Nonabelian Algebraic Topology

by Ronald Brown, Philip J. Higgins, and Rafael Sivera

The following account is adapted from a Proposal for a Leverhulme Emeritus Fellowship for Brown, 2002-2004; the chosen referees were Professor A. Bak (Bielefeld) and Professor J.P. May (Chicago), and the proposal was fully funded.

The Fellowship supported the finishing of the revised version of 'Topology and Groupoids' (see www.bangor.ac.uk/r.brown/topgpds.html), attendance at a number of conferences, visits from other mathematicians (including Sivera), and the development and publication of 16 or so research items from [133-159] in Brown's publication list. A considerable part of the work for this book (e.g. typing in LaTeX, discussions with Sivera) was done in this period with this support.

Structure of the book

One major theme of the book is the *theory and applications of crossed complexes.* The *fundamental crossed complex* ΠX_* *of a filtered space* X_* is defined using relative homotopy groups $\pi_n(X_n, X_{n-1}, x), n \geqslant 2, x \in X_0$ and the fundamental groupoid $\pi_1(X_1, X_0)$ and its actions. It contains nonabelian information in dimensions 1 and 2.

There are two major theorems presented and applied in Part II of the book:

1. *Higher Homotopy van Kampen Theorem*: the functor Π from filtered topological spaces to crossed complexes preserves certain colimits; as an example application, the Relative Hurewicz Theorem is seen as a special case of a *homotopical excision theorem*, involving induced modules and crossed modules;
2. *Homotopy Classification Theorem*: there is a bijection (which generalises classical work of Eilenberg-Mac Lane):

$$[X, BC] \cong [\Pi X_*, C]$$

where X_* is the skeletal filtration of a CW-complex, and BC is the classifying space of the crossed complex. Thus the left hand side is topological, and the right hand side is algebraic.

This second theorem requires for its proof knowledge of the complicated monoidal closed structure on the category of crossed complexes.

The foundation for the proofs of these theorems is given in Part III. This gives another theme of the book, the construction and properties of the *cubical higher homotopy* ω*-groupoid* ρX_* *of a filtered space* X_*. A crucial and highly nontrivial fact is that the algebra of this construction is equivalent to the classical notion of *crossed complex*, and under this equivalence ρX_* becomes ΠX_*.

Because of the novelty (even after so many years!) of these ideas for most algebraic topologists, and in order to emphasise the intuitive basis, Part I of the book deals entirely with dimensions 1 and 2, gives the necessary background in crossed module theory, elaborates on the nonabelian applications of the Higher Homotopy van Kampen Theorem in dimension 2, and proves the theorem using the construction of the homotopy double groupoid of a pair (X, X_1) of spaces with a set of base points $X_0 \subseteq X_1$.

The structure of this set of ideas is quite intricate, and to obtain a consistent presentation has taken a number of years.

Background to the book

Here is some background to the notion of *higher homotopy groupoid.*

In the early part of the 20th century, topologists were aware that in the connected case the first homology group of a space was the fundamental group made abelian, and that homology groups existed in all positive dimensions. Further, the nonabelian nature of the fundamental group gave more information in geometric and analytic contexts than did the first homology group. They were therefore interested in seeking higher dimensional versions of the non abelian fundamental group.

E. Čech submitted to the 1932 ICM at Zurich a paper on higher homotopy groups, using maps of spheres. However these groups were quickly proved to be abelian in dimensions > 1, and Čech was asked (by Alexandrov and Hopf) to withdraw his paper, so that only a short paragraph appeared in the Proceedings. Thus the dream of these topologists seemed to fail, and since then this dream has widely been felt to have been a mirage. Of course higher homotopy groups have since been attributed to Hurewicz.

Work of J.H.C. Whitehead in the 1940s introduced the notion of *crossed module*, using the boundary of the second relative homotopy group of a pair and the action of the fundamental group. He and Mac Lane showed that crossed modules classified (connected, pointed, weak) homotopy 2-types. Crossed modules are indeed more complicated than groups, and there is a case for regarding them as '2-dimensional groups'. They are being increasingly used in combinatorial group theory, homological algebra, algebraic topology, and differential geometry. The book gives new presentations of some less recognised but deep work of Whitehead.

