What is and what should be 'Higher Dimensional Group Theory'? Liverpool

Ronnie Brown

December 4, 2009

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Optimistic answer:

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What might be the logic of 2-dimensional (or 17-dimensional) formulae?

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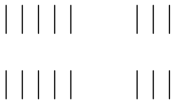

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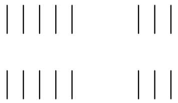

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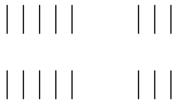

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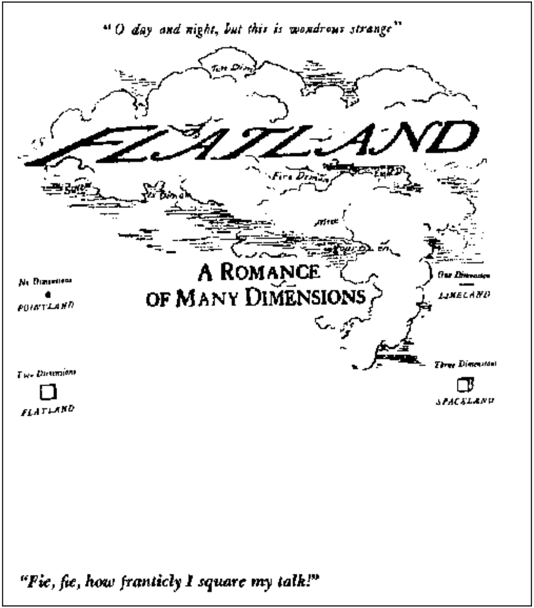

Published in 1884, available on the internet.

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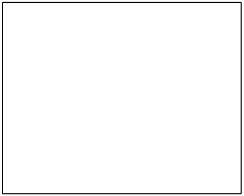

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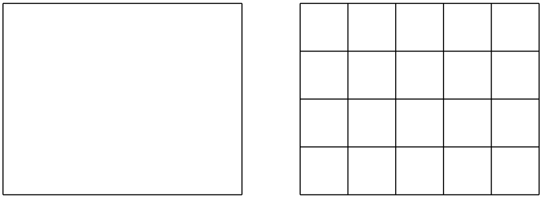

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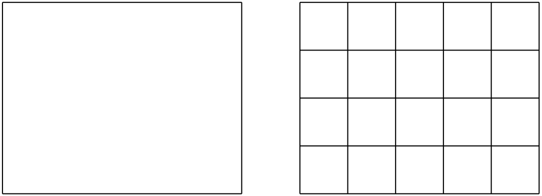

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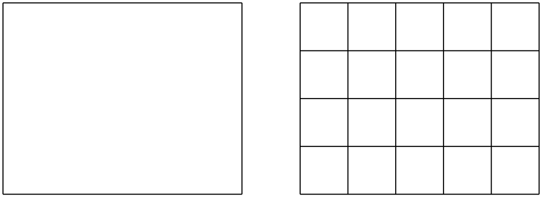

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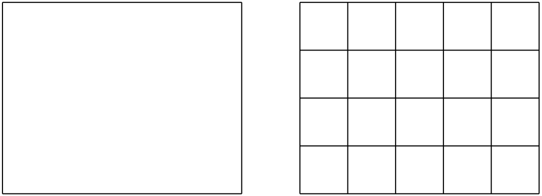

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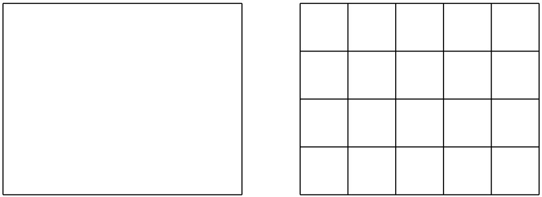

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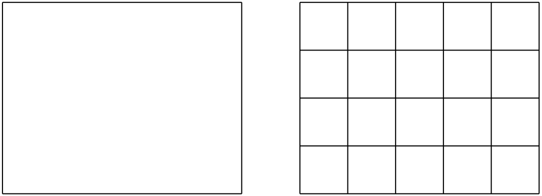

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What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?

Look towards

Look towards
higher dimensional,

Look towards
higher dimensional,
noncommutative methods

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and contributing to the unification of mathematics.

