ON A METHOD OF P. OLUM

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We present another proof that $\pi_1(S^1) = Z!$. Actually our main purpose is to show that the techniques used by P. Olum in [1] also allow one to prove the following result.

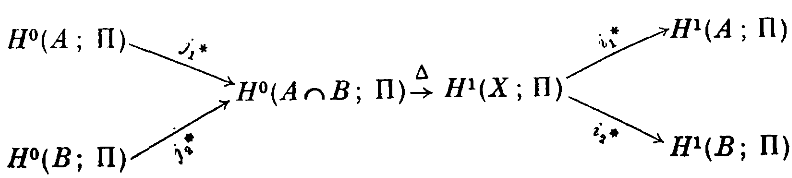
Let $X = A \cup B$ be a topological space such that (i) the interiors of A and B cover X, (ii) A and B are 1-connected, and (iii) $A \cap B$ has exactly $n+1$ path-components. (Thus X is clearly path-connected.)

THEOREM. *$\pi_1(X)$ is a free group on n generators.*

This shows that we can derive by a uniform method all the facts necessary to compute the fundamental group of quite general spaces, including, for example, all CW-complexes. (The fact that $\pi_1(S^n) = 0$, $n > 1$, follows easily from Van Kampen's theorem.)

The method of P. Olum is to construct a Mayer-Victoris sequence for cohomology with coefficients in a non-abelian group Π. Our theorem follows from a study of the bottom end of this sequence.

We consider spaces with base point, and abbreviate $H^i(X, *; \Pi)$ $(i = 0, 1)$ to $H^i(X; \Pi)$; the base point of $A \cup B$ is $* \in A \cap B$. From now on, we make the assumption (i). Then we have, by Theorem 1 (a) of [1], a diagram



in which i_1, i_2, j_1, j_2 are injections and

(1) $\mathit{Image}\,\Delta = \operatorname{Ker} i_1{}^* \cap \operatorname{Ker} i_2{}^*$.

We recall that the definition of Δ is not symmetrical in A and B. This is reflected in the following lemma, which describes the amount of exactness at $H^0(A \cap B; \Pi)$.

Let us suppose that X satisfies the following condition: each point of $A \cap B$ can be joined by a path in A to $*$. Let c, d in $H^0(A \cap B; \Pi)$ be such that $\Delta c = \Delta d$.

LEMMA 1. *There is an element b in $H^0(B; \Pi)$ such that*

$$c = d + j_2{}^* b.$$

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This is proved by simple calculations with singular cochains. An immediate corollary of Lemma 1 is the following:

(2) *If A and B are path-connected, then Δ is mono.*

For any path-connected X, there is a natural bijection [(1.3) of **1**]

$$H^1(X;\Pi)\to\mathrm{Hom}\big(\pi_1(X),\Pi\big).$$

So from (1) and (2) we deduce:

(3) *If A and B are 1-connected, then there is a natural bijection*

$$H^0(A\cap B;\Pi)\to\mathrm{Hom}\,\big(\pi_1(X),\Pi\big).$$

We now make the assumption (iii). Then $H^0(A\cap B;\Pi)$ is naturally isomorphic to Π^n, the direct product of n copies of Π. So the theorem follows from (3) and the next lemma, for whose proof I am indebted to J. F. Adams.

LEMMA 2. *Let Φ be a group such that for any group Π there is a natural bijection*

$$\Pi^n\to\mathrm{Hom}\,(\Phi,\Pi).$$

Then Φ is a free group on n generators.

Proof. Let F be a free group on n generators. It is well known that there is a natural bijection

$$\Pi^n\to\mathrm{Hom}\,(F,\Pi).$$

So we deduce a natural bijection

$$\lambda:\mathrm{Hom}\,(\Phi,\Pi)\to\mathrm{Hom}\,(F,\Pi). \tag{4}$$

We define $f: F\to\Phi$ by setting $\Pi=\Phi$, $f=\lambda(1_\Phi)$ in (4); and $g:\Phi\to F$ by setting $\Pi=F$, $g=\lambda^{-1}(1_F)$ in (4). It is easy to check, using naturality, that $fg=1_\Phi$, $gf=1_F$. This proves the lemma.

There remains the determination of generators of $\pi_1(X)$. In each path-component of $A\cap B$ (other than that containing $*$) let a point x_i be chosen, $i=1,\ldots,n$. Let λ_i in $\pi_1(X)$ be represented by the composite of a path in A joining $*$ to x_i and a path in B joining x_i to $*$. Then the inverse μ of the bijection of (3) is determined by

$$\mu(f)(x_i)=f(\lambda_i)\quad i=1,\ldots,n \tag{5}$$

for any $f\in\mathrm{Hom}\big(\pi_1(X),\Pi\big)$. It follows from this that $\lambda_1,\ldots,\lambda_n$ is a set of generators of $\pi_1(X)$.

Reference

1. P. Olum, "Non-abelian cohomology and van Kampen's theorem", *Annals of Math.*, 68 (1958), 658–667.

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