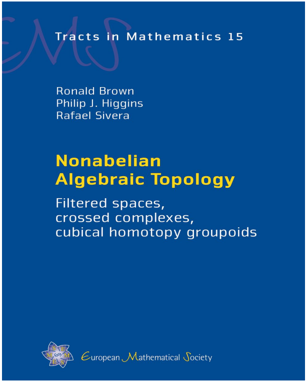
Filtered spaces crossed complexes and cubical higher homotopy groupoids: a new foundation for algebraic topology

Ronnie Brown

March 28, 2011
Tbilisi
Conference on Homotopy Theory and Non Commutative Geometry



xxxiii+ 643pp

To appear (2011)

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as their interaction has been very much my pursuit for 45 years and still is so.

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Evaluation! Analysis!
Einstein (1917): What is essential and what is based only on the accidents of development? ...... It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little...

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when $M = K \cap L$ is connected and $(K, M), (L, M)$ are 1-connected and cofibred;
8) numerous explicit calculations of homotopy 2-types given by crossed modules, unobtainable otherwise;

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Axioms are those universally satisfied by this example!

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THINK: skeletal filtration of a CW-complex.

Standard use of coequalisers

Let X_* be a filtered space, and $\mathcal{U} = \{U^\lambda \mid \lambda \in \Lambda\}$ an open cover of X. For $\zeta \in \Lambda^n$ let

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$\pi_* : \mathsf{FTop} \to \mathsf{Crs}$ commutes with $\bigsqcup$, disjoint union.

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Against all traditions of algebraic topology??!!

The HHSvKT includes nonabelian information in dimensions 1 and 2, and information on operations of the fundamental group(oid) on relative homotopy groups. It is convenient for a certain range of calculations.

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It took 12 years and collaborations with Chris Spencer and Philip Higgins to get the above HHSvK Theorem!

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which takes ρX_* to $\pi_* X_*$.

From weak structures to strict

X_* be a filtered space,
I^n_* the filtered space of the standard n-cube.
$R_n X_* = \mathsf{FTop}(I^n_*, X_*)$.
Then RX_* becomes a cubical set with composition.
$\alpha, \beta \in R_n X_*$: a thin homotopy $h_t : \alpha \equiv \beta$ is a map
$h : I^n \times I \to X$ such that
h_t is a filtered map, $h_0 = \alpha, h_1 = \beta$ and h_t is a constant homotopy on I^n_0, i.e. is rel vertices.

Lax and strict ω-groupoids

Define $\rho X_* = (RX_*)/\equiv$.

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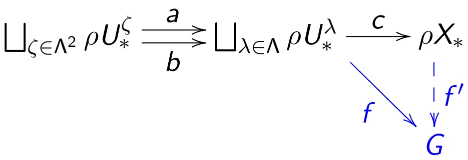
We need strict structures to calculate exactly using strict colimits!

We prove the following is a coequaliser:

$$\bigsqcup_{\zeta \in \Lambda^2} \rho U_*^{\zeta} \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \rho U_*^{\lambda} \xrightarrow{c} \rho X_*$$

under the connectivity assumptions (and we also need the connections!).

Proof goes by verifying the universal property of a coequaliser!



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The proof does not assume colimits exist for cubical ω-groupoids.

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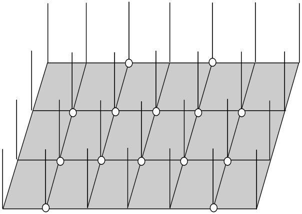

The harder part is proving independence of choices made. A key aspect is the connections defined on RX_* using the monoid structure max on I; this gives new kinds of 'degenerate' cubes given by operators

$$\Gamma_i : R_n X_* \to R_{n+1} X_*.$$

An element of $\rho_n X_*$ is algebraically thin if it is a multiple composition of elements of the form of repeated negatives $-_i$ of degenerate elements ε_i or Γ_j; it is geometrically thin if it has a representative α such that $\alpha(I^n) \subseteq X_{n-1}$.

Theorem

Algebraically thin is equivalent to geometrically thin.

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Subdivide $I^n \times I$ into lots of $(n+1)$-cubes each contained in a set of $\mathcal{U}$ and then use the connectivity conditions to deform h to another homotopy $h' : \alpha \equiv \beta$ whose class in $\rho_{n+1} X_*$ is a composite of thin elements.

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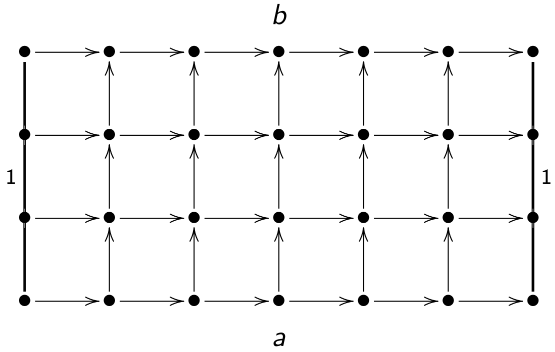
So the image in G is thin and has similar properties to those of the class of h; that turns out to be enough.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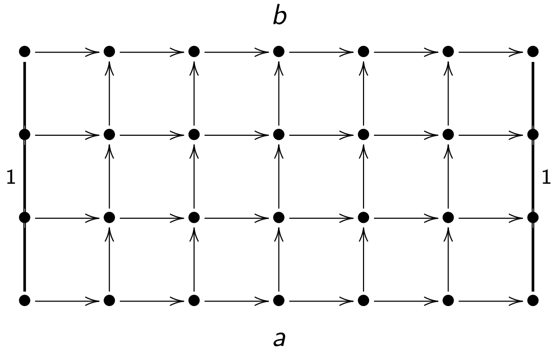
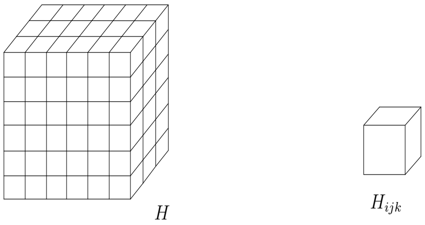


The problem is to lift this argument to dimensions 2 and higher!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5) Nonabelian colimit calculations of some homotopy n-types.

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In a letter dated 02/05/1983 Alexander Grothendieck wrote:
Don't be surprised by my supposed efficiency in digging out the right kind of notions–I have just been following, rather let myself be pulled ahead, by that very strong thread (roughly: understand non commutative cohomology of topoi!) which I kept trying to sell for about ten or twenty years now, without anyone ready to "buy" it, namely to do the work. So finally I got mad and decided to work out at least an outline by myself.