Non-abelian cohomology and the homotopy classification of maps

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To a filtered space

$$\underline{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

we can associate the *homotopy crossed complex* $\pi\underline{X}$, which consists for $n = 1$ of the fundamental groupoid $\pi_1\underline{X} = \pi_1(X_1, X_0)$, and for $n \geq 2$ of the family $\pi_n\underline{X}$ of relative homotopy groups $\pi_n(X_n, X_{n-1}, v)$, $v \in X_0$, with the usual boundaries $\delta : \pi_n\underline{X} \to \pi_{n-1}\underline{X}$ and action of $\pi_1\underline{X}$ on $\pi_n\underline{X}$. The formal properties satisfied by $\pi\underline{X}$ define the notion of *crossed complex*, and we have a category XC of crossed complexes. Note that crossed complexes generalise chain complexes C (with $C_i = 0$ for $i < 1$), and they also generalise groups, groupoids, and crossed modules. A brief survey of their use in topology and algebra is given in [6]. See also [4, 5, 7].

The category XC of crossed complexes has a convenient notion of homotopy [10, 6, 7]. So for crossed complexes D, C we can define the set

$$[D, C]$$

of homotopy classes of morphisms $D \to C$.

The object of this talk is to advertise the definition (suggested in §5 of [6])

$$H^0(X; C) = [\pi\underline{X}, C]$$

for CW-complex X with skeletal filtration $\underline{X}$, and for a crossed complex C. That is, we take $[\pi\underline{X}, C]$ as the *cohomology of* X *with coefficients in* C.

The definition makes sense, because $\pi\underline{X}$ is a homotopy invariant of X. The proof of this is not entirely trivial. One proof is given by J.H.C. Whitehead in [10] another is given in [7]. (Here we mean $X \simeq Y$ implies $\pi\underline{X} \simeq \pi\underline{Y}$.)

The point of the definition is that we expect cohomology to have something to do with the sets $[X, Y]$ of homotopy classes of maps of spaces. From [7] we take:

Theorem 1. *There is a functor* $B : XC \to \mathrm{Top}$ *assigning to a crossed complex* C *a CW-complex* BC *with the property that there is a natural bijection*

$$[X, BC] \cong H^0(X; C)$$

for CW-complexes X.

Two special cases are of interest:

(i) If C is a group G in dimension n (where G is abelian if $n \geq 2$) and zero otherwise, then $BC = K(G, n)$, and Theorem 1 generalise a classical result of Eilenberg-MacLane. Note that the non-abelian case $n = 1$ is also included.

(ii) If C_1 is a group G, C_n is a G-module M, $C_i = 0$ for $i \neq 1, n$ and all boundaries are zero then

$H^0(X; C)$ is a kind of twisted cohomology of X with coefficients in the G-module M, and so we have a twisted homotopy classification theorem.

There are three obvious questions about Theorem 1:

Q1. How do you prove it?

Q2. What use is it in tackling the *general* problem of listing the elements of the set $[X, Y]$ of homotopy classes of maps $X \to Y$?

Q3. How do you compute $H^0(X; C)$?

All these have interesting answers which we can only outline here. More details are given in [4, 5, 7].

The construction of the "classifying space" BC is done cubically. So we construct a cubical complex NC, the *nerve* of C, by setting

$$(NC)_n = XC(\pi \underline{I}^n, C)$$

where $\underline{I}^n$ is the standard skeletal filtration of the n-cube. We then set $BC = |NC|$, the geometric realisation of the cubical complex NC. (There is also a simplicial, and homotopy equivalent, version $B^{\Delta}C$; see the Introduction to [3], which includes the relevant theses [1, 8].)

The first part of the proof of Theorem 1 is to note that it is sufficient to restrict to the case when X is the realisation $|K|$ of a cubical complex K, and then to use an equivalence to homotopy categories to obtain

$$[|K|, BC] \cong [K, NC] .$$

For this we need to know NC is a Kan complex. In fact, NC has a lot of extra structure, since it turns out to be an example of an *ω-groupoid*, which is a complicated algebraic structure defined in [4]. Any ω-groupoid is a Kan complex, and hence NC is a Kan complex. We write (as in [4, 5]) λC for NC with its structure of ω-groupoid.

Because λC is an ω-groupoid, we have a bijection

$$[K, NC] \cong [\rho K, \lambda C]$$

where the latter set of homotopy classes is taken in the category of ω-groupoids, and ρK denotes the *free ω-groupoid* on K. But it also turns out that there is an equivalence, of categories with homotopy, between ω-groupoids and crossed complexes, and that this equivalence takes ρK to $\pi|\underline{K}|$, and λC to C. So

$$[\rho K, \lambda C] \cong [\pi|\underline{K}|, C]$$

and we are done.

