What is and what should be 'Higher Dimensional Group Theory'? Liverpool

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What should be higher dimensional group theory?

Optimistic answer:
Real analysis $\subseteq$ many variable analysis
Group theory $\subseteq$ higher dimensional group theory
What is 1-dimensional about group theory?
We all use formulae on a line (more or less):

$$w = ab^2a^{-1}b^3a^{-17}c^5$$

subject to the relations $ab^2c = 1$, say.
Can we have 2-dimensional formulae?
What might be the logic of 2-dimensional (or 17-dimensional) formulae?

The idea is that we may need to get away from 'linear' thinking in order to express intuitions clearly.
Thus the equation

$$2 \times (5 + 3) = 2 \times 5 + 2 \times 3$$

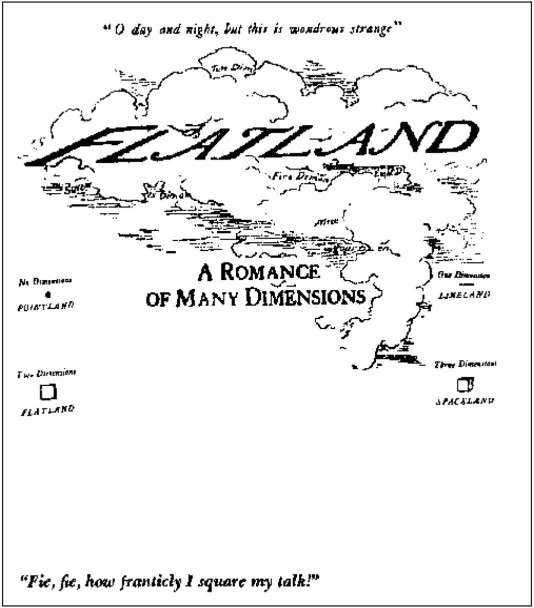
is more clearly shown by the figure

$$| \, | \, | \, | \, | \qquad | \, | \, |$$

$$| \, | \, | \, | \, | \qquad | \, | \, |$$

But we seem to need a linear formula to express the general law

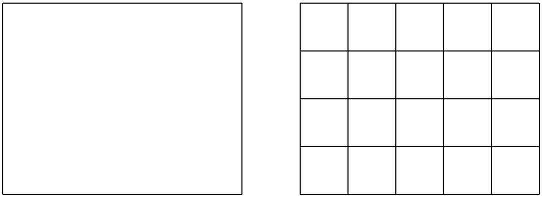
$$a \times (b + c) = a \times b + a \times c.$$



Published in 1884, available on the internet.

The linelanders had limited interaction capabilities!

Consider the figures:



From left to right gives subdivision.
From right to left should give composition.
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?

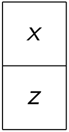
Look towards
higher dimensional,
noncommutative methods
for local-to-global problems
and contributing to the unification of mathematics.

Higher dimensional group theory cannot exist (it seems)!

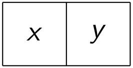
First try: A 2-dimensional group should be a set G with two group operations $\circ_1, \circ_2$ each of which is a morphism

$$G \times G \to G$$

for the other.
Write the two group operations as:



$x \circ_1 z$



$x \circ_2 y$

That each is a morphism for the other gives the interchange law:

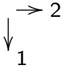
$$(x \circ_2 y) \circ_1 (z \circ_2 w) = (x \circ_1 z) \circ_2 (y \circ_1 w).$$

This can be written in two dimensions as

x	y
z	w

can be interpreted in only one way, and so may be written:

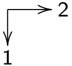
$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$



This is another indication that a '2-dimensional formula' can be more comprehensible than a 1-dimensional formula!

Theorem Let X be a set with two binary operations $\circ_1, \circ_2$, each with identities e_1, e_2, and satisfying the interchange law. Then the two binary operations coincide, and are commutative and associative.

Proof



$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix} \qquad e_1 = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix} = e_2.$$

We write then e for e_1 and e_2.

$$\begin{bmatrix} x & e \\ e & w \end{bmatrix} \qquad x \circ_1 w = x \circ_2 w.$$

So we write $\circ$ for each of $\circ_1, \circ_2$.

$$\begin{bmatrix} e & y \\ z & e \end{bmatrix} \qquad y \circ z = z \circ y.$$

We leave the proof of associativity to you. This completes the proof.

