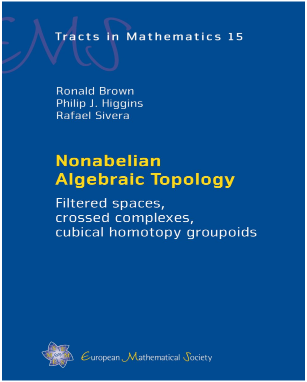
Filtered spaces crossed complexes and cubical higher homotopy groupoids: a new foundation for algebraic topology

Ronnie Brown

March 28, 2011
Tbilisi
Conference on Homotopy Theory and Non Commutative Geometry



xxxiii+ 643pp

To appear (2011)

Very pleased to be in Tbilisi for my 4th visit, and especially this conference involving

Homotopy Theory and Non Commutativity

as their interaction has been very much my pursuit for 45 years and still is so.

You may well say:
Why the need for a new foundation for algebraic topology?
We have had contributions from
Poincaré, Alexander, Eilenberg, the axioms of
Eilenberg-Steenrod, excellent texts, ...
Answer: Look for anomalies, and places where the expositions seem embarrassingly tricky, or ad hoc, or avoided altogether.
Evaluation! Analysis!
Einstein (1917): What is essential and what is based only on the accidents of development? ...... It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little...

Problems in algebraic topology exposition: the initial border between homology and homotopy, and dealing with **nonabelian** algebraic models of homotopy types.
Traditional arguments of, say,
transversality, simplicial approximation, general position,
are not completely modelled by algebra, except partially in abelian terms (chain complexes).
Rational homotopy is largely the simply connected case.
Aim: to realise the vision of the workers in topology of the early 20th century (Dehn, *et al*) to find and apply
higher dimensional versions of the fundamental group!
Start question (1965):
are groupoids useful in higher homotopy theory?
Inspiration: Work of Crowell and Fox on the Seifert-van Kampen Theorem, Higgins on groupoids, Ehresmann on higher categories, and crucially Henry Whitehead on 'Combinatorial Homotopy II'.

Central Role in the Theory:

a Higher Homotopy Seifert-van Kampen Theorem, proved without using singular homology or simplicial approximation, and which includes or implies:

1) The Seifert-van Kampen Theorem for the fundamental groupoid on a set of base points;

2) the Relative Hurewicz Theorem in the form that: pointed pair (X, A) is $(n-1)$-connected, then the natural morphism

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x)$$

factors the action of $\pi_1(A, x)$ on $\pi_n(X, A, x)$;

3) the Brouwer Degree Theorem: $\pi_n(S^n) \cong \mathbb{Z}$;

4) if X is a CW-complex, and $n \geqslant 3$ then $\pi_n(X_n, X_{n-1}, X_0)$ is a free $\pi_1(X, X_0)$-module on the characteristic maps of the n-cells; i.e. good handling of higher relative homotopy groups as modules over π_1;

Nonabelian results such as:
5) Whitehead's Theorem that
$\pi_2(X \cup \{e^2_\lambda\}, X, x)$ is a free crossed $\pi_1(X, x)$-module;
6) a generalisation of Whitehead's Theorem to describe the crossed module

$$\pi_2(X \cup_f CA, X, x) \to \pi_1(X, x)$$

as induced by the morphism $f_* \colon \pi_1(A, a) \to \pi_1(X, x)$ from the identity crossed module $\pi_1(A, a) \to \pi_1(A, a)$; and
7) a coproduct description of the crossed module

$$\pi_2(K \cup L, M, x) \to \pi_1(M, x)$$

when $M = K \cap L$ is connected and $(K, M), (L, M)$ are 1-connected and cofibred;
8) numerous explicit calculations of homotopy 2-types given by crossed modules, unobtainable otherwise;

9) As an example of 8) we get lots of specific computations of homotopy 2-types of spaces X given by a homotopy pushout of classifying spaces of groups, for example:

$$\begin{array}{ccc} BS_3 & \xrightarrow{Bf} & BS_4 \\ {\scriptstyle Bp}\downarrow & & \downarrow \\ BC_2 & \longrightarrow & X. \end{array}$$

Computer calculations by Chris Wensley: the 2-type of X is given by
a crossed module $SL(2,3) \to S_4$ with kernel C_2.
These confirm that π_2 as a module over π_1 is commonly
a pale shadow of the homotopy 2-type,
which is represented by a crossed module.

Filtered spaces

A key is to work with filtered spaces:
A *filtered space* X_* is simply a topological space X and a sequence of subspaces:

$$X_* \colon \; X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X.$$

So we get a category FTop of filtered spaces.
Many spaces come with a natural or convenient filtration: CW-complexes, free topological monoids (filtered by length), manifolds with a Morse function, stratified spaces, ...

Crossed complexes

There is homotopically defined functor

$$\pi_* : \mathsf{FTop} \to \mathsf{Crs}$$

where Crs is the category of crossed complexes, using relative homotopy groups and the fundamental groupoid, giving a crossed complex: so if $C = \pi_* X_*$ then C is of the form

$$\cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} C_{n-2} \to \cdots \to C_2 \xrightarrow{\delta_2} C_1$$

where
$C_1 = \pi_1(X_1, X_0)$ - fundamental groupoid;
$C_n(x) = \pi_n(X_n, X_{n-1}, x)$ for $n \geqslant 2$ and $x \in X_0$;
Note that C_1 operates on C_n for $n \geqslant 2$
and $\delta_2 : C_2 \to C_1$ is a crossed module (over a groupoid);
Axioms are those universally satisfied by this example!

The aim is direct **colimit** calculations of this homotopically defined functor

$$\pi_* : \mathsf{FTop} \to \mathsf{Crs}$$

without using singular homology or simplicial approximation. But only under certain connectivity conditions.

A crucial feature of the functor π_* is that crossed complexes have structure in a range of dimensions from 0.

The filtered space X_* is said to be *connected* if

(ϕ_0): The function $\pi_0 X_0 \to \pi_0 X_r$ induced by inclusion is surjective for all $r \geqslant 0$; and, for all $i \geqslant 1$,

$(\phi_i) : \pi_i(X_r, X_i, v) = 0 \text{ for all } r > i \text{ and } v \in X_0$.

THINK: skeletal filtration of a CW-complex.

Standard use of coequalisers

Let X_* be a filtered space, and $\mathcal{U} = \{U^\lambda \mid \lambda \in \Lambda\}$ an open cover of X. For $\zeta \in \Lambda^n$ let

$$U^\zeta = \bigcap_{i=1}^{n} U^{\zeta_i}; \quad U^\zeta_n = U^\zeta \cap X_n.$$

So we have a coequaliser of filtered spaces:

$$\bigsqcup_{\zeta \in \Lambda^2} U^\zeta_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} U^\lambda_* \xrightarrow{c} X_*$$

Here c is determined by the inclusions $U^\lambda \to X$ and
a, b are determined by the inclusions $U^\zeta \to U^\lambda, U^\zeta \to U^\mu$ for $\zeta = (\lambda, \mu) \in \Lambda^2$.
Advantage of the groupoid approach:
$\pi_* : \mathsf{FTop} \to \mathsf{Crs}$ commutes with $\bigsqcup$, disjoint union.

Higher Homotopy Seifert–van Kampen Theorem

Theorem (Brown-Higgins 1981)

Suppose for all finite intersections U^ζ of the elements of the cover $\mathcal{U}$ the filtered space U^ζ_* is connected. Then X_* is connected and

$$\bigsqcup_{\zeta \in \Lambda^2} \pi_* U^\zeta_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \pi_* U^\lambda_* \xrightarrow{c} \pi_* X_* \qquad (!)$$

is a coequaliser diagram of crossed complexes.

This Theorem gives when it applies complete computations of $\pi_* X_*$ in terms of gluing information on the filtered space X_*. Colimits are usually more precise than exact sequences, particularly in nonabelian situations.

Against all traditions of algebraic topology??!!

The HHSvKT includes nonabelian information in dimensions 1 and 2, and information on operations of the fundamental group(oid) on relative homotopy groups. It is convenient for a certain range of calculations.

Impossible to conjecture directly?!

1965 Initial idea: the proof of the groupoid Seifert-van Kampen Theorem generalised to all dimensions if one had a cubical higher groupoid homotopical gadget which allowed for:

1) algebraic inverses to subdivision,

2) compositions of ‘commutative cubes’ are commutative.

An idea for a proof in search of a theorem!

It took 12 years and collaborations with Chris Spencer and Philip Higgins to get the above HHSvK Theorem!

A cubical higher homotopy groupoid

The proof goes through another
homotopically defined and very clear and 'obvious' construction

$$\rho : \mathsf{FTop} \to \omega\text{-}\mathsf{Gpds}$$

to cubical ω-groupoids with connections
(explain connections later).
Major fact:

Theorem (Brown-Higgins 1981)

There is an equivalence of algebraic categories

$$\gamma : \omega\text{-}\mathsf{Gpds} \leftrightarrows \mathsf{Crs} : \lambda$$

which takes ρX_* to $\pi_* X_*$.

From weak structures to strict

X_* be a filtered space,
I^n_* the filtered space of the standard n-cube.
$R_n X_* = \mathsf{FTop}(I^n_*, X_*)$.
Then RX_* becomes a cubical set with composition.
$\alpha, \beta \in R_n X_*$: a thin homotopy $h_t : \alpha \equiv \beta$ is a map
$h : I^n \times I \to X$ such that
h_t is a filtered map, $h_0 = \alpha, h_1 = \beta$ and h_t is a constant homotopy on I^n_0, i.e. is rel vertices.

Lax and strict ω-groupoids

Define $\rho X_* = (RX_*)/\equiv$.

