}{ccc} & B^W G & \\ \rotatebox{45}{$\simeq$} & & \rotatebox{-45}{$\simeq$} \\ B^\Delta G & & B^\square G \end{array}$$

<u>Step 2</u> *Definition of the functor* μ *from simplicial T-complexes to crossed complexes.*

We denote the category of crossed complexes by XC. μ is a modification of Ashley's functor $N: \Delta T\mathcal{C} \to XC$ (see [3]).

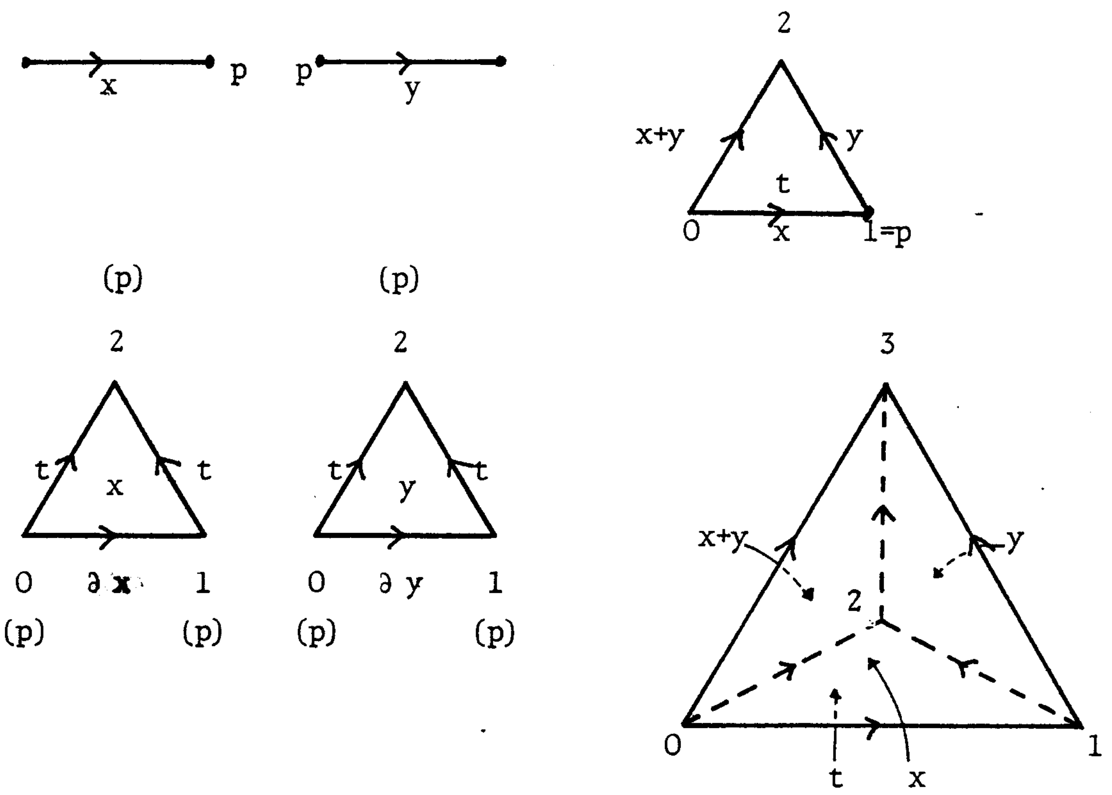
For L a simplicial complex, set

$$(\mu L)_0 = L_0 \quad , \quad (\mu L)_1 = L_1 \quad ;$$

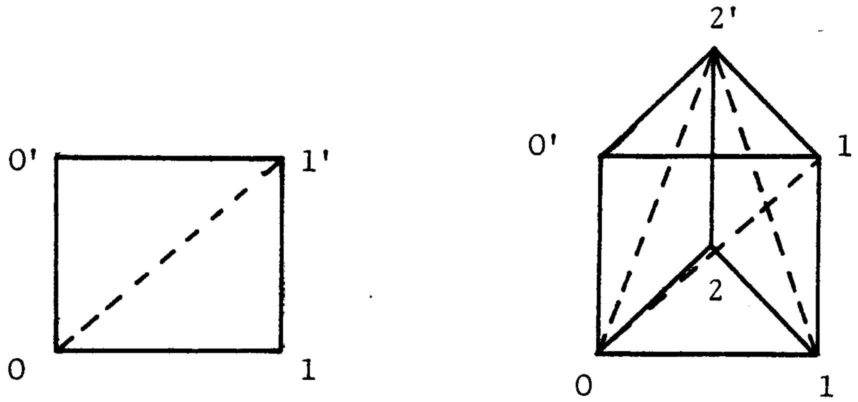
and, for $n > 1$ and $p \in L_0$,

$$(\mu L)_n(p) = \{x \in L_n \,|\, d_i x = s_0^{n-1} p \ , \ i=0,1,\ldots,n-1\} \ .$$

The obvious boundary maps are used. The groupoid structure on $(\mu L)_1$ and the group structure on $(\mu L)_n(p)$ are defined as follows. For suitable elements x, y of $(\mu L)_1$ or $(\mu L)_n$ let $x + y = d_1 M(x,y)$ where $M(x,y)$ is an $(n+1)$-simplex built using thin elements in the manner of Ashley.