Dreams shattered!

Back to basics!

How does group theory work in mathematics?

Symmetry

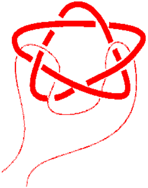
An abstract algebraic structure, e.g. in number theory, geometry.

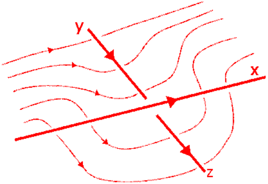
Paths in a space: fundamental group

Algebra structuring space

F.W. Lawvere: The notion of space is associated with representing motion.
How can algebra structure space?
[The following graphics were accompanied by the tying of string on a copper pentoil knot. Then a member of the audience was invited to help take the loop off the knot!]

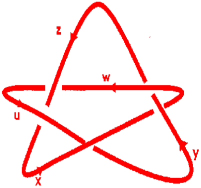
Moving in the space around a knot



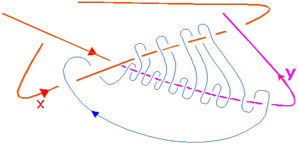


$y = x z x^{-1}$

Relation at a crossing



$x\, y\, x\, y\, x\, y^{-1}\, x^{-1}\, y^{-1}\, x^{-1}\, y^{-1} = 1$



Local and global issue.
Use rewriting of relations.
Classify the ways of pulling the loop off the knot!

Groupoids to the rescue

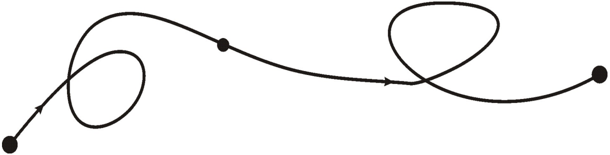
Groupoid: underlying geometric structure is a graph

$$G_0 \xrightarrow{i} G \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

such that $si = ti = 1$. Write $a : sa \to ta$.
Multiplication $(a, b) \mapsto ab$ defined if and only if $ta = sb$;
so it is a partial multiplication, assumed associative.
ix is an identity for the multiplication: $(isa)a = a = a(ita)$
So G is a small category, and we assume all $a \in G$ are invertible.

$$(\text{groups}) \subseteq (\text{groupoids})$$

The notion of groupoid first arose in number theory, generalising work of Gauss from binary to quaternary quadratic forms.
Groupoids clearly arise in the notion of composition of paths, giving a geography to the intermediate steps.



The objects of a groupoid add a spatial component to group theory.
Groupoids have a partial multiplication, and this opens the door into the world of partial algebraic structures.
Higher dimensional algebra: algebra structures with partial operations defined under geometric conditions.
Allows new combinations of algebra and geometry, new kinds of mathematical structures, and new ways of describing their inter-relations.

Theorem Let G be a set with two groupoid compositions satisfying the interchange law (a double groupoid). Then G contains a family of abelian groups.

Double groupoids are more nonabelian than groups.

n-fold groupoids are even more nonabelian!

Masses of algebraic and geometric examples, linking with classical themes, particularly crossed modules. Rich algebraic structures!

Are there applications in geometry? in physics? in neuroscience?

Credo:

Any simply defined and intuitive mathematical structure is bound to have useful applications, eventually!

Search on the internet for "higher dimensional algebra".

344,000 hits recently

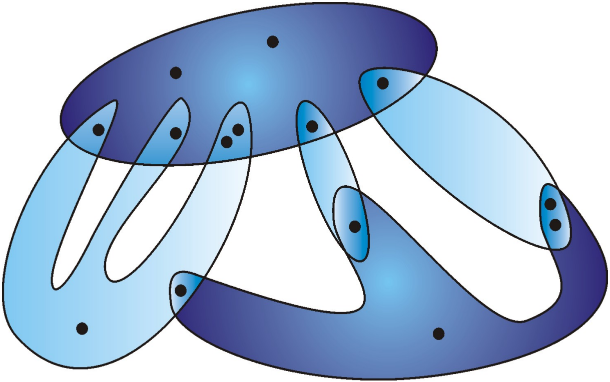
How did I get into this area?
Fundamental group $\pi_1(X, a)$ of a space with base point.
van Kampen Theorem: Calculate the fundamental group of a union.