Unfortunately, the details of the above are strenuous. However, the pattern of argument parallels the case $BC = K(G, n)$ $(n \geq 2)$, which uses the simplicial abelian group structure on $K(G, n)$. We are using ω-groupoid structures instead, and this is what allows for non-abelian results.

Something needs to be said about the homotopy type of BC. For convenience we restrict to the reduced case, i.e. when C_0 is a point. Then $\pi_1(BC, v)$ is the quotient group $G = C_2/\delta C_1$, while for $n \geq 2$ $\pi_n(BC, v)$ is the homology of C, i.e. $\mathrm{Ker}\delta/\mathrm{Im}\delta$, together with the action of G. Further, there is a fibration $BC \to K(G, 1)$ whose fibre is 1-connected and is of the homotopy type of a product of Eilenberg-MacLane spaces. (This observation is due to J-L. Loday. I am not too clear about the classification of such non-principal fibrations.)

Now let Y be a reduced CW-complex with cellular filtration $\underline{Y}$. We can form the homotopy crossed complex $\pi\underline{Y}$ and the classifying space $B\pi\underline{Y}$. In this case $\pi_1(B\pi\underline{Y}, v) \cong \pi_1(Y, v)$ and for $n \geq 2$ $\pi_n(B\pi\underline{Y}, v)$ is isomorphic to $H_n(\tilde{Y})$, the homology of the universal cover $\tilde{Y}$ of Y. Further there is a map $q : Y \to B\pi\underline{Y}$ which induces, on homotopy groups π_n, an isomorphism for $n = 1$, and for $n \geq 2$ a morphism equivalent to the Hurewicz homomorphism $\pi_n(Y, v) \xrightarrow{\omega} H_n(\tilde{Y})$.

These facts are deducible from results of §8, 9 of [5], but are not explicit there, so it should prove useful to explain the procedure.

For any filtered space $\underline{Y}$ there are cubical complexes and maps

$$\begin{array}{ccc} R\underline{Y} & \xrightarrow{i} & KY \\ {\scriptstyle p}\downarrow & & \\ \rho\underline{Y} & & \end{array}$$

where KY is the cubical singular complex of Y, and i is the inclusion of the *filtered singular complex* $R\underline{Y}$ of $\underline{Y}$; that is $R\underline{Y}$ consists in dimension n of all filtered maps $\underline{I}^n \to \underline{Y}$. The mapping p is a quotient mapping. It identifies two filtered maps $\underline{I}^n \to \underline{Y}$ if and only if they are homotopic, relative to the vertices of $\underline{I}^n$, and through filtered maps. (This definition is not exactly the same as that given in [5], but the two definitions agree if $\pi_0 Y_0 = Y_0$, which is sufficient for our purposes.)

The cubical complex $\rho\underline{Y}$ has the structure of ω-groupoid, and its associated crossed complex is $\pi\underline{Y}$. That is, $\rho\underline{Y}$ is isomorphic as ω-groupoid to $\lambda\pi\underline{Y}$.

In [5] it was shown that $p : R\underline{Y} \to \rho\underline{Y}$ is a fibration in the sense of Kan. This result was found to be an important technical tool in the proofs of the main results of [5], since it helped in proving $\rho\underline{Y} \cong \lambda\pi\underline{Y}$, and in establishing a crucial property of "thin elements" in $\rho\underline{Y}$. We can now give this fibration property of p another rôle.

The cubical complexes $R\underline{Y}$ and KY are known to be Kan complexes. (The corresponding property for $\rho\underline{Y}$ is not so simple to prove.) The inclusion $i : R\underline{Y} \to KY$

is a homotopy equivalence if the functions induced by inclusion $\pi_0 Y_r \to \pi_0 Y$ are surjective for $r = 0$ and bijective for $r > 0$, and the based pairs (Y, Y_m, v) are m-connected for all $m \geq 1$ and $v \in Y_0$. In particular, i is a homotopy equivalence if $\underline{Y}$ is the skeletal filtration of a CW-complex Y. For such a Y, the realisation $|KY|$ has the same homotopy type as Y, and in this way we obtain the map $q : Y \to B\pi\underline{Y}$ with the properties set out above.

Let X be a CW-complex. We have an induced function

$$q_* : [X, Y] \longrightarrow [X, B\pi\underline{Y}] .$$

This function is bijective if $\dim X \leq m$ and $q : Y \to B\pi\underline{Y}$ has m-connected homotopy fibre. This will be true if, for example, $\pi_i Y = 0$ for $1 < i < m$. In these circumstances we obtain a bijection

$$[X, Y] \longrightarrow H^0(X; \pi\underline{Y}) .$$

So we can see the relevance of this non-abelian cohomology to some general homotopy classification problems, particularly in the non-simply connected case.

How do we compute $H^0(X; C)$? For this we generalise some ideas of Whitehead in [10].