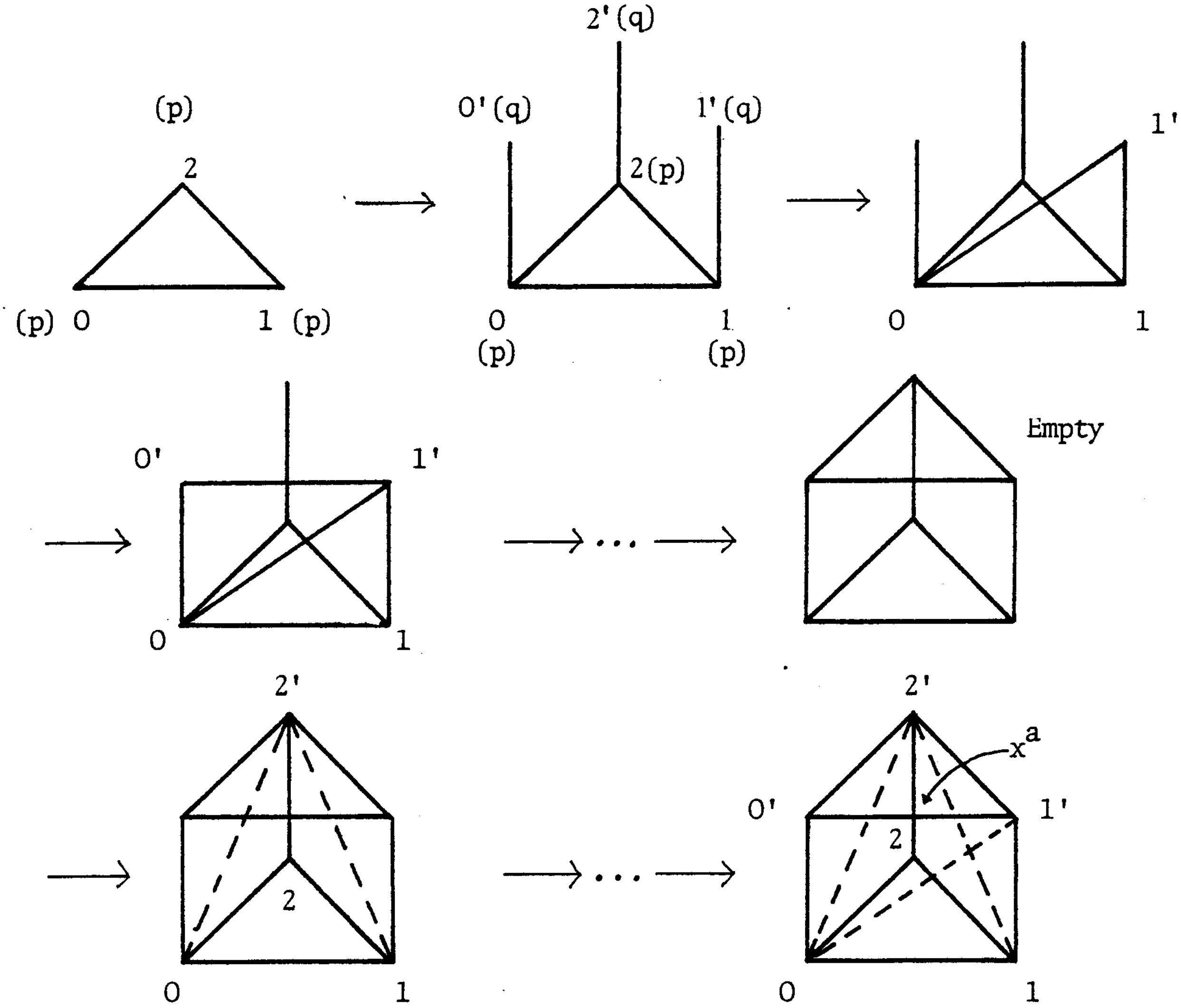


The groupoid action $(x,a) \to x^a$ $(x \in (\mu L)_n(p)$, $a \in (\mu L)_1(p,q))$ is defined using a triangulated prism $01 \ldots n0'1' \ldots n'$ whose $(n+1)$-simplices are $01 \ldots nn'$, $01 \ldots (n-1)(n-1)'n'$, $\ldots$, $00'1' \ldots n'$.



Let $01 \ldots n$ be x and let each edge $00'$, $11'$, ..., nn' be a. For $k \geq 1$ and each k-face $i_0 i_1 \ldots i_k$ of $01 \ldots n$, in order of increasing dimension, fill the $(k+1)$-simplices $i_0 i_1 \ldots i_k i_k'$, $i_0 i_1 \ldots i_{(k-1)} i_{(k-1)}' i_k'$, ..., $i_0 i_1' \ldots i_k'$ thinly.

Take the face $0'1' \ldots n'$ to be x^a .



Step 3 *The identity* $\gamma' = \mu \circ \sigma$

$$\begin{array}{ccc} & \square T\text{ - complexes} & \\ \sigma \swarrow & & \downarrow \gamma' \\ \Delta T\text{-complexes} & & \\ \mu \downarrow & & \\ & \text{crossed complexes} & \end{array}$$

Brown and Higgins [10] have constructed an adjoint equivalence

$$\gamma: \omega\text{-groupoids} \rightleftarrows \text{crossed complexes} : \lambda .$$

They have also obtained [12] an isomorphism

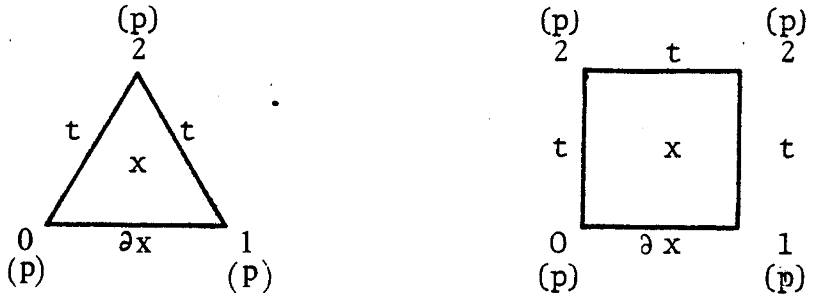
$$\sigma_G: \text{cubical T-complexes} \to \omega\text{-groupoids}.$$

We write γ' for $\gamma \circ \sigma_G$ and λ' for $\sigma_G^{-1} \circ \lambda$.

Our definition of μ is such that, for K a cubical T-complex,

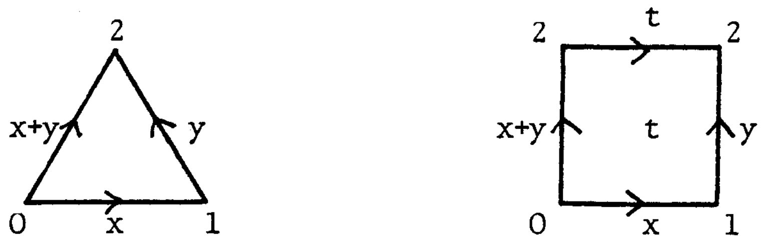
$$(\mu \circ \sigma K)_n = (\gamma' K)_n , \quad n = 0, 1$$

$$(\mu \circ \sigma K)_n(p) = (\gamma' K)_n(p) , \quad p \in K_0 , \quad n \geq 2 .$$

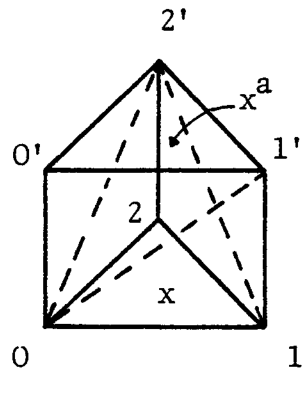


Also, the groupoid and group structures of $\mu \circ \sigma K$ and $\gamma' K$ coincide:

in $\mu \circ \sigma K$ in $\gamma' K$



To check that the groupoid actions in $\mu \circ \sigma K$ and $\gamma' K$ coincide, consider the prism $01 \ldots n0'1' \ldots n'$ used to define the action $(x,a) \to (x^a)_{\mu\sigma}$ $(x \in (\mu \circ \sigma K)_n(p)$, $a \in (\mu \circ \sigma K)_1(p,q))$
in $\mu \circ \sigma K$.



The $(n+1)$-simplices intersect in the n-simplices $01 \ldots (n-1)n'$, $01 \ldots (n-1)'n'$, ..., $01' \ldots n'$. By applying the cubical homotopy addition lemma [10, 7.1] in K to each $(n+1)$-simplex in turn, and bearing in mind that all n-simplices other than $01 \ldots n$, $01 \ldots (n-1)n'$, ..., $0'1' \ldots n'$ are thin, we find that

$$\begin{aligned}\Phi(x^a)_{\mu\sigma} = \Phi(0'1' \ldots n') &= \Phi(01' \ldots n') \\ &\vdots \\ &= \Phi(01 \ldots (n-1)n') \\ &= \Phi(x^a)_{(\gamma')} ,\end{aligned}$$

where Φ is the 'folding operator'. But $(x^a)_{\mu\sigma}$ and $(x^a)_{(\gamma')}$ are elements in $\gamma'K$ so $\Phi(x^a)_{\mu\sigma} = (x^a)_{\mu\sigma}$ and $\Phi(x^a)_{(\gamma')} = (x^a)_{(\gamma')}$. Hence $(x^a)_{\mu\sigma} = (x^a)_{(\gamma')}$ and the groupoid actions of $\mu\ \sigma K$ and $\gamma'K$ coincide.

This gives $\gamma' = \mu \circ \sigma$.

<u>Step 4</u> *'Proofs' of* 5.4 *and* 5.10 .

To obtain 5.4 we show that

$$\rho_{\square} : PTC \to \square TC \quad , \quad e_P' \circ \sigma : \square TC \to PTC$$

are inverse equivalences.