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \longrightarrow & \pi_1(V, x) \\ \downarrow & & \downarrow \\ \pi_1(U, x) & \longrightarrow & \pi_1(U \cup V, x) \end{array} \qquad \text{pushout}$$

OK if U,V are open and $U \cap V$ is path connected.
This does not calculate the fundamental group of a circle S^1.
If $U \cap V$ is not connected, where to choose the basepoint?

Fundamental group $\pi_1(X, A)$ on a set A of base points.
Alexander Grothendieck

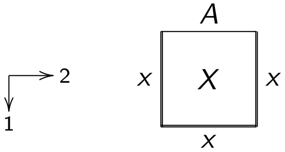
......people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care - or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids.



For all of 1-dimensional homotopy theory, the use of groupoids gives more powerful theorems with simpler proofs.

Groupoids in higher homotopy theory?

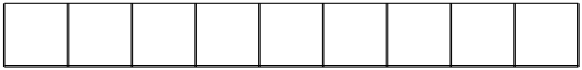
Consider second relative homotopy groups $\pi_2(X, A, x)$:



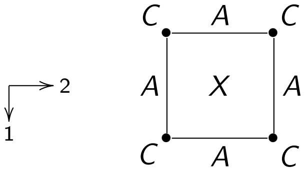
where thick lines show constant paths.

Definition involves choices, and is unsymmetrical w.r.t. directions. Unaesthetic!

All compositions are on a line:



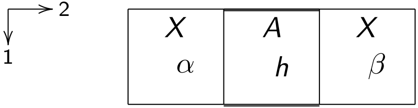
Brown-Higgins 1974 $\rho_2(X, A, C)$: homotopy classes rel vertices of maps $[0,1]^2 \to X$ with edges to A and vertices to C



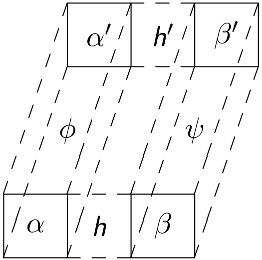
$$\rho_2(X, A, C) \Rrightarrow \pi_1(A, C) \rightrightarrows C$$

Childish idea: glue two square if the right side of one is the same as the left side of the other. Geometric condition

There is a horizontal composition $\langle\langle\alpha\rangle\rangle +_2 \langle\langle\beta\rangle\rangle$ of classes in $\rho_2(X, A, C)$, where thick lines show constant paths.

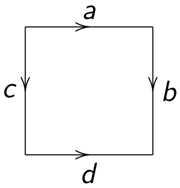


To show $+_2$ well defined, let $\phi : \alpha \equiv \alpha'$ and $\psi : \beta \equiv \beta'$, and let $\alpha' +_2 h' +_2 \beta'$ be defined. We get a picture in which dash-lines denote constant paths. Can you see why the 'hole' can be filled appropriately?



Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

In dimension 1, we still need the 2-dimensional notion of commutative square:



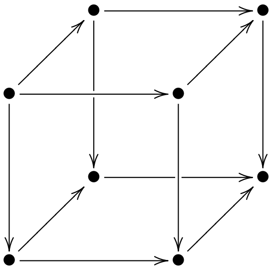
$ab = cd \quad a = cdb^{-1}$

Easy result: any composition of commutative squares is commutative.
In ordinary equations:

$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

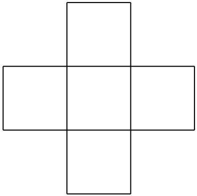
The commutative squares in a category form a double category!
Compare Stokes' theorem! Local Stokes implies global Stokes.

What is a commutative cube?

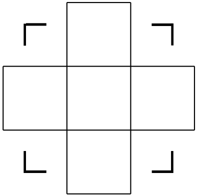


We want the faces to commute!

we might say the top face is the composite of the other faces: so fold them flat to give:



which makes no sense! Need fillers:



To resolve this, we need some special squares called thin:
First the easy ones:

$$\begin{pmatrix} 1 & \begin{matrix} 1 \\ 1 \end{matrix} & 1 \end{pmatrix} \qquad \begin{pmatrix} a & \begin{matrix} 1 \\ 1 \end{matrix} & a \end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix} b \\ b \end{matrix} & 1 \end{pmatrix}$$

$$\square \qquad \overline{\underline{}} \text{ or } \varepsilon_2 a \qquad | \, | \text{ or } \varepsilon_1 b$$

$$\text{laws} \qquad \begin{bmatrix} a & \overline{\underline{}} \end{bmatrix} = a \qquad \begin{bmatrix} b \\ | \, | \end{bmatrix} = b$$

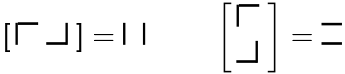
Then we need some new ones:

$$\begin{pmatrix} a & \begin{matrix} a \\ 1 \end{matrix} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & \begin{matrix} 1 \\ a \end{matrix} & a \end{pmatrix}$$

These are the connections

$$\lrcorner \qquad \ulcorner$$

What are the laws on connections?



(cancellation)

(transport)

These are equations on turning left or right, and so are a part of 2-dimensional algebra.
The term transport law and the term connections came from laws on path connections in differential geometry.
It is a good exercise to prove that any composition of commutative cubes is commutative.

Rotations in a double groupoid with connections

To show some 2-dimensional rewriting, we consider the notion of rotations σ, τ of an element u in a double groupoid with connections:

$$\sigma(u) = \begin{bmatrix} \mid\mid & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \mid\mid \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \mid\mid \\ \ulcorner & u & \lrcorner \\ \mid\mid & \llcorner & = \end{bmatrix}.$$

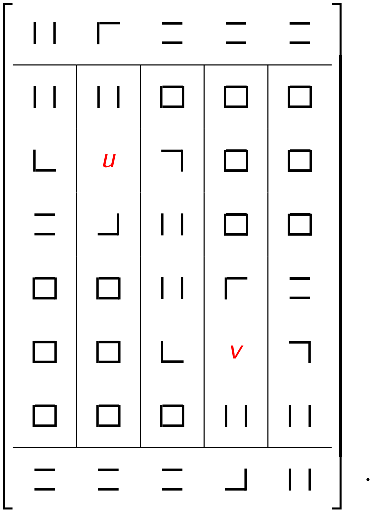
For any $u, v, w \in G_2$,

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u]$$

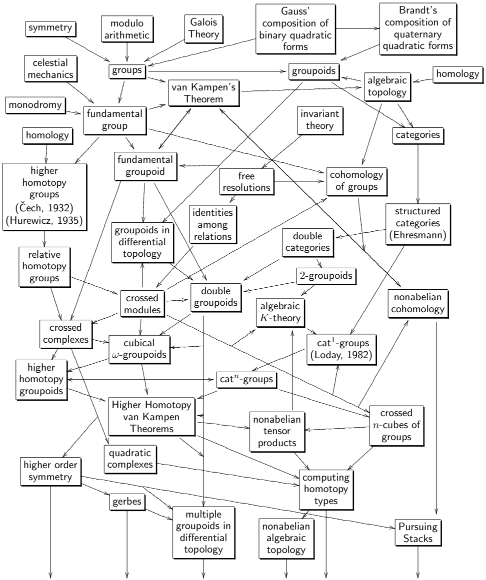
$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.
Further $\sigma^2\alpha = -_1 -_2 \alpha$, and $\tau\sigma = 1$.

To prove the first of these one has to rewrite $\sigma(u +_2 v)$ until one ends up with an array, shown on the next slide, which can be reduced in a different way to $\sigma u +_2 \sigma v$. Can you identify σu, σv in this array? This gives some of the flavour of this 2-dimensional algebra of double groupoids.
When interpreted in $\rho(X, A, C)$ this algebra implies the existence, even construction, of certain homotopies which may be difficult to do otherwise.


u
v

Some Historical Context for Higher Dimensional Group Theory



The end

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