ELSEVIER

Topology and its Applications 115 (2001) 125–138

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Lie local subgroupoids and their holonomy and monodromy Lie groupoids

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Received 6 August 1998; received in revised form 18 February 2000

Abstract

The notion of local equivalence relation on a topological space is generalized to that of local subgroupoid. The main result is the construction of the holonomy and monodromy groupoids of certain Lie local subgroupoids, and the formulation of a monodromy principle on the extendability of local Lie morphisms. © 2001 Elsevier Science B.V. All rights reserved.

AMS classification: 58H05; 22A22; 18F20

Keywords: Local equivalence relation; Local subgroupoid; Holonomy groupoid; Monodromy groupoid; Monodromy principle

Introduction

It has long been recognized that the notion of Lie group is inadequate to express the local-to-global ideas inherent in the investigations of Sophus Lie, and various extensions have been developed, particularly the notion of Lie groupoid, in the hands of Ehresmann, Pradines, and others.

Another set of local descriptions have been given in the notion of foliation (due to Ehresmann) and also in the notion of local equivalence relation (due to Grothendieck and Verdier).

Pradines in [16] also introduced the notion of what he called 'morceau d'un groupoïde de Lie' and which we have preferred to call 'locally Lie groupoid' in [5]. This is a groupoid G with a subset W of G containing the identities of G and with a manifold structure on W making the structure maps 'as smooth as possible'. It is a classical result that in the case

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PII: S0166-8641(00)00062-6

G is a group the manifold structure can be transported around G to make G a Lie group. This is false in general for groupoids, and this in fact gives rise to the *holonomy groupoid* for certain such (G, W).

In [6] it is shown that a foliation on a paracompact manifold gives rise to a locally Lie groupoid. It is part of the theory of Lie groupoids that a Lie algebroid gives rise, under certain conditions, to a locally Lie groupoid. Thus a locally Lie groupoid is one of the ways of giving a useful expression of local-to-global structures.

The notion of *local equivalence relation* was introduced by Grothendieck and Verdier [10] in a series of exercises presented as open problems concerning the construction of a certain kind of topos. It was investigated further by Rosenthal [17,18] and more recently by Kock and Moerdijk [13,14]. A local equivalence relation is a global section of the sheaf $\mathcal{E}$ defined by the presheaf E where $E(U)$ is the set of all equivalence relations on the open subsets U of X, and E_{UV} is the restriction map from $E(U)$ to $E(V)$ for $V \subseteq U$. The main aims of the papers [10,13,14,17,18] are towards the connections with sheaf theory and topos theory. Any foliation gives rise to a local equivalence relation, defined by the path components of local intersections of small open sets with the leaves.

An equivalence relation on a set U is just a wide subgroupoid of the indiscrete groupoid $U \times U$ on U. Thus it is natural to consider the generalization which replaces the indiscrete groupoid on the topological space X by any groupoid Q on X. So we define a *local subgroupoid* of the groupoid Q to be a global section of the sheaf $\mathcal{L}$ associated to the presheaf L_Q where $L(U)$ is the set of all wide subgroupoids of $Q|U$ and L_{UV} is the restriction map from $L(U)$ to $L(V)$ for $V \subseteq U$. Examples of local subgroupoids, generalizing the foliation example, are given in [4].

Our aim is towards local-to-global principles and in particular the monodromy principle, which in our terms is formulated as the globalization of local morphisms (compare [7,16, 5]). Our first formulation is for the case Q has no topology, and this gives our 'weak monodromy principle' (Theorem 2.3).

In the case Q is a Lie groupoid we expect to deal with Lie local subgroupoids s and the globalization of local smooth morphisms to a smooth morphism $M(s) \to K$ on a 'monodromy Lie groupoid' $M(s)$ of s. The construction of the Lie structure on $M(s)$ requires extra conditions on s and its main steps are:

- the construction of a locally Lie groupoid from s and a strictly regular atlas for s,
- applying the construction of the holonomy Lie groupoid of the locally Lie groupoid, as in [1,5],
- the further construction of the monodromy Lie groupoid, as in [6].

For strictly regular atlases $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ for s this leads to a morphism of Lie groupoids

$$\zeta : Mon(s, \mathcal{U}_s) \to Hol(s, \mathcal{U}_s)$$

each of which contains the $H_i,\ i \in I$, as Lie subgroupoids, and which are in a certain sense maximal and minimal, respectively for this property. This morphism ζ is étale on stars. Further, a smooth local morphism $\{f_i : H_i \to K,\ i \in I\}$ to a Lie groupoid K extends uniquely to a smooth morphism $Mon(s, \mathcal{U}_s) \to K$. This is our strong monodromy principle (Theorem 3.13).

It should be noticed that this route to a monodromy Lie groupoid is different from that commonly taken in the theory of foliations. For a foliation $\mathcal{F}$ it is possible to define the monodromy groupoid as the union of the fundamental groupoids of the leaves, and then to take the holonomy groupoid as a quotient groupoid of this, identifying classes of paths which induce the same holonomy.

However there seem to be strong advantages in seeing these holonomy and monodromy groupoids as special cases of much more general constructions, in which the distinct universal properties become clear. In particular, this gives a link between the monodromy groupoid and the important monodromy principle, of extendability of local morphisms. In the Lie case, this requires moving away from the étale groupoids which is the main emphasis in [13,14].