Since λ' and γ' are inverse equivalences and $\gamma' = \mu \circ \sigma$ (Step 3) we have

$$\lambda' \circ \mu \circ \sigma \simeq 1_{\square TC} .$$

Thus

$$\begin{aligned}\rho_{\square} \circ e_P' \circ \sigma &\simeq \lambda' \circ \mu \circ \sigma \circ \rho_{\square} \circ e_P' \circ \sigma \\ &\simeq \lambda' \circ \mu \circ \rho_{\Delta} \circ e_P' \circ \sigma \qquad \text{(Step 1)} \\ &\simeq \lambda' \circ \mu \circ \sigma \qquad \text{(5.3)} \\ &\simeq 1_{\square TC} .\end{aligned}$$

Further, $e_P' \circ \sigma \circ \rho_\square \simeq e_P' \circ \rho_\Delta$ (Step 1)

$\simeq 1_{PTC}$. (5.3)

This gives 5.4 .

To obtain 5.10 we have to show that

$$\sigma : \square TC \to \Delta TC , \quad \tau : \Delta TC \to \square TC$$

are inverse equivalences.

Now $\tau = \rho_\square \circ e_P'$ (5.6)

so $\tau \circ \sigma = \rho_\square \circ e_P' \circ \sigma$

$\simeq 1_{\square TC}$. (see above)

Also $\sigma \circ \tau = \sigma \circ \rho_\square \circ e_P'$

$\simeq \rho_\Delta \circ e_P'$ (Part 1)

$\simeq 1_{\Delta TC}$. (5.3)

This gives 5.10 .

CHAPTER V

COMMENTS AND POSSIBILITIES FOR FURTHER WORK

In this final chapter we make some remarks about the work of the thesis and discuss possible future developments. Throughout, M denotes a model category belonging to the class Γ defined in I 6.1. For some of the chapter, we use the terms *poly-set*, *poly* T-*complex* as generic terms for M-sets and MT-complexes for various M .

1. One major area which requires further work is to determine which model categories M are such that the category of M-sets is convenient for the purposes of algebraic topology. The category Δ-sets of simplicial sets has certain features which would be desirable in a category of M-sets. For instance:

(i) Δ-sets is Cartesian-closed;

(ii) if $|\,|$: Δ-sets $\to$ Top is the realization functor and K , L are simplicial sets then $|K \times L| \cong |K| \times |L|$ with the weak topology on the right hand side;

(iii) there is an equivalence of homotopy categories

$$|\,| : \text{Ho(Kan } \Delta\text{-sets)} \to \text{Ho CW}$$

We know from I §1 that for any M there is a singular functor

$$s_M : \text{Top} \to M\text{-sets}$$

and a realization functor

$$| \, |_M : M\text{-sets} \to \text{Top} .$$

such that $| \, |_M$ is a left adjoint of s_M. There is also a notion of *Kan* M-*set*. It would be very interesting to characterize those model categories M such that the category of Kan M-sets has a homotopy notion for which $| \, |_M$ gives an equivalence

$$\text{Ho(Kan M-sets)} \to \text{Ho CW} .$$

It may be that specific categories M are useful for specific purposes. We have noted earlier (IV §6) how the models W_n are useful for relating simplicial and cubical theories. Other models may have other uses. We hope that we have set up a basis for further work and exploitation.

2. In this respect, the proof of the equivalence between MT-complexes and simplicial T-complexes has been found to be a useful test-bed for basic definitions and techniques. In particular, the need for strong collapsibility conditions led by a happy chance to an understanding of the benefits of shellability notions, and hence to our equivalence between S-shellable polycells and S-posets (II §§3,4).

3. It would be interesting to obtain a classification of those categories $M \in \Gamma$ for which the equivalence MT-complexes → simplicial T-complexes holds. We have shown that the equivalence holds for an infinite class of categories in Γ but it is unlikely that this can be extended to all members of Γ.

For instance, consider the category G of *globes* (I 6.6). Intuitively, for the equivalence GTC → ΔTC to hold, it ought to be possible to define some non-trivial composition in a GT-complex K using the T-complex axioms. (bear in mind the groupoid structures which can be defined in simplicial and cubical T-complexes and the role of these structures in the equivalences T-complexes $\to$ crossed complexes [3,10,12].) However, an n-dimensional globe has only two (n-1)-faces so a box in K has one top-dimensional face. This means that the T-complex axioms can not be used to define any composites of elements in K.

It is doubtful that there is an equivalence $\square_I TC \to \Delta TC$, where $\square_I$ is the wide subcategory with injective morphisms of the usual cubical model category $\square$. It was shown in IV 1.2 that a $\square_I$ T-complex does not, in general, admit a degeneracy structure. Hence there is a significant difference between $\square_I$T-complexes and $\square$T-complexes.

4. For P a skeleton of the category SPoly of S-polycells we have defined an equivalence PT-complexes $\to$ simplicial T-complexes and a functor PT-complexes $\to$ cubical T-complexes which we believe to be an equivalence (IV §5). It should be possible to obtain direct equivalences between PT-complexes and the categories of *crossed complexes* (XC) and *∞-groupoids* (∞-Gpd) studied by Brown and Higgins [10,13].

Given a PT-complex K, one way of constructing an ∞-groupoid $r_\infty K$ and a crossed complex $r_{XC}K$ is to take certain elements of K and define the required structure using the T-complex axioms. Intuitively, the elements of an ∞-groupoid are 'globular' so we could set $(r_\infty K)_n = K(G^n)$ $(n \geq 0)$. Also feasible is $(r_{XC}K)_n$ = the set of elements $x \in K(G^n)$ with all faces (of all dimensions) thin except x

and $\partial_{(G^n_*)}x$. (See I §6 and the equivalence ∞-groupoids → crossed complexes defined in [13] .)

In order to define ∞-groupoid and crossed complex structures on $r_\infty K$ and $r_{XC}K$ non-globular elements of K are required. Hence, although the elements of $r_\infty K$ are the elements of a GT-complex (the restriction of K) it is not obvious how to define a functor GT-complexes → ∞-groupoids.

5. Let $\underset{\sim}{X} : X_0 \subset X_1 \subset X_2 \subset \ldots$ be a filtered space. A cubical or simplicial Kan complex $R(\underset{\sim}{X})$ can be associated with $\underset{\sim}{X}$ in a natural way (see Brown-Higgins [11] and Ashley [3]). When $R(\underset{\sim}{X})$ is factored by the relation of *filtered homotopy* a quotient map $p: R(\underset{\sim}{X}) \to \rho(\underset{\sim}{X})$ is obtained, where $\rho(\underset{\sim}{X})$ is also a Kan complex. If $\underset{\sim}{X}$ is a J_0-*filtered space* (that is, each loop in X_0 is contractible in X_1) then p is a Kan fibration and $\rho(\underset{\sim}{X})$ is a T-complex.

A Kan M-set $R^M(\underset{\sim}{X})$ can be constructed and, on taking quotients, a Kan M-set $\rho^M(\underset{\sim}{X})$ with quotient map $p_M: R^M(\underset{\sim}{X}) \to \rho^M(\underset{\sim}{X})$. There is a question: for which M is it true that $\underset{\sim}{X}$ is a J_0-filtered space implies p_M is a Kan fibration? If the implication holds then it should follow that $\rho^M(\underset{\sim}{X})$ is an MT-complex, giving an excellent geometric example of such a gadget.

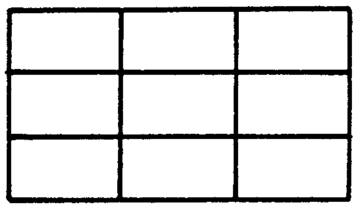
One point here is that the proofs of the cubical and simplicial versions of the implication rely on strong collapsibility properties of I^n and Δ^n (see 6(iii)).

The case M = G = the category of globes is of interest. It should be possible to define an ∞-groupoid structure on

$\rho^G(\underset{\sim}{X})$ (see 4 above). If so, the ∞-groupoid $\rho^G(\underset{\sim}{X})$ is probably injected into the ω-groupoid $\rho(\underset{\sim}{X})$ defined in [11]. None of this has been proved, however, so we cannot say whether $\rho^G(B^n)$ is the free ∞-groupoid on one generator of dimension n.

6. A proof of the Brown-Higgins Union theorem [11] could be attempted using poly T-complexes. Several points arise:

(i) A crucial factor in the success of cubical methods in proving the Union theorem is that an array such as



can be composed. Simplicial theory lacks a suitable notion of multiple composition and there is as yet no simplicial proof of the theorem. As explained in 7 below, multiple composition is easily handled in a poly T-complex.

(ii) Another key factor in the Brown-Higgins proof is the relationship between thin elements and degenerate elements in a cubical T-complex; in particular, the fact that degenerate elements may be characterized as thin elements with certain thin faces. Our definition of a degenerate element in a poly T-complex as a special thin element (IV §4) therefore seems suitable.

(iii) For there to be a proof of the Union theorem using MT-complexes, M-cells may have to satisfy stronger shellability/collapsibility conditions than S-shellability which was sufficient for the equivalence MTC → ΔTC . An important technical tool in the work of Brown and Higgins is the *deformation theorem* [11, 3.2] . (The result, mentioned in 5 , that p: $R(\underset{\sim}{X}) \to \rho(\underset{\sim}{X})$ is a Kan fibration is a corollary.) The proof of the deformation theorem uses a strong shellability-collapsibility property of cubes (expressed in terms of *partial boxes*).

There is a resemblance between the shellability part of this property and the notion of *(recursive) shellability* of a regular complex due to Bjorner-Wachs [6]. Let X by a pure n-dimensional regular complex. An ordering $F_1, F_2, \ldots, F_t$ of the n-faces of X is said to be a *shelling* if $n = 0$ or if $n > 0$, BdF_1 is shellable and for $j = 2, 3, \ldots, t$, $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is a pure (n-1)-complex having a shelling which extends to a shelling of BdF_j . (That is, BdF_j has a shelling in which the (n-1)-faces of $F_j \cap \bigcup_{i=1}^{j-1} F_i$ come first).

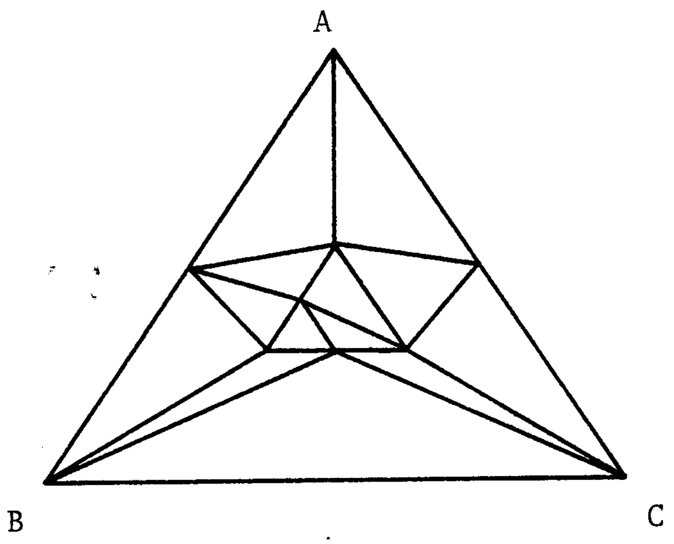
There is a question whether a condition based on recursive shellability can be imposed on polycells to give the equivalent of the deformation theorem in a polyhedral proof of the union theorem. One requirement is: if the objects of $M \in \Gamma$ are polycells with the extra condition there must be an equivalence MT-complexes → simplicial T-complexes.

7. A feature of poly T-complexes is that they provide a theory of 'general compositions'. We quote R. Brown, who writes in his Introduction to work of Dakin and Ashley [9] .

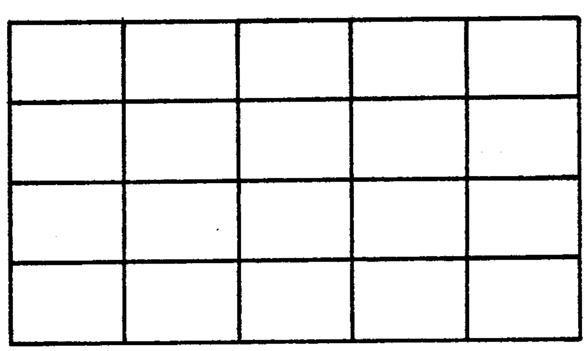
"The study of [categories equivalent to crossed complexes] has a basic motivation:

Determine an algebraic operation inverse to subdivision.

"Subdivision is of course an old technique in topology. The idea is to study a space by cutting it up into small, manageable bits. Intuitively, one then studies *cycles* and *boundaries* as certain 'composites' of these bits. Eventually, it was found convenient to treat these 'composites' as formal sums, and this is the formulation of homology theory that we know today. It is, indeed, difficult to know how one can 'really compose' all the bits of the following subdivision of the triangle ABC in order to form the big triangle ABC . Simplicial theory lacks suitable composition operations.



By contrast, in cubical theory, such compositions are easy to manage, since in a diagram such as



one composes rows first and then columns, and this is a well-defined operation. The interchange law allows one to carry out these operations in the other order, or by computing blocks in a partitioned matrix.

"A theory of general compositions, including simplicial, cubical, or polyhedral 'pieces' or 'bits', has to do three things:

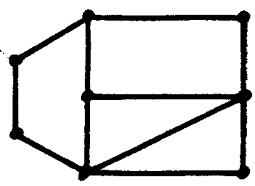
<u>Composition 1.</u> Define the 'bits' and the circumstances under which they are 'composable'.

<u>Composition 2.</u> Given 'composable bits', define their 'composite'.

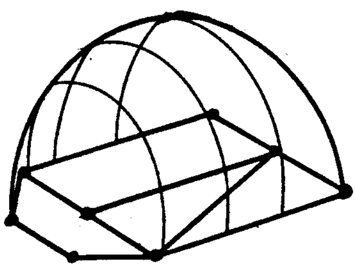
<u>Composition 3.</u> Give all relations among various 'compositions'.

"It seems likely that all three of these requirements are metby the theory of poly T-complexes. Thus the notion of T-complex looks as if it will continue to have wide ramifications."

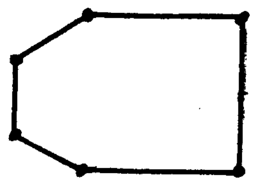
The 'bits' referred to are the elements of a poly T-complex K . Elements are 'composable' if they form a box B in K and the 'composite' is the free face of the unique thin filler of B (axiom T2) .



Composable elements (making up a box)

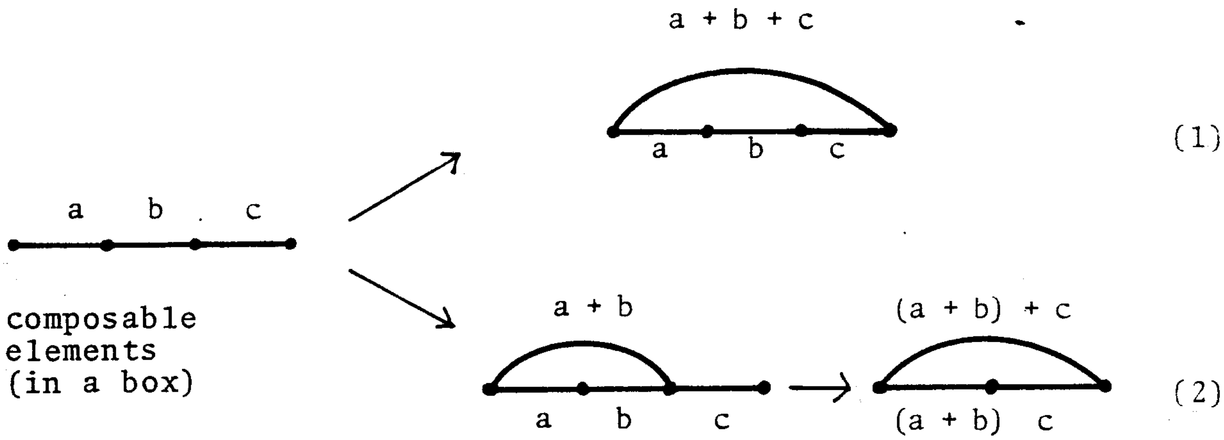


Unique thin filler

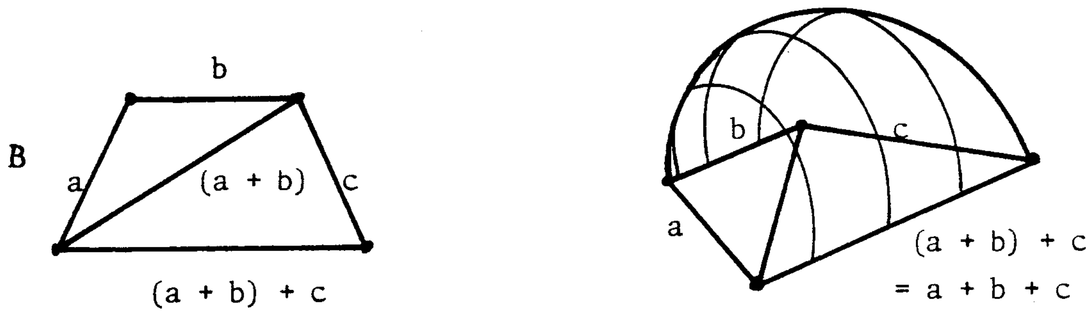


Composite

Relations among 'compositions' follow from axiom T3 . For instance, consider the following two cases



That a + b + c = (a + b) + c follows from filling the box B below:

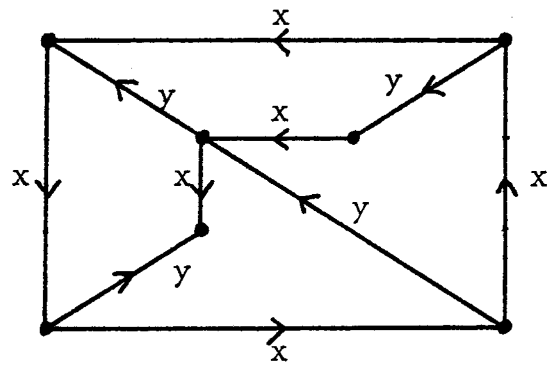


Since all 2-faces of B are thin the free face of the filler is also thin. Hence a + b + c = (a + b) + c .

It is interesting to compare the notion of 'general compositions' with the idea that the poly T-complex structure is a version of 'higher dimensional group theory' (see the Introduction). Taking this further, we can think of the degenerate elements defined in IV §4 as 'higher dimensional identity elements'.

8. Another intriguing feature of poly T-complexes is the link with the notion of *van Kampen diagrams* which occurs in combinatorial group theory. This point, noted by R. Brown, provided some of the initial motivation for our study.

Van Kampen diagrams for a group $G = \langle X | R \rangle$ express geometrically the deduction of new relators from old (see Johnson [28]). The are defined to be finite, connected, planar graphs whose edges are oriented and labelled by the generators of G . If the word assigned to the boundary of every face belongs to R then the boundary label of the diagram is equal to the identity in G . For example, the diagram below shows that the relation $x^4 = 1$ holds in the quaternion group $\langle x,y \mid xyxy^{-1} = yxyx^{-1} = 1 \rangle$.



We can think of the 2-faces of a diagram D as 2-dimensional elements of a poly T-complex K whose thin 2-elements are faces with boundary label equal to the identity in G . Then D is a box in K and the unique thin filler has free face F with boundary identical to BdD . The fact that the label on BdD = BdF is the identity in G follows immediately from axion T3 .

The theory of poly T-complexes might thus be regarded as a version of 'higher dimensional combinatorial group theory'. However, this idea needs to be clarified and developed.

APPENDIX

S-SHELLABILITY OF CONE-COMPLEXES

Shellable simplicial complexes were discussed in II §1 and the notion of an S-*shellable cone-complex* was introduced in II 1.6. Here, proofs are given of statements concerning S-shellability made in Chapters II and IV, namely:

The (CC-) dome, cone and cylinder constructions preserve S-shellability (II §1; A2, A4, A6). The construction VZ preserves S-shellability (II §2 ; A8).
The rectifier RJ on a pseudocylinder J(X) is S-shellable (IV §3 ; A9).

Also, in support of a statement made in II §2 and for use here, we give the following well-known result.

<u>A1 Proposition</u> *For* $n \geq 0$, $Sd\Delta^n$ *is shellable; that is, a simplicial complex is* S-*shellable.*

<u>Proof</u> See the proof of Proposition 1 of [16]. □

The definition of the *dome* DX on a cone-cell X is given in I 3.4.

<u>A2 Proposition</u> *If* X *is* S-*shellable then so is* DX.

<u>Proof</u> Let $\dim X = n$. Recall that DX is an $(n+1)$-cell whose boundary consists of two copies X^+, X^- of X.

The result follows if Sd DX is shellable which, since Sd DX can be identified with CSdBdDX , is true if there is a shelling of SdBdDX .

For F an n-simplex of SdX^+ , $F \cap SdX^- = F \cap SdBdX^+ =$ an (n-1)-face of F . A shelling of SdBdDX is thus obtained if we shell first SdX^- then SdX^+ . □

In order to deal with the cone and cylinder constructions we need the result below.

Recall (II §1) that a simplicial cone-complex is a cone-complex CC-isomorphic to a simplicial complex. From A1 , a simplicial cone-complex with a marked face structure becomes a simplicial $\vec{SC}$-complex. Pseudocylinder structures on $\vec{SC}$-complexes are discussed in IV §2 .

<u>A3 Proposition</u> *Let* J(X) *be a pseudocylinder with no trivial stacks and let* X *and* UJ *be simplicial* $\vec{SC}$*-complexes. If there is a shelling of* X *such that condition* (ii) *of* II 1.1 *holds then* UJ *is shellable.*

<u>Proof</u> For dim X = n we define a shelling *S* of UJ which goes through the stacks on the n-faces of X in turn, following a shelling F1 , F2 , ..., Ft of X which satisfies II 1.1 (ii) .

For A an n-face of X let the stack J_A be $\{i^o(A) = A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q = i^1(A)\}$. Recall that the union of the faces of J_A is denoted by $\underset{\sim}{J}_A$ (IV 2.3) . We require two facts about $\underset{\sim}{J}_A$:

1. For $j = 1,2,\ldots,q$, A_{j-1} and A_j are n-faces of the (n+1)-simplex $\tilde{A}_j$ so that $A_{j-1} \cap A_j$ is an (n-1)-face of $\tilde{A}_j$.

There is an (n-1)-face B of A such that $\tilde{A}_j \cap \underset{\sim}{J}_B = A_{j-1} \cap A_j$. For any other (n-1)-face G of A , $\tilde{A}_j \cap \underset{\sim}{J}_G$ = an n-face of $\tilde{A}_j$ other than A_{j-1} or A_j . (See IV 2.1 .)

2. For any (n-1)-face B of A there exists p , $1 \le p \le q$, such that $\tilde{A}_p \cap \underset{\sim}{J}_B$ = the (n-1)-face $A_{p-1} \cap A_p$. (Assume that $\tilde{A}_j \cap \underset{\sim}{J}_B$ is an n-face for j = 1, 2, ..., q . Then, denoting the vertex in A - B by v , $J_v = \{v\}$ is trivial, which does not satisfy the conditions of the Proposition.)

Suppose that S has been defined on $\underset{\sim}{J}_{F1} \cup \underset{\sim}{J}_{F2} \cup \dots \cup \underset{\sim}{J}_{F(k-1)}$. We have

$$Fk \cap \bigcup_{i=1}^{k-1} Fi = \bigcup_{G \in \Phi} G ,$$

where Φ is a set of (n-1)-faces of the Fk which does not include all such faces. From IV 2.5 ,

$$\underset{\sim}{J}_{Fk} \cap \bigcup_{i=1}^{k-1} \underset{\sim}{J}_{Fi} = \bigcup_{G \in \Phi} \underset{\sim}{J}_G .$$

Let $(\tilde{Fk})_p$ be the first (n+1)-face in the stack J_{Fk} such that $(Fk)_{p-1} \cap (Fk)_p = (\tilde{Fk})_p \cap \underset{\sim}{J}_B$ where B is an (n-1)-face of Fk not contained in Φ . (Such an $(Fk)_p$ exists by Fact 2.) By Fact 1 , $(\tilde{Fk})_p \cap \bigcup_{G \in \Phi} \underset{\sim}{J}_G$ = a union of n-faces of $(Fk)_p$ not including $(\tilde{Fk})_{p-1}$, $(Fk)_p$. We can therefore take $(\tilde{Fk})_p$ to be the next (n+1)-simplex in the shelling S .

The remaining (n+1)-faces of J_{Fk} are now ordered

$$(\tilde{Fk})_{p-1}, (\tilde{Fk})_{p-2}, \dots, (\tilde{Fk})_1, (\tilde{Fk})_{p+1}, (\tilde{Fk})_{p+2}, \dots, (\tilde{Fk})_q$$

to define S on $\bigcup_{i=1}^{k} \underset{\sim}{J}_{Fi}$.

We check S at the face $(\tilde{Fk})_j$, $1 \le j < p$. It follows from IV 2.1(a) that $(\tilde{Fk})_j \cap ((\tilde{Fk})_p \cup \ldots \cup (\tilde{Fk})_{j+1})$ is the n-face $(Fk)_j$. By Fact 1 , $(\tilde{Fk})_j \cap \bigcup_{G\in\Phi} \underset{\sim}{J}_G$ is either a union of n-faces of $(\tilde{Fk})_j$ or $((Fk)_{j-1} \cap (Fk)_j) \cup$ n-faces of $(\tilde{Fk})_j$. Also, $(\tilde{Fk})_j \cap \bigcup_{G\in\Phi} \underset{\sim}{J}_G$ does not contain $(Fk)_{j-1}$ or $(Fk)_j$. Hence $(\tilde{Fk})_j \cap ((\tilde{Fk})_p \cup \ldots \cup (\tilde{Fk})_{j+1} \cup \bigcup_{G\in\Phi} \underset{\sim}{J}_G) =$ a union of n-faces of $(\tilde{Fk})_j$ not including $(Fk)_{j-1}$ and we may take $(\tilde{Fk})_j$ to be the (n+1)-simplex following $(\tilde{Fk})_{j+1}$ in the shelling S .

The reasoning in the case of $(\tilde{Fk})_j$, $p < j \le q$, is similar. □

In what follows, for any cone-cell A , we identify SdA with CSdBdA (the barycentre bA of A becomes the cone point). To avoid confusion with other cones, we write $SdA = C_A SdBdA$.

<u>A4 Proposition</u> *If* Z *is an* S-*shellable cone-complex then so is the* (CC-)*cone* CZ *on* Z .

<u>Proof</u> We have to show that if SdX is shellable, for X a cone-cell, then so is SdCX . Since X is a ball any shelling of SdX satisfies condition (ii) of II 1.1 . Thus the result follows from Proposition A3 if we show that there is a pseudocylinder structure J(SdX) on SdCX (having equipped SdX , SdCX with marked faces to obtain simplicial $\vec{SC}$-complexes).

The case X = an 0-cell is obvious.

Let $\dim X = n \ge 1$. A marked face structure can be defined on SdX as shown in II 2.1 . We identify $X \times \{0\} \subset CX$

with X and specify a pseudocylinder $J(SdX)$ with $UJ = SdCX$, $i^0(SdX) = SdX$ and $i^1(SdX)$ = a subcomplex Y of $SdCX$ which is defined by induction on the skeleta of X.

Let Y_0 be the union of the barycentres of the faces CB for B an 0-face of X. Assume Y_{k-1} $(1 \leq k \leq n-1)$ has been defined. For each k-face A of X (using $SdCA = C_{CA}SdBdA$) let $Y_k \cap SdCA = C_{CA}(Y_{k-1} \cap SdBdCA)$.

This gives Y_{n-1}.

Each vertex of Y_{n-1} is the barycentre of CA for some face A of X. Hence, for $k \geq 0$ and any k-simplex D of Y_{n-1}, there is a $(k+1)$-simplex Dv of $SdCX$ containing D and the cone-point v of CX. We take Y to be the union of the simplices Dv for $D \subset Y_{n-1}$. Thus Y is the (simplicial) cone on Y_{n-1} with cone point v.

The construction of Y through taking successive cones gives a CC-isomorphism $\nu: SdX \to Y$. We define a structure of marked faces on Y to make ν an $\overrightarrow{SC}$-isomorphism.

The stacks of $J(SdX)$ are defined by induction on the skeleta of SdX. For each face F of SdX we give the subcomplex $\underset{\sim}{J}_F$ of $SdCX$: this specifies the stack J_F in an obvious way.

If F is a vertex of SdX then F is the barycentre bA of some face A of X. We set $\underset{\sim}{J}_{bA} = C_{CA}bA$ for $A \neq X$ and $\underset{\sim}{J}_{bX} = C_X(bX \cup v)$. Assume the stacks on $(k-1)$-faces of SdX have been defined. Let F be a k-face of SdX and let A be the highest-dimensional face of X such that bA is a vertex of F. Denote the $(k-1)$-face of F not containing bA by E. If $A \neq X$ we set $\underset{\sim}{J}_F = C_{CA}(F \cup J_E)$; if $A = X$ we set $\underset{\sim}{J}_F = C_{CX}(F \cup \underset{\sim}{J}_E \cup \nu(F))$.

We specify the marked faces of the simplices in the stack $J_F = \{F_o, \tilde{F}_1, \ldots, \tilde{F}_q, F_q\}$ as follows. For $j = 1, 2, \ldots, q$, $(\tilde{F}_j)_* = F_{j-1}$ and $(F_j)_* =$ the $(k-1)$-face of F_j which belongs to J_{F*} $(\dim F = k)$.

We now have that SdCX is a simplicial $\vec{SC}$-complex and J(SdX) is a pseudocylinder with no trivial stacks. □

Before going on to the cylinder construction we prove the following result, which is used in A9 .

A5 Lemma *If* X *is an* $\vec{SC}$*-complex and* J(X) *is a pseudocylinder with no trivial stacks there exists a pseudocylinder* SJ(SdX) *with no trivial stacks such that* USJ = SdUJ , $(SdX)^o_{SJ} = SdX^o_J$ *and* $(SdX)^1_{SJ} = SdX^1_J$.

Proof We let $i^\alpha_{SJ} : SdX \to (SdX)^\alpha_{SJ}$ be the $\vec{SC}$-isomorphism induced by $i^\alpha_J : X \to X^\alpha_J$ $(\alpha = 0,1)$.

Induction on skeleta of SdX is used to define the stacks of SJ(SdX) . For each face F of SdX we give the subcomplex $\underset{\sim}{SJ}_F$ of USJ = SdUJ ; this specifies the stack SJ_F .

For A a face of X , let the stack J_A be $\{i^o_J(A) = A_o, \tilde{A}_1, A_1, \ldots, \tilde{A}_q, A_q = i^1_J(A)\}$. Recall (IV 2.1) that A_{j-1} and A_j are faces of $\tilde{A}_i$ and there is an SPoly-isomorphism $\nu_j : A \to A_j$ for $j = 0, 1, \ldots, q$. Let $s\nu_j : SdA \to SdA_j$ be the $\vec{SC}$-isomorphism induced by ν_j .

If F is a vertex of SdX then F is the barycentre bA of some face A of X . We set

$$\underset{\sim}{SJ}_F = \bigcup_{1 \le j \le q} C_{\tilde{A}_j} (s\nu_{j-1}(F) \cup s\nu_j(F)) .$$

(setting $Sd\tilde{A}_j = C_{\tilde{A}_j} SdBd\tilde{A}_j$) . Assume the stacks on $(k-1)$-faces of SdX have been defined and consider a k-face F of SdX . Take A to be the highest-dimensional face of X such that bA is a vertex of F and denote the (k-1)-face of F not containing bA by E . We set

$$\underset{\sim}{SJ}_F = \bigcup_{1\le j\le q} C_{\tilde{A}_j}(s\nu_{j-1}(F) \cup (\underset{\sim}{SJ}_E \cap Sd\tilde{A}_j) \cup s\nu_j(F)) .$$

Routine checking shows that we obtain a pseudocylinder SJ(SdX) as required. □

<u>A6 Proposition</u> *If* Z *is an* S-*shellable cone-complex then so is the cylinder* $Z \times I$

<u>Proof</u> We have to show that if SdX is shellable, for X a cone-cell, then $Sd(X \times I)$ is shellable.

Marked face structures can be defined on SdX , $Sd(X \times I)$ (see II 2.1) to give simplicial $\vec{SC}$-complexes. Although we cannot define a pseudocylinder structure on $X \times I$ without S-shellability we can define a pseudocylinder SJ(SdX) with $UJ = Sd(X \times I)$, $(SdX)^o_{SJ} = Sd(X \times \{0\})$, $(SdX)^1_{SJ} = Sd(X \times \{1\})$ in a way precisely analogous to the proof of A5 . Since SJ(SdX) has no trivial stacks and any shelling of the ball SdX satisfies condition (ii) of II 1.1 , $Sd(X \times I)$ is shellable by A3 . □

In order to show that the construction VZ preserves S-shellability we need the result below. The notion of a *(general) subdivision* of a cone-complex is used in the proof. Let U and V be cone complexes. We say V is a *subdivision* of U if the underlying spaces of V and U are identical and each open cell of V is contained in an open cell of U . The notion of barycentric subdivision SdU is a special case of general subdivision.

<u>A7 Proposition</u> *Let* (Y,Z) *be a* CC-*pair and let* sY *be the complex obtained by replacing* Z *by* SdZ . *If* Y *is* S-*shellable then so is* sY .

<u>Proof</u> Consider an n-cell X of Y $(n \geq 1)$ with $X \cap Z \neq \emptyset$.

If $X \subset Z$ then X is replaced by SdX in sY . Since the faces of SdX are simplices SdX is an S-shellable subcomplex of sY .

If $X \not\subset Z$ then $X \cap Z$ is a subcomplex of BdX and X is replaced by the n-cell sX in sY . We have to show that Sd(sX) is shellable.

Let α be an n-simplex of SdX such that $\alpha \cap SdZ \neq \emptyset$. Denote the subcomplex $\alpha \cap SdZ$ of α by B and let $q = \dim B$. Since $X \not\subset Z$ we have $0 \leq q \leq n - 1$. Now α may be characterized as $(bA_o, bA_1, \ldots, bA_n)$ where bA_i is the barycentre of the i-face A_i of X and $A_o \subset A_1 \subset \ldots \subset A_n = X$. If B contains any vertex bA_r of α such that $r > q$ then $A_r \subset Z$ so that the r-face $(bA_o, bA_1, \ldots, bA_r) \subset B$, which is a contradiction. On the other hand, B must contain at least one q-face of α . Hence $B = (bA_o, bA_1, \ldots, bA_q)$.

The face B of sX is replaced by SdB in Sd(xX) . There is a subcomplex A of Sd(sX) such that A is a (general) subdivision of α and there is a CC-isomorphism $A \to \underbrace{CC\ldots C}_{n-q}\, SdB$ (= C^{n-q} SdB) which maps bA_k onto the cone point of $C(C^{k-q-1}SdB)$. We identify A with $C^{n-q}SdB$.

We can define a shelling of Sd(sX) which models a shelling of SdX (replacing each n-simplex α such that $\alpha \cap SdZ \neq \emptyset$ with a sequence of n-simplices of $C^{n-q}SdB$) if the following holds:

Claim If $W = \bigcup_{V \in \mathcal{W}} V$, where $\mathcal{W}$ is a proper subset of the set of $(n-1)$-faces of $\alpha = (bA_o, bA_1, \ldots, bA_n)$, there is a linear ordering $F_1, F_2, \ldots, F_t$ of the n-faces of $C^{n-q}SdB$ which satisfies $(*)_{\mathcal{W}}$: for $1 \leq i \leq t$,
$F_i \cap (W \cup \bigcup_{j=1}^{i-1} F_j)$ is a non-empty union of $(n-1)$-faces of F_i which does not include every such face.

The claim is proved by induction. SdB is shellable by A1 and the intersection of each q-face of SdB with BdB is a $(q-1)$-face. Assume that, for $q < k \leq n$ and each proper subset $\mathcal{W}_{k-1}$ of the set of $(k-2)$-faces of $(bA_o, bA_1, \ldots, bA_{k-1})$, there is a shelling of $C^{k-q-1}SdB$ which satisfies $(*)_{\mathcal{W}_{k-1}}$. Consider a proper subset $\mathcal{W}_k$ of the set of $(k-1)$-faces of $(bA_o, bA_1, \ldots, bA_k)$. We define a shelling $F_1, F_2, \ldots, F_t$ of $C^{k-q}SdB$ which satisfies $(*)_{\mathcal{W}_k}$. There are three cases:

1. $\mathcal{W}_k$ is the set of all $(k-1)$-faces of $(bA_o, bA_1, \ldots, bA_k)$ other than $(bA_o, \ldots, bA_{k-1})$.

It is easily shown that $F_i \cap W_k$ is a union of $(k-1)$-faces of F_i $(1 \leq i \leq t)$. Thus we can take any shelling $E_1, E_2, \ldots, E_t$ of $C^{k-q-1}SdB$ and set $F_i = CE_i$.

2. $\mathcal{W}_k$ is a proper subset of the set of $(k-1)$-faces of $(bA_o, \ldots, bA_k)$ other than $(bA_o, \ldots, bA_{k-1})$.

Here there exists a proper subset $\mathcal{W}_{k-1}$ of the set of $(k-2)$-faces of $(bA_o, \ldots, bA_{k-1})$ such that $W_k = CW_{k-1}$. Let $E_1, E_2, \ldots, E_t$ be a shelling of C^{k-q-1} satisfying $(*)_{\mathcal{W}_{k-1}}$ and take $F_i = CE_i$ for $i = 1, 2, \ldots, t$.

3. $(bA_o, \ldots, bA_{k-1}) \in \mathcal{W}_k$.

For any k-face F of $C^{k-q}SdB$, $F \cap (bA_o, \ldots, bA_{k-1})$ is a (k-1)-face of F . We let $F_i = CE_i$ $(1 \le i \le t)$, where $E_1, E_2, \ldots, E_t$ is a shelling $C^{k-q-1}SdB$. If $\mathcal{W}_k = \{(bA_o, \ldots, bA_{k-1})\}$ any shelling of C^{k-q-1} may be used. Otherwise, $E_1, E_2, \ldots, E_t$ is obtained as in 2 above. □

From Propositions A6 and A7 there follows immediately:

<u>A8 Proposition</u> *If* Z *is an* S-*shellable marked cone-complex then* VZ *is also* S-*shellable.* □

Finally, we have (see IV 3.4 , 3.5) :

<u>A9 Proposition</u> *For* Z *an* $\overrightarrow{SC}$-*complex, let* J(Z) *be a pseudocylinder. A rectifier* RJ *on* J(Z) *is* S-*shellable.*

<u>Proof</u> The notation of IV §§2, 3 is used. In view of IV 3.5 it is sufficient to show that if X is an S-polycell and J(X) is a pseudocylinder then $SdRJ = SdRJ_X$ is shellable.

Since the pseudocylinder EJ(X) has no trivial stacks there exists, by A5 , a pseudocylinder structure SEJ(SdX) on Sd UEJ with $(SdX)^o_{SEJ} = SdX^o_{EJ} = SdX^o_J$ and $(SdX)^1_{SEJ} = SdX^1_{EJ} = Sd(X \times \{0\})$.

By IV 3.5 (iii) , $rJ = rJ_X$ is $\overrightarrow{SC}$isomorphic to $X \times I$. Thus (IV 2.2 (iii)) there is a canonical pseudocylinder structure $\Pi(X)$ with $U\Pi = rJ$, $X^o_\Pi = X^o_J$ and $X^1_\Pi = X \times \{0\}$. Since $\Pi(X)$ has no trivial stacks, there is also a pseudocylinder $S\Pi(SdX)$ with $US\Pi = SdrJ$, $(SdX)^o_{S\Pi} = SdX^o_J$ and $(SdX)^1_{S\Pi} = Sd(X \times \{0\})$.

For each face F of SdX we define a subcomplex Q_F of $SdRJ$. (As usual, we identify SdW with $C_W SdBdW$ for W a face of RJ.) If F is a vertex of SdX then F is the barycentre bA of a face A of X. Set $Q_F = C_{RJ_A}(\underline{\underline{SEJ}}_F \cup \underline{\underline{S\Pi}}_F)$. Assume that Q_G has been defined for each face G of SdX with $\dim G \leq k - 1$. Let F be a k-simplex of SdX. Take A to be the highest-dimensional face of X such that bA is a vertex and let D by the $(k-1)$-face of F not containing bA. Set
$Q_F = C_{RJ_A}(\underline{\underline{SEJ}}_F \cup Q_D \cup \underline{\underline{S\Pi}}_F)$.

For each face F of SdX a shelling S_F of Q_F is defined by induction on $\dim F$. Since Q_F is a cone on a complex Y we can specify S_F by giving a shelling of Y. In the case F = a vertex, $Y = \underline{\underline{SEJ}}_F \cup \underline{\underline{S\Pi}}_F$ is shelled as follows. Let $SEJ_F = \{F_o, \tilde{F}_1, \ldots, \tilde{F}_q, F_q\}$, $S\Pi_F = \{F'_o, \tilde{F}'_1, F'_1, \ldots, \tilde{F}'_r, F'_r\}$ and proceed: $\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_q, \tilde{F}'_1, \ldots, \tilde{F}'_r$. Assume S_G has been defined for $\dim G = k - 1$ and consider the k-face F. Here the shelling of $Y = \underline{\underline{SEJ}}_F \cup Q_D \cup \underline{\underline{S\Pi}}_F$ proceeds: $\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_q$; the shelling S_D of Q_D; $\tilde{F}'_1, \tilde{F}'_2, \ldots, \tilde{F}'_r$.

Let $\dim X = n$ so that $\dim RJ = n + 2$. We have $SdRJ = \bigcup_{F \subset SdX} Q_F$. Thus a linear order S on the set of $(n+2)$-faces of $SdRJ$ is defined by following a shelling $F1, F2$ $F1, F2, \ldots, Ft$ of SdX, replacing Fi $(i = 1, 2, \ldots, t)$ by the sequence S_{Fi} of $(n+2)$-faces of Q_{Fi}.

To see that S is a shelling, consider Q_{Fi}. Since Fi is an n-face of SdX, the barycentre bX is a vertex of Fi. Denote the $(n-1)$-face of Fi not containing bX by D.

Then $D = Fi \cap BdSdX$ and we have $Fi \cap \bigcup_{j=1}^{i-1} Fj = \bigcup_{G \in \Phi} G$, where Φ is a set of (n-1)-faces of Fi which does not include D. It can be shown that if U, V, W are simplices of SdX such that $U \cap V = W$ then $Q_U \cap Q_V = Q_W$. Hence $Q_{Fi} \cap \bigcup_{j=1}^{i-1} Q_{Fj} = \bigcup_{G \in \Phi} Q_G$.

Let the shelling S_{Fi} be $Z_1, Z_2, \ldots, Z_m$. We find (by an argument using induction on dim Fi) that if $G \neq D$ is an (n-1)-face of Fi then $Z_k \cap Q_G \subset \bigcup_{j=1}^{k-1} Z_j$ or $Z_k \cap Q_G$ is an (n+1)-face of Z_k $(1 < k \leq m)$. Hence

$$Z_k \cap \left(\bigcup_{j=1}^{k-1} Z_j \cup \bigcup_{j=1}^{i-1} Q_{Fj}\right) = \left(Z_k \cap \bigcup_{j=1}^{k-1} Z_j\right) \cup \left(Z_k \cap \bigcup_{G \in \Phi} Q_G\right)$$

is a union of (n+1)-faces of Z_k and S is a shelling of SdRJ . □

GLOSSARY OF SYMBOLS

Standard notation used without comment

B^n	standard n-cell
S^n	standard n-sphere
I	unit interval
Bd) ∂)	boundary of a cell or manifold
Int	interior " " " " "
$\bar{e}$	closed cell
cl	closure

Categories

Set	sets	I/2
Top	topological spaces	I/2
Reg	regular complexes	I/4
CC	cone-complexes	I/6
$\overrightarrow{CC}$	marked cone-complexes	I/11
Poly	polycells	I/12
$\overrightarrow{SC}$	S-shellable marked cone-complexes	II/6
SPoly	S-shellable polycells	II/6
P	skeleton of SPoly	II/11
SPos	S-posets	II/14
P'	skeleton of SPos	II/23
M	model category	I/2
Δ	simplicial model category	I/2
Δ_I	wide subcategory of Δ; Poly-simplices	I/24
$\Box$	cubical model category	I/25
$\Box_I$	wide subcategory of $\Box$; Poly-cubes	I/25
G	globes	I/24
SC_i	i = 0, 1, ...	II/8
Conv	convex cone-cells	II/11

Standard polycells

Constructions of cone-complexes

Collapses

Pseudocylinders

Miscellaneous

h_λ	characteristic map for cell	I/5
$\hat{a}$) bA)	barycentre of cell	I/7
A_*	marked face	I/11
$\zeta(X)$	ordering of faces of a polycell	I/19
$\zeta_o(X)$	vertex-ordering of polycell	I/20
$\zeta_s(X)$	ordering of cells of Int SdX	III/13
$\zeta_s(BdX)$	ordering of cells of SdBdX	III/13
∂_A	face map in M-sets	I/23
ε_J	degeneracy map in an MT-complex	IV/18
d_i, s_j	simplicial face, degeneracy maps	III/5
$\partial_i^\alpha, \varepsilon_j$	cubical face, degeneracy maps	IV/2
[a,b]	interval in poset	II/13
$\rho(a)$	rank of element of graded poset	II/12
$\Delta(Q)$	order complex of a poset	II/12
Q_*	maximal subtree of poset Q	II/14

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