In 1967, Brown introduced the fundamental groupoid of a space on a set of base points, and the writing of his 1968 book on topology suggested to him that all of 1-dimensional homotopy theory was better expressed in terms of groupoids rather than groups. This raised the question of the

potential use of groupoids in higher homotopy theory,

based on squares and cubes rather than paths. A positive answer in all dimensions took 11 years.

An initial question was the potential extent of an algebraic theory of double groupoids. Relations of certain double groupoids to crossed modules were worked out with C.B. Spencer in the early 1970s, [20,21].

A definition of a homotopy double groupoid of a pair of pointed spaces was made with P.J. Higgins in 1974, and shown to yield a 2-dimensional Van Kampen type theorem which could compute some homotopy 2-types. This work was published in 1978 [25]. Extensions of this to all dimensions were worked out and announced in 1977, 1978, and a full account of the basic theory was published in 1981 [31,32]. Successive years gave further developments and applications, particularly tensor products and homotopies for these higher dimensional objects [48,57,59], relation with chain complexes [61], and the notion and applications of a (simplicial) classifying space [71]. A crucial part of this work was the technically difficult proof of the major properties of the cubical higher homotopy groupoid of a filtered space.

It is this structure which enabled the proof of a Higher Homotopy Van Kampen Theorem, based on the slogan of 'algebraic inverses to subdivision', and to apply tensor products to topological situations. For example, two key technical and surprising results are that if X_* is a filtered space, then (i) the compositions on $R(X_*)$ are inherited by $\rho(X_*)$, and (ii) the natural map $p : R(X_*) \to \rho(X_*)$ is a Kan fibration of cubical sets. The proofs use methods of collapsing for subcomplexes of cubes (ideas of Whitehead).

New applications of the technology of crossed complexes are regularly being discovered. We feel that these ideas stem from the intuitive roots of algebraic topology.

For recent references, see
http://www.bangor.ac.uk/r.brown/fields-art3.pdf
http://www.bangor.ac.uk/r.brown/noncommut-at.pdf

Aims of the book

The aim is the first monograph to deal with this area of 'higher dimensional algebra (HDA)'. It will be at a level appropriate for postgraduate students as well as advanced researchers. The content will mainly be work of R. Brown with P.J. Higgins and others since 1970, and which is scattered over various journals and not in a common format. It will start with the intuitive background and the 2-dimensional case, which allows for pictures. Its timeliness reflects the recent explosion of interest in HDA across mathematics, physics and computer science (for more information, do a web search on 'Higher Dimensional Algebra').

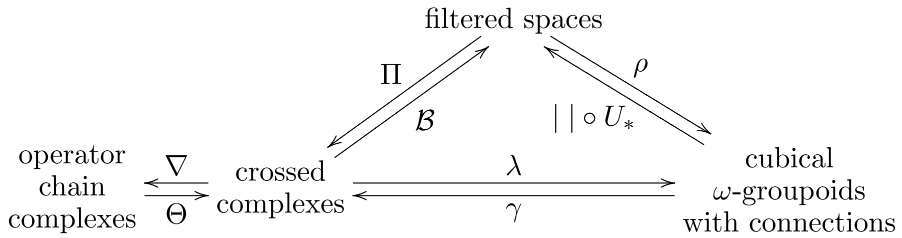
The monograph will give an exposition of some basic algebraic topology at the border of homotopy and homology using algebraic tools which closely model the geometry, and give new results and calculations. One slogan is 'higher dimensional nonabelian tools for local-to-global problems'.

The monograph will enable students and researchers to understand these new methods in algebra and topology and the way they relate to basic intuitions of using algebraic structures whose operations are defined under geometric conditions. This relaxation enables the algebra to more closely reflect the geometry, leads to the right algebraic context of classical results such as the Relative Hurewicz Theorem, and has the serendipitous effect of leading to more calculability. Indeed, the Higher Homotopy van Kampen theorems generalise exceptional features of the van Kampen theorem for the fundamental groupoid on a set of base points, and even give precise homotopical calculations in dimension 2 not possible by classical means such as homology and exact sequences. This is seen in Part I of the book.