We plan to investigate elsewhere the relation of these ideas to questions on fibre bundles and transformation groups.

We would like to thank a referee for helpful comments.

1. Local subgroupoids

Consider a groupoid Q on a set X of objects, and suppose also X has a topology. For any open subset U of X we write $Q|U$ for the full subgroupoid of Q on the object set U. Let $L_Q(U)$ denote the set of all wide subgroupoids of $Q|U$. For $V \subseteq U$, there is a restriction map $L_{UV} : L_Q(U) \to L_Q(V)$ sending H in $L_Q(U)$ to $H|V$. This gives L_Q the structure of presheaf on X.

We first interpret in our case the usual construction of the sheaf $p_Q : \mathcal{L}_Q \to X$ constructed from the presheaf L_Q.

For $x \in X$, the stalk $p_Q^{-1}(x)$ of $\mathcal{L}_Q$ has elements the germs $[U, H_U]_x$ where U is open in X, $x \in U$, H_U is a wide subgroupoid of $Q|U$, and the equivalence relation $\sim_x$ yielding the germs at x is that $H_U \sim_x K_V$, where K_V is wide subgroupoid of $Q|V$, if and only if there is a neighbourhood W of x such that $W \subseteq U \cap V$ and $H_U|W = K_V|W$.

Definition 1.1. A *local subgroupoid* of Q on the topological space X is a global section of the sheaf $p_Q : \mathcal{L}_Q \to X$ associated to the presheaf L_Q.

An *atlas* $\mathcal{U}_s = \{(U_i, H_i): i \in I\}$ for a local subgroupoid s of Q consists of an open cover $\mathcal{U} = \{U_i: i \in I\}$ of X, and for each $i \in I$ a wide subgroupoid H_i of $Q|U_i$ such that for all $x \in X$, $i \in I$, if $x \in U_i$ then $s(x) = [U_i, H_i]_x$.

Two standard examples of Q are $Q = X$, $Q = X \times X$. In the first case, L_X is a sheaf and $\mathcal{L}_X \to X$ is a bijection. In the case Q is the indiscrete groupoid $X \times X$ with multiplication $(x, y)(y, z) = (x, z)$, $x, y, z \in X$, the local subgroupoids of Q are the local equivalence relations on X, as mentioned in the Introduction. It is known that $L_{X\times X}$ is in general not a sheaf [17].

In the following, we show that many of the basic results obtained by Rosenthal in [17, 18] extend conveniently to the local subgroupoid case.

The set $L_Q(X)$ of wide subgroupoids of Q is a poset under inclusion. We write $\leqslant$ for this partial order.

Let $\mathrm{Loc}(Q)$ be the set of local subgroupoids of Q. We define a partial order $\leqslant$ on $\mathrm{Loc}(Q)$ as follows.

Let $x \in X$. We define a partial order on the stalks $p_Q^{-1}(x) = \mathcal{L}_x^Q$ by $[U', H']_x \leqslant [U, H]_x$ if there is an open neighbourhood W of x such that $W \subseteq U \cap U'$ and $H'|W$ is a subgroupoid of $H|W$. Clearly this partial order is well defined. It induces a partial order on $\mathrm{Loc}(Q)$ by $s \leqslant t$ if and only if $s(x) \leqslant t(x)$ for all $x \in X$.

We now fix a groupoid Q on X, so that $L_Q(X)$ is the set of wide subgroupoids of Q, with its inclusion partial order, which we shall write $\leqslant$.

We define poset morphisms

$$loc_Q : L_Q(X) \to \mathrm{Loc}(Q) \quad \text{and} \quad glob_Q : \mathrm{Loc}(Q) \to L_Q(X)$$

as follows. We abbreviate loc_Q, $glob_Q$ to $loc, glob$.

Definition 1.2. If H is a wide subgroupoid of the groupoid Q on X, then $loc(H)$ is the local subgroupoid defined by

$$loc(H)(x) = [X, H]_x.$$

Let s be a local subgroupoid of Q. Then $glob(s)$ is the wide subgroupoid of Q which is the intersection of all wide subgroupoids H of Q such that $s \leqslant loc(H)$.

We think of $glob(s)$ as an approximation to s by a global subgroupoid.

Proposition 1.3.
(i) *loc and glob are morphisms of posets.*
(ii) *For any wide subgroupoid H of Q, $glob(loc(H)) \leqslant H$.*

The proofs are clear.

However, $s \leqslant loc(glob(s))$ need not hold. Examples of this are given in Rosenthal's paper [17] for the case of local equivalence relations.

Here is an alternative description of *glob*. Let $\mathcal{U}_s = \{(U_i, H_i)\colon i \in I\}$ be an atlas for the local subgroupoid s. We define $glob(\mathcal{U}_s)$ to be the subgroupoid of Q generated by all the H_i, $i \in I$.

An atlas $\mathcal{V}_s = \{(V_j, s_j)\colon j \in J\}$ for s is said to refine $\mathcal{U}_s$ if for each index $j \in J$ there exists an index $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$ and $s_{i(j)}|V_j = s_j$.

Proