Applications of a non abelian tensor product of groups

Ronnie Brown

Göttingen, May 5, 2011

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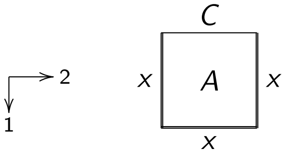

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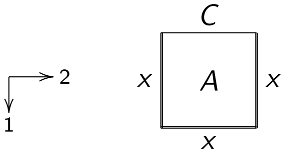

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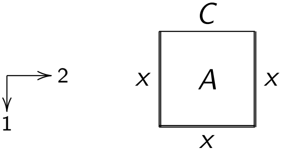

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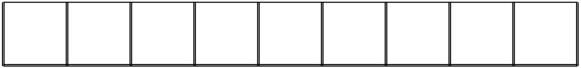

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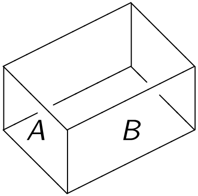

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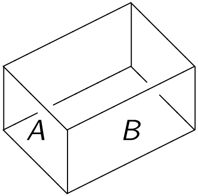

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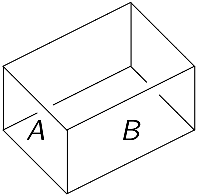

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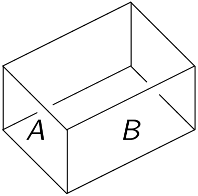

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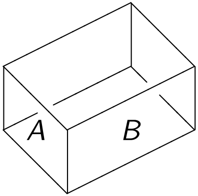

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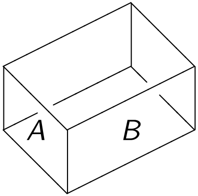

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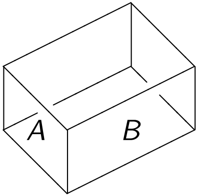

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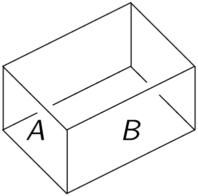

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Theorem (Blakers-Massey, 1953)

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Currently, homotopy theory tries to move away from the fundamental group and nonabelian group methods.

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I.e., bimultiplicative maps are boring!

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Mystic statement: Here be groupoids!

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Grothendieck liked to call this integration of homotopy types.

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Corollary If M is a group, then

$$\pi_3 SK(M, 1) \cong \mathrm{Ker}(M \otimes M \to M).$$

This was the first time this homotopy group was calculated!
Proof of Corollary: Exact sequences of triads and pairs.

Example Calculation:

$M = N = D_{2n}$ the dihedral group of order $2n$ with presentation $\langle x, y : x^n, y^2, xyxy \rangle$. Then $M \otimes M$ is isomorphic to:

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For n even the elements

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This calculation by hand. Lots more calculations available, some by computer (see bibliography). The case M is infinite and non commutative is quite hard (L-C. Kappe and students).

For more computer calculations see Graham Ellis:

http://hamilton.nuigalway.ie/Hap/www/SideLinks/About/aboutTensorSquare.html

Back to formalities: Expand $mm' \otimes nn'$ in two ways? After some reduction and manipulation you get

$$({}^{mn}m' \otimes {}^{mn}n')(m \otimes n) = (m \otimes n)({}^{nm}m' \otimes {}^{nm}n').$$

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Crossed squares may be thought of as 3-dimensional groups!

Back to formalities: Expand $mm' \otimes nn'$ in two ways? After some reduction and manipulation you get

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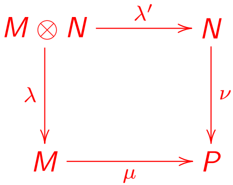

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for all $l \in L, m, m' \in M, n, n' \in N, p \in P$.

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This is related to the pushout of squares of spaces when $X = A \cup B, C = A \cap B$:

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Also true for homotopy n-types! (Loday, 1982).

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This is the start of work of Ellis to obtain many new results on the classical Schur Multiplier.

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We have given examples of precise higher dimensional nonabelian methods for local-to-global problems.

Additional points were made on the blackboard and in answer to questions.

1. The usual idea is that we want invariants of topological spaces. However a space has to be specified in some way, by some kind of data, and this data usually has some kind of structure. It can be expected that this structure is reflected somehow in the space. So we should have invariants of structured spaces and these should lead to structured algebraic invariants.