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First try: A 2-dimensional group should be a set G with two group operations $\circ_1, \circ_2$ each of which is a morphism

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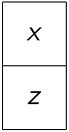
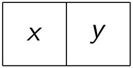
Write the two group operations as:



$x \circ_1 z$



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This is another indication that a '2-dimensional formula' can be more comprehensible than a 1-dimensional formula!

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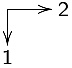

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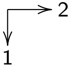

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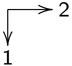

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We write then e for e_1 and e_2.

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$$\begin{bmatrix} e & y \\ z & e \end{bmatrix} \qquad y \circ z = z \circ y.$$

We leave the proof of associativity to you. This completes the proof.

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Back to basics!

How does group theory work in mathematics?

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An abstract algebraic structure, e.g. in number theory, geometry.

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Paths in a space: fundamental group

Algebra structuring space

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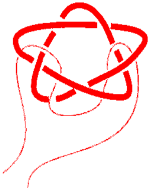
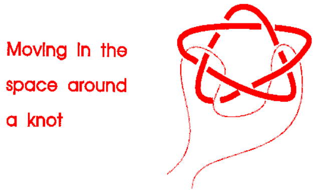
How can algebra structure space?
[The following graphics were accompanied by the tying of string on a copper pentoil knot. Then a member of the audience was invited to help take the loop off the knot!]

Moving in the space around a knot

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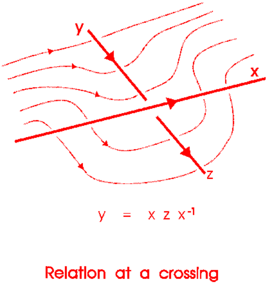
y
x
z
y = x z x-1
Relation at a crossing

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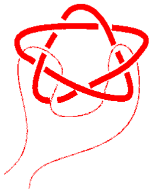

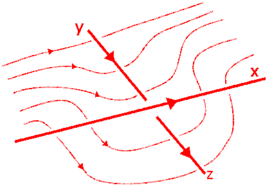

$y = x z x^{-1}$

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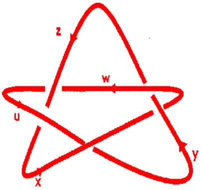
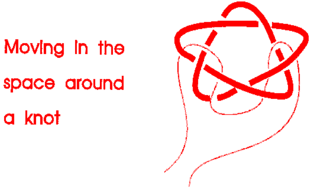
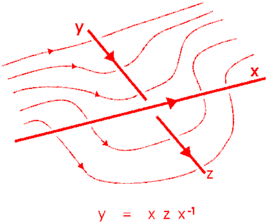


$x y x y x y^{-1} x^{-1} y^{-1} x^{-1} y^{-1} = 1$





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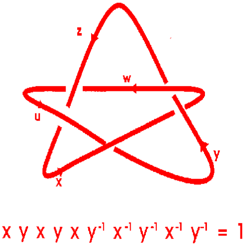

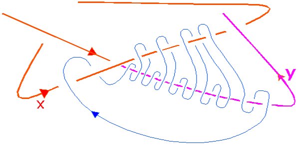

Local and global issue.

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Classify the ways of pulling the loop off the knot!

Groupoids to the rescue

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$$\text{(groups)} \subseteq \text{(groupoids)}$$

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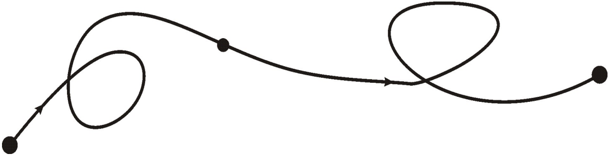

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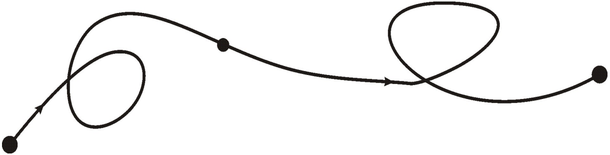

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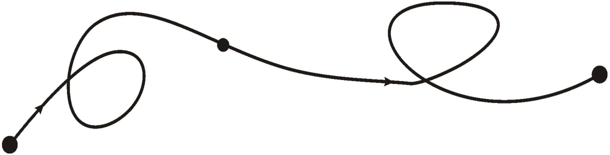