The novelty of the ideas for most mathematicians makes it important to treat in full detail this special case of higher dimensional algebra, since it has had and will continue to have a role as a model for future work ([31] has inspired recent work on concurrency). The book will be of the order of 550 pages. Some of the key papers, such as [31,32,48,61], are written in a clear but sparse style and need more explanation and pointers to the structure of the proofs, for example of the sophisticated methods of proof using cubical collapsing.

The major work has been in retyping and reorganising where necessary the published work, improving proofs and the structure of the exposition, giving a common style and viewpoint, and providing a full bibliography to this large and expanding area. As an example of improvement, the understanding of the tensor product $C \otimes D$ of crossed complexes is greatly helped in dimensions > 2 by the relation with chain complexes with a groupoid of operators. Historical notes will also be provided. The first part of the work will cover aspects of the papers [20,21,25,35,43,92,95] referred to below, and some of [50], while the later parts will cover [31,32,48,57,61,71,72,104,120]. The term 'higher dimensional algebra' was coined by Brown in 1987 and has caught on, as a web search shows.

The complete and intricate story has its main facts as summarised in the diagram:



(Main Diagram)

The power behind crossed complexes comes from the equivalence with the ω-groupoids hinted at in the diagram, since it is in that context one can conjecture and prove some subtle theorems. For more details on this, see the survey [132].

The book does not cover work with Loday (see references [42,62,74]), for lack of space. So there is more to be written under this title.

Major changes in the exposition since the original proposal

The project aims to give complete and comprehensible proofs, except for the large item of the equivalence of homotopy categories of CW-complexes, and of Kan cubical sets. Originally, it was intended to use a simplicial classifying space as in [71]: this created problems in giving a complete concise exposition. It was realised in 2007 that the original (unpublished) cubical approach to the classifying space fitted in much better with the remaining exposition and enabled a cleaner and better organisation of the material. At an earlier stage, it was realised that the understanding of the tensor product of crossed complexes could be clarified by another reorganisation, an earlier description of the relation with chain complexes with a groupoid of operators.

The proof and applications of the homotopy classification theorem employ most of the techniques of the book. The payoff of these techniques are however much wider.

The authors

P.J. Higgins is retired and devoted to music, so wishes to take no part in the writing of this text. However the results of this book are imbued with his expertise, imagination and expository skills, so it is only right that his name should be on the text.

R. Brown is the other instigator of this research, carried out over many years and publications.

R. Sivera has worked on presenting this complicated material since the 1990s. He has contributed many points of exposition, which is a key research value of this book.

Major Publications of Brown related to the area of research

(Discussion and a wider list of publications on the area can be found in the web article 'Higher dimensional group theory' www.bangor.ac.uk/r.brown/hdaweb2.htm)

BOOK 'Topology and Groupoids' Booksurge 2006 (xxv+512pp)ISBN 1-4196-2722-8. Updated, extended and revised edition of: Elements of modern topology, McGraw Hill, Maidenhead, 1968. Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid, Ellis Horwood, Chichester (1988) 460 pp. It is reviewed in Bull. London Math. Soc. 39 (2007) 867-868. (This seems to be still the only text on basic topology to use groupoids fully in the area of 1-dimensional homotopy theory, and the topics of the Van Kampen theorem, covering spaces, orbit spaces, and subgroup theorems in group theory. Writing this text set the scene for all the following research.)

The numbers in the following refer to Brown's full publication list.

20 (with C.B. SPENCER), "$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group", *Proc. Kon. Ned. Akad. v. Wet.* 7 (1976) 296-302

21. (with C.B. SPENCER), "Double groupoids and crossed modules", Cah. Top. Géom. Diff. 17 (1976) 343-362.

22. (with P.J. HIGGINS), "Sur les complexes croisés, ω-groupoïdes et T-complexes", *C.R. Acad. Sci. Paris Sér. A.* 285 (1977) 997-999.

23. (with P.J. HIGGINS), "Sur les complexes croisés d'homotopie associés a quelques espaces filtrés", *C.R. Acad. Sci. Paris Sér. A.*286 (1978) 91-93.

25. (with P.J. HIGGINS), "On the connection between the second relative homotopy groups of some related spaces", Proc. London Math. Soc. (3) 36 (1978) 193-212.

30. "On the second relative homotopy group of an adjunction space: an exposition of a theorem of J.H.C. Whitehead", *J. London Math. Soc.* (2) 22 (1980) 146-152.
31. (with P.J. HIGGINS), "On the algebra of cubes", J. Pure Appl. Algebra 21 (1981) 233-260
32. (with P.J. HIGGINS), "Colimit theorems for relative homotopy groups", J. Pure Appl. Algebra 22 (1981) 11-41.
33. (with P.J. HIGGINS), "The equivalence of ω-groupoids and cubical T-complexes", Cah. Top. Géom. Diff. 22 (1981) 349-370.
34. (with P.J. HIGGINS), "The equivalence of ∞-groupoids and crossed complexes", Cah. Top. Gom. Diff. 22 (1981) 371-386.
35. (with J. HUEBSCHMANN), "Identities among relations", in Low dimensional topology, London Math. Soc. Lecture Note Series 48 (ed. R. Brown and T.L. Thickstun, Cambridge University Press, 1982), pp. 153-202.
36. "Higher dimensional group theory", in Low dimensional topology, London Math Soc. Lecture Note Series 48 (ed. R. Brown and T.L. Thickstun, Cambridge University Press, 1982), pp. 215-238.
42. (with J.-L. LODAY), "Excision homotopique en basse dimension", C.R. Acad. Sci. Paris Sr. I 298 (1984) 353-356.
43. "Coproducts of crossed P-modules: applications to second homotopy groups and to the homology of groups", Topology 23 (1984) 337-345.
48. (with P.J. HIGGINS), "Tensor products and homotopies for ω-groupoids and crossed complexes", J. Pure Appl. Alg. 47 (1987) 1-33.
50. "From groups to groupoids: a brief survey", Bull. London Math. Soc. 19 (1987) 113-134.
57. (with M. GOLASINSKI), "A model structure for the homotopy theory of crossed complexes", Cah. Top. Gom. Diff. Cat. 30 (1989) 61-82.
59. (with N.D. GILBERT), "Algebraic models of 3-types and automorphism structures for crossed modules", Proc. London Math. Soc. (3) 59 (1989) 51-73.
61. (with P.J. HIGGINS), "Crossed complexes and chain complexes with operators", Math. Proc. Camb. Phil. Soc. 107 (1990) 33-57.
62. "Some problems in non-Abelian homotopical and homological algebra", *Homotopy theory and related topics, Proceedings Kinosaki, 1988*, ed. M. Mimura, Springer Lecture Notes in Math., 1418 (1990) 105-129.
71. (with P.J.HIGGINS), "The classifying space of a crossed complex", Math. Proc. Camb. Phil. Soc. 110 (1991) 95-120.
72. (with H.J.BAUES), "On the relative homotopy groups of the product filtration and a formula of Hopf", J. Pure Appl. Algebra 89 (1993) 49-61.
74. "Computing homotopy types using crossed n-cubes of groups", Adams Memorial Symposium on Algebraic Topology, Vol 1, edited N. Ray and G Walker, Cambridge University Press, 1992, 187-210.
78. "Out of line", Royal Institution Proceedings, 64 (1992) 207-243.
82. (with O. MUCUK), "Covering groups of non-connected topological groups revisited", *Math. Proc. Camb. Phil. Soc*, 115 (1994) 97-110.
92. (with C.D.WENSLEY), "On finite induced crossed modules and the homotopy 2-type of mapping cones", Theory and Applications of Categories 1(3) (1995) 54-71.
95. (with C.D.WENSLEY), "Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types", Theory and Applications of Categories 2 (1996) 3-16.

97. (with T. PORTER), “On the Schreier theory of non-abelian extensions: generalisations and computations”. Proceedings Royal Irish Academy 96A (1996) 213-227.

102. ‘Groupoids and crossed objects in algebraic topology’, Homology, homotopy and applications, 1 (1999) 1-78.

104. (with A. RAZAK SALLEH), ‘Free crossed crossed resolutions of groups and presentations of modules of identities among relations’, LMS J. Comp. and Math. 2 (1999) 28-61.

105. (with G.H. MOSA), ‘Double categories, 2-categories, thin structures and connections’, Theory and Applications of Categories 5 (1999) 163-175.