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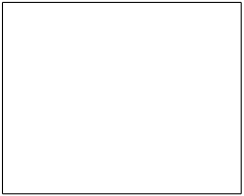

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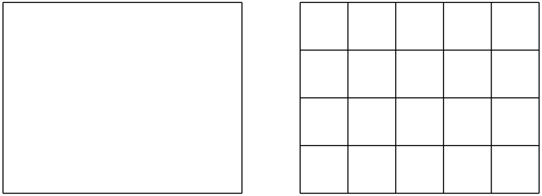

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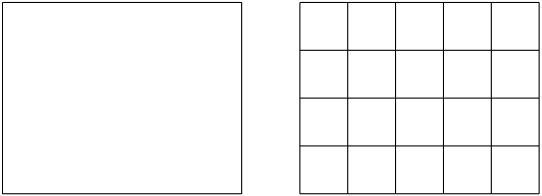

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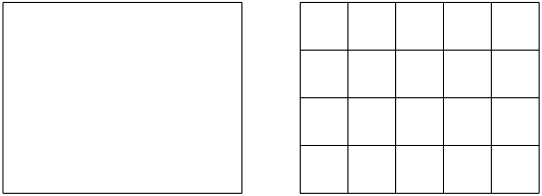

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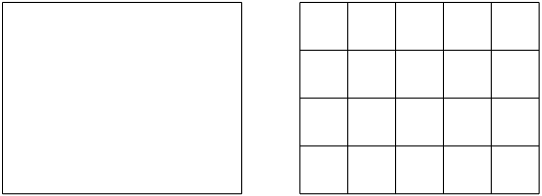

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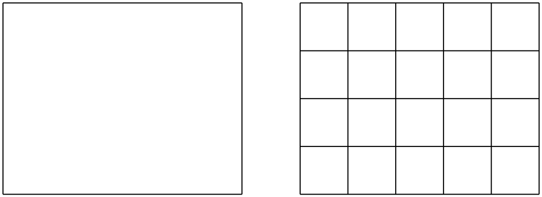

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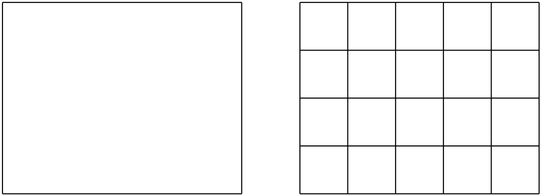

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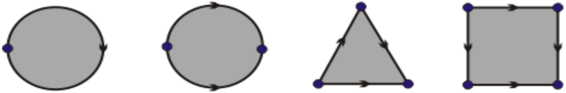
What we need for local-to-global problems is:
Algebraic inverses to subdivision.
We know how to cut things up, but how to control algebraically putting them together again?
These figures suggest the advantage of a cubical approach.

3. In moving from 1-dimensional compositions to higher dimensional ones it seems to be necessary to choose the basic geometric objects. But there are an infinite number of compact convex subsets of $\mathbb{R}^n$ for $n \geqslant 2$. With some cell structure they may be seen for $n = 2$ as cell, globe, simplex, cube, etc. It is common to see higher category theory in a globular, or sometimes simplicial context, but we use mainly a cubical approach.



4. The term 'biderivation' may also be used in the context of Lie algebras, and then the bracket of the Lie algebra

$$[\ ,\ ] : L \times L \to L$$

is seen as a biderivation. This leads to a nonabelian tensor product for Lie algebras.

5. In the calculation of $D_{2n} \otimes D_{2n}$ for n even, the elements $y \otimes y, x \otimes x$ represent composition with the Hopf map, and $(x \otimes y)(y \otimes x)$ represents a Whitehead product, when x, y and these tensor products are interpreted in π_2, π_3 of $SK(D_{2n}, 1)$.

6. If $x \in A \cap B \subseteq A, B \subseteq X$ it is natural to consider the space Φ of maps $I^2 \to X$ which map the vertices to x, edges in direction 1 to A, and edges in direction 2 to B. This set also has compositions $\circ_1, \circ_2$ in 2 directions making it a weak double category. However it seems that this structure is not inherited by $\pi_0\Phi$ except under extra conditions, for example that the images of $\pi_2(A, x), \pi_2(B, x)$ in $\pi_2(X, x)$ coincide, which happens of course if $A = B$. However the fundamental group $\pi_1(\Phi, *)$ does inherit these compositions to become a cat^2-group. This and its generalisations are due to J.-L. Loday, 1982.