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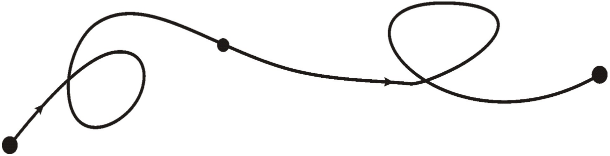

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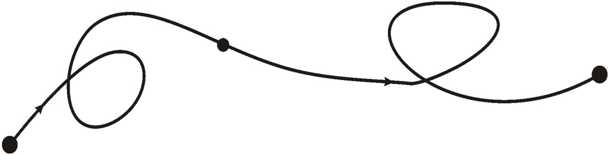

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Higher dimensional algebra: algebra structures with partial operations defined under geometric conditions.

Allows new combinations of algebra and geometry, new kinds of mathematical structures, and new ways of describing their inter-relations.

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If $U \cap V$ is not connected, where to choose the basepoint?

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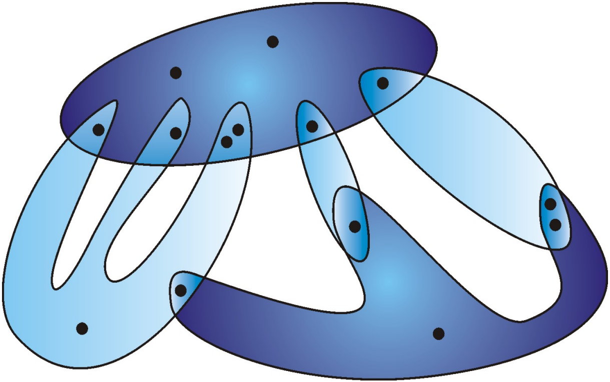
Fundamental group $\pi_1(X, A)$ on a set A of base points. Alexander Grothendieck

......people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care - or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids.



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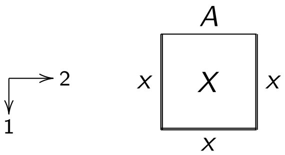

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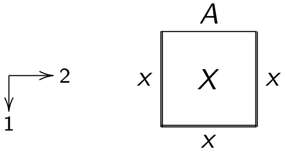

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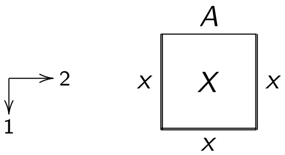

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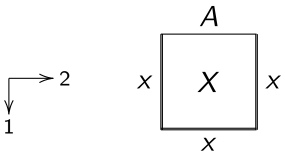

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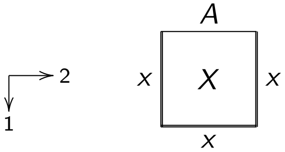

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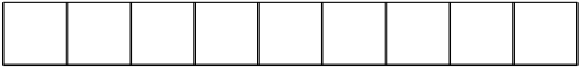

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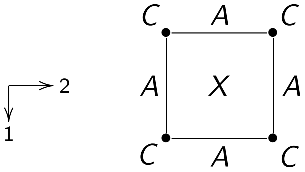

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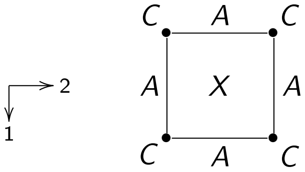

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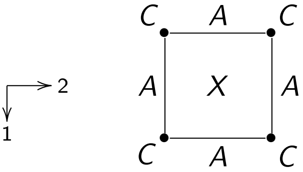

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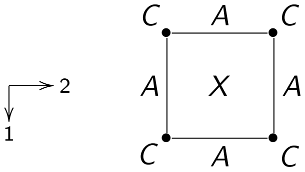

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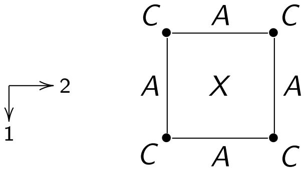

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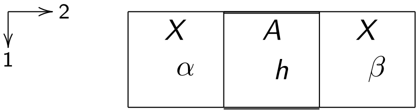

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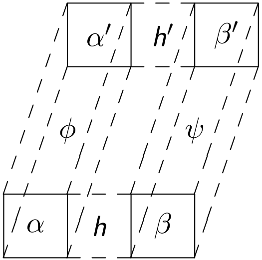

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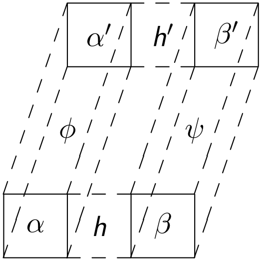

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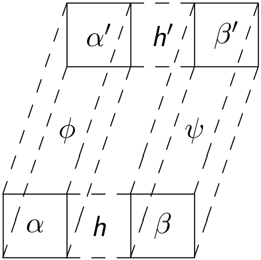

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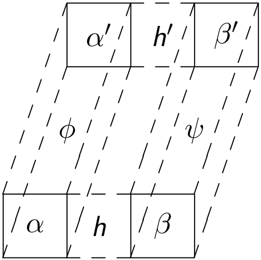

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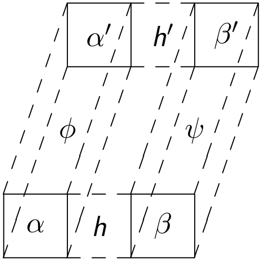

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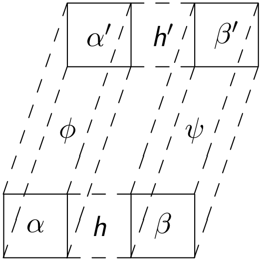

Thus $\rho(X, A, C)$ has in dimension 2 compositions in directions 1,2 satisfying the interchange law and is a double groupoid, containing as a substructure $\pi_2(X, A, x), x \in C$ and $\pi_1(A, C)$.

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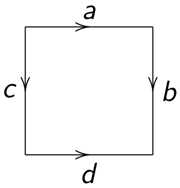

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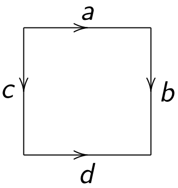

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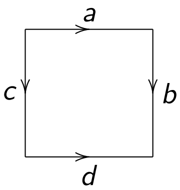

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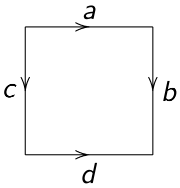

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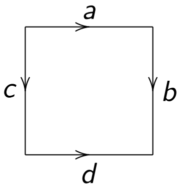

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The commutative squares in a category form a double category! Compare Stokes' theorem! Local Stokes implies global Stokes.

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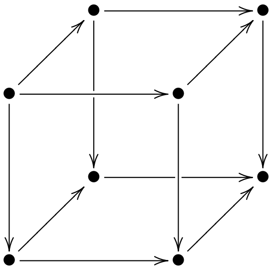

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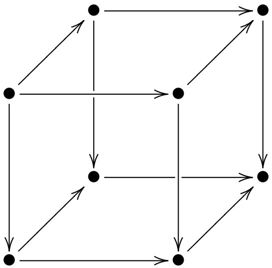

We want the faces to commute!

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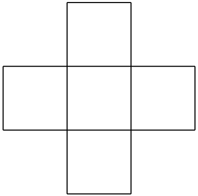

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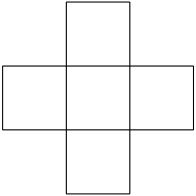

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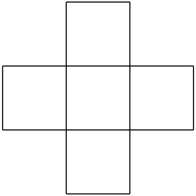

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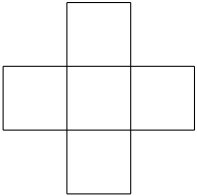

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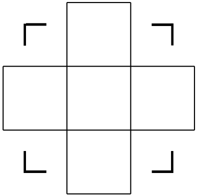

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These are the connections

$$\lrcorner \qquad \ulcorner$$

What are the laws on connections?

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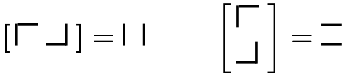
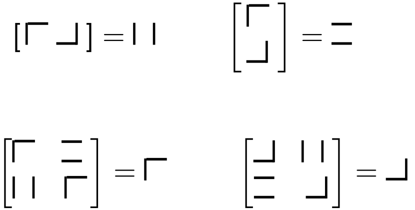


(cancellation)

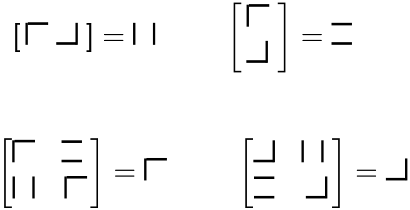
What are the laws on connections?