107. (with ANNE HEYWORTH), ‘Using rewriting systems to compute left Kan extensions and induced actions of categories’, J. Symbolic Computation 29 (2000) 5-31.

114. (with M. GOLASINSKI, T.PORTER and A.P.TONKS), “On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case”, K-Theory 23 (2001)129-155.

115. (with I. Ien), ‘Automorphisms of crossed modules of groupoids’, Applied Categorical Structures 11 (2003) 185-206.

116. (with F.A. AL-AGL and R. STEINER), “Multiple categories: the equivalence between a globular and cubical approach”, Advances in Mathematics 170 (2002) 71-118.

118. (with K.HARDIE, H.KAMPS, T. PORTER), ‘ The homotopy double groupoid of a Hausdorff space’, Theory and Applications of Categories, 10 (2002) 71-93.

122. (with İ. Içen, and O. Mucuk), ‘Local subgroupoids II: Examples and properties’, Topology and its Applications 127 (2003) 393-408.

123. (with I.ICEN) ‘Towards a 2-dimensional notion of holonomy’, Advances in Math. 178 (2003) 141-175. math.DG/0009082

124. (with C.D.WENSLEY), ‘Induced crossed modules and computational group theory’, J. Symbolic Computation 35 (2003) 59-72.

125. (with M. BULLEJOS and T.PORTER),‘Crossed complexes, free crossed resolutions and graph products of groups’, Proceedings Workshop Korea 2000, J. Mennicke, Moo Ha Woo (eds.) Recent Advances in Group Theory, Heldermann Verlag Research and Exposition in Mathematics 27 (2002) 8–23.

126. (with E. MOORE, T.PORTER, C.D.WENSLEY), ‘Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups’, Georgian Math. J. 9 (2002) 623-644.

129. (with G. JANELIDZE), ‘A new homotopy double groupoid of a map of spaces’, Applied Categorical Structures 12 (2004) 63-80.

130. (with T. PORTER), ‘The intuitions of higher dimensional algebra for the study of structured space’, Revue de Synthèse, 124 (2003) 174-203.

131. (with HIGGINS, P.J.), ‘Cubical abelian groups with connections are equivalent to chain complexes’, Homology, Homotopy and Applications, 5(1) (2003) 49-52.

132. ‘Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems’, Proceedings of the Fields Institute Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories, September 23-28, Fields Institute Communications 43 (2004) 101-130. math.AT/0212274

147. ‘Three themes in the work of Charles Ehresmann: Local-to-global; Groupoids; Higher dimensions’, Proceedings of the 7th Conference on the Geometry and Topology of Manifolds: The

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7. A. RAZAK SALLEH, Union theorems for groupoids and double groupoids, (1976).
8. M.K. DAKIN, Kan complexes and multiple groupoids, (1977).
9. N.K. ASHLEY, T-complexes and crossed complexes, (1978).
12. D.W. JONES Poly-T-complexes, (1984).
15. G.H. MOSA, Higher dimensional algebroids and crossed complexes, (1987).
19. O. MUCUK, Coverings, and monodromy, for Lie groupoids, (1993). Lecturer, Erciyes University, Turkey.
20. A. P. TONKS, Theory and applications of crossed complexes, (1993).
21. I. İÇEN, A 2-dimensional notion of holonomy, (1996).
22. Anne HEYWORTH, (Supervised with C.D. Wensley), Applications of Rewriting Systems and Groebner Bases to Computing Kan Extensions and Identities Among Relations, (1999).
23. Emma MOORE, (Supervised with C.D. Wensley), Graphs of groups: word computations and free crossed resolutions, (2001).

Directly related papers in preparation

1. (with P.J.Higgins and R. Sivera), 'The cubical classifying space of a crossed complex'.
2. (with R. Sivera), 'Computation of colimits in fibred and opfibred categories: applications to modules and crossed modules of groupoids'
3. (with R. Street), 'Universal coverings of tensor products of crossed complexes and cubical omega-groupoids'.

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Baues, H.-J. and Tonks, A., On the twisted cobar construction. Math. Proc. Cambridge Philos. Soc. 121 (2) (1997) 229–245.
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