Theorem (Brown-Higgins 1981)

The compositions on RX_* are inherited to make ρX_* a strict cubical ω-groupoid with connections, and the projection

$$RX_* \to \rho X_*$$

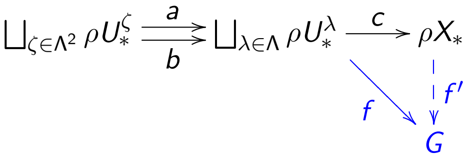
is a cubical Kan fibration.

Proof uses collapsing methods (on cubes) due to Henry Whitehead.
We need strict structures to calculate exactly using strict colimits!

We prove the following is a coequaliser:

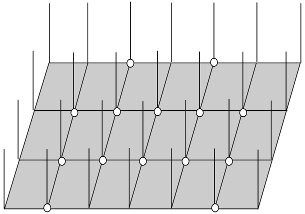
$$\bigsqcup_{\zeta\in\Lambda^2} \rho U_*^{\zeta} \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho U_*^{\lambda} \xrightarrow{c} \rho X_*$$

under the connectivity assumptions (and we also need the connections!).
Proof goes by verifying the universal property of a coequaliser!



The whole point is the exquisite match between the geometry from ρX_* and the algebra of the cubical ω-groupoid G.
The proof does not assume colimits exist for cubical ω-groupoids.

Reason for the connectivity conditions:
If $\alpha : I^n_* \to X_*$ is a filtered map, then under subdivision, the induced map $\alpha_{(r)}$ of a little cube maps into X_n but is not usually a filtered map. So it has to be deformed using the connectivity condition.
Here is a sample picture:



The harder part is proving independence of choices made. A key aspect is the connections defined on RX_* using the monoid structure max on I; this gives new kinds of 'degenerate' cubes given by operators

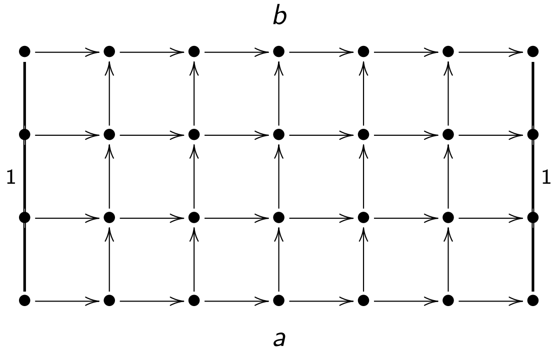
$$\Gamma_i : R_n X_* \to R_{n+1} X_*.$$

An element of $\rho_n X_*$ is algebraically thin if it is a multiple composition of elements of the form of repeated negatives $-_i$ of degenerate elements ε_i or Γ_j; it is geometrically thin if it has a representative α such that $\alpha(I^n) \subseteq X_{n-1}$.

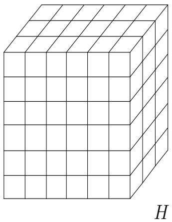
Theorem

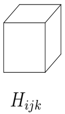
Algebraically thin is equivalent to geometrically thin.

Key point: given a thin homotopy $h\colon \alpha \equiv \beta$ where $\alpha, \beta \in R_n X_*$:
then $h : I^n \times I \to X$ s.t. $h(I^n_r \times I) \subseteq X_r$ for $0 \leqslant r \leqslant n$.
Subdivide $I^n \times I$ into lots of $(n+1)$-cubes each contained in a set of $\mathcal{U}$ and then use the connectivity conditions to deform h to another homotopy $h' : \alpha \equiv \beta$ whose class in $\rho_{n+1} X_*$ is a composite of thin elements.
So the image in G is thin and has similar properties to those of the class of h; that turns out to be enough.



The problem is to lift this argument to dimensions 2 and higher!


H


H_{ijk}

Further directions

1) Monoidal closed structure on Crs.
2) Crossed complexes as a related but more powerful tool than chain complexes with a group(oid) of operators.
3) Classifying space $\mathbb{B} : \text{Crs} \to FTop$, and homotopy classification results: X a CW-complex, C a crossed complex implies

$$[X, BC] \cong [\pi_* X_*, C].$$

Non simply connected homotopy classification theorem. Includes work of Whitehead, Olum.
4) Multifiltered spaces, n-cubes of spaces, n-adic SvKT and Hurewicz Theorems (RB/Loday).
5) Nonabelian colimit calculations of some homotopy n-types.

Conclusion

New developments in algebraic topology (Hopf formula) led to the development of homological algebra.

There are still questions as to what is and what should be nonabelian homological algebra. Maybe this should start from further developments of nonabelian algebraic topology.

In a letter dated 02/05/1983 Alexander Grothendieck wrote:

Don't be surprised by my supposed efficiency in digging out the right kind of notions–I have just been following, rather let myself be pulled ahead, by that very strong thread (roughly: understand non commutative cohomology of topoi!) which I kept trying to sell for about ten or twenty years now, without anyone ready to "buy" it, namely to do the work. So finally I got mad and decided to work out at least an outline by myself.