For simplicity, we restrict to the reduced case. Let GC_* be the category with objects the triples (K, G, v) in which G is a group, K is a chain complex of G-modules (with $K_i = 0$ for $i < 0$), and K_0 is a free G-module with basis the element $v \in K_0$. The morphisms of GC_* are to be pairs $(f, \theta) : (K, G, v) \longrightarrow (K', G', v')$ where $\theta : G \to G'$ is a morphism of groups, $f : K \to K'$ is a chain map and an operator morphism over θ, and $f(v) = v'$.

Let XC_* be the category of reduced crossed complexes. There is a functor $\Delta : XC_* \to GC_*$ in which if $(K, G, v) = \Delta C$, then $G = C_1/\delta C_2$; $K_n = C_n$ as a G-module for $n \geq 3$; K_2 is C_2 made abelian; K_1 is the G-module induced from the augmentation ideal IC_1 by the quotient morphism $C_1 \to G$; and K_0 is the free G-module on the element $v \in C_0$. (This construction is given in [7] and extends a construction given in [10] for the case C_1 is free. A further result proved in [7] is that Δ has a right adjoint, and so preserves colimits.) This functor Δ transforms homotopies to homotopies, for a suitable definition of homotopy in GC_*. So for reduced crossed complexes C, D we have a function

$$\Delta_* : [D, C] \longrightarrow [\Delta D, \Delta C] .$$

Now Whitehead proves (but does not state) that if C_1 and D_1 are free groups and D_2 is a free crossed D_1-module, then Δ_* is a bijection. Also, he notes that if $\underline{X}$ is the skeletal filtration of a reduced CW-complex X, then $\Delta\pi\underline{X}$ consists of the cellular chains $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ of X, these chains being taken as modules over the fundamental group of X. That is, we have a bijection

$$H^0(X;\ C) \cong [C_*(\tilde{X}),\ \Delta C]\ .$$

This gives a reasonable computational description of $H^0(X;\ C)$, and so of $[X,\ BC]$. For example, it leads to the homotopy classification of maps from a surface to the projective plane [2].

Consider again the bijection

$$[X,\ Y] \cong [C_*(\tilde{X}),\ C_*(\tilde{Y})]$$

given when $\dim X \le m$ and $\pi_i Y = 0$ for $1 < i < m$. If also $\pi_1 Y = 0$, then $\tilde{Y} = Y$ and the definition of morphism and chain homotopy in GC_* implies that

$$[C_*(\tilde{X}),\ C_*(\tilde{Y})] \cong [C_*(X),\ C_*(Y)]$$

where $C_*(X)$ is the usual cellular chain complex of X . Since $C_*(Y)$ is a chain complex of free abelian groups there is a chain map $\phi : C_*(Y) \to H_*(Y)$ (where the latter has zero differential) inducing an isomorphism in homology. So we obtain

$$\begin{aligned}[X,\ Y] &\cong [C_*(X),\ H_*(Y)] \\ &\cong H^0(X;\ H_*(Y)) \\ &\cong H^m(X;\ H_m(Y))\ .\end{aligned}$$

This result includes the Hopf classification theorem (which is the case $Y = S^m$) . Thus the non-abelian results reduce to classical abelian results.

All these results give point to a remark of Whitehead in the Introduction to [10], which reads in our terminology:

The crossed complex $\pi\underline{X}$ *appears to be more useful than the chain complex* $C_*(\tilde{X})$ *in problems concerning geometric realisability. On the other hand, the chain complex* $C_*(\tilde{X})$ *is useful in studying concrete problems.*

References

1. N.K. Ashley, *Crossed complexes and T-complexes*, Ph.D. Thesis, University of Wales, (1978).
2. R. Brown, 'The homotopy classification of maps from a surface to the projective plane', Bangor Preprint 82.5, (1982).
3. R. Brown, (ed.), *Simplicial T-complexes*, Esquisses Math. (to appear).
4. R. Brown and P.J. Higgins, 'The algebra of cubes', J. Pure Appl. Alg. 21 (1981), 233-260
5. R. Brown and P.J. Higgins, 'Colimit theorems for relative homotopy groups', J. Pure Appl. Alg. 22 (1981), 11-41.
6. R. Brown and P.J. Higgins, 'Crossed complexes and non-abelian extensions', *Proc. Int. Conf. on Category Theory, Gummersbach, 1981* (ed. H. Kamps, D. Pumplün, M Tholen) Springer L.N.M. (to appear).
7. R. Brown and P.J. Higgins, 'Crossed complexes and chain complexes with operators', (in preparation).
8. M.K. Dakin, *Kan complexes and multiple groupoids*, Ph.D. Thesis, University of Wales, (1977).
9. G. Segal, 'Classifying spaces and spectral sequences', Publ. Math. I.H.E.S. 34 (1968), 105-112.
10. J.H.C. Whitehead, 'Combinatorial homotopy II', Bull. Amer. Math. Soc. 55 (1949) 453-496.