(cancellation)

(transport)

What are the laws on connections?



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These are equations on turning left or right, and so

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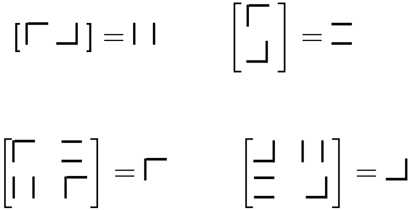

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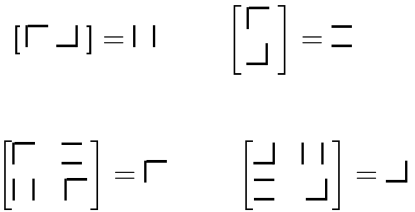

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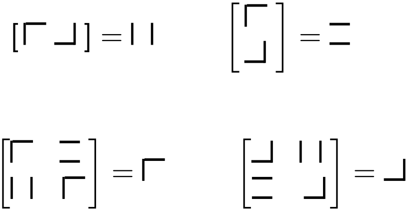

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The term transport law and the term connections came from laws on path connections in differential geometry.

It is a good exercise to prove that any composition of commutative cubes is commutative.

Rotations in a double groupoid with connections

To show some 2-dimensional rewriting, we consider the notion of rotations σ, τ of an element u in a double groupoid with connections:

$$\sigma(u) = \begin{bmatrix} \mid\mid & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \mid\mid \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \mid\mid \\ \ulcorner & u & \lrcorner \\ \mid\mid & \llcorner & = \end{bmatrix}.$$

For any $u, v, w \in G_2$,

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\sigma w, \sigma u]$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau\left(\begin{bmatrix} u \\ w \end{bmatrix}\right) = [\tau u, \tau w]$$

whenever the compositions are defined.
Further $\sigma^2\alpha = -_1 -_2 \alpha$, and $\tau\sigma = 1$.

To prove the first of these one has to rewrite $\sigma(u +_2 v)$ until one ends up with an array, shown on the next slide, which can be reduced in a different way to $\sigma u +_2 \sigma v$. Can you identify σu, σv in this array? This gives some of the flavour of this 2-dimensional algebra of double groupoids.

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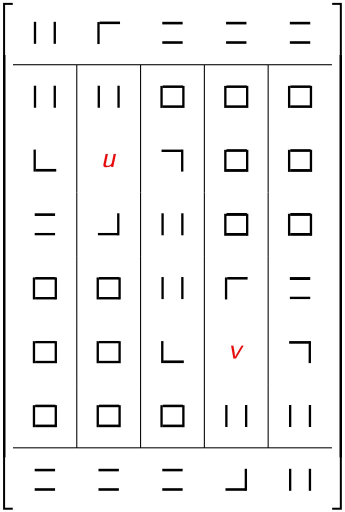
When interpreted in $\rho(X, A, C)$ this algebra implies the existence, even construction, of certain homotopies which may be difficult to do otherwise.


u
v

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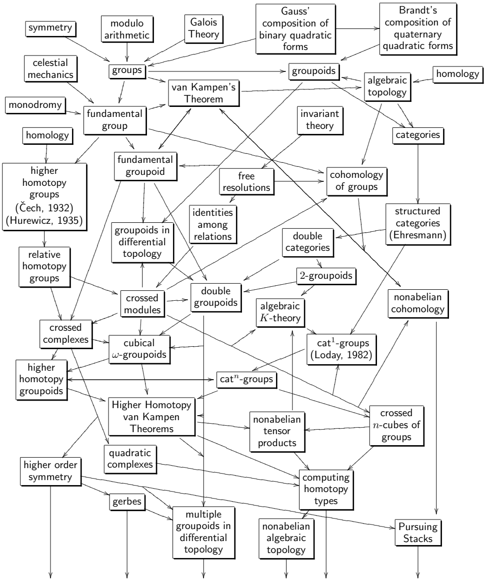
Some Historical Context for Higher Dimensional Group Theory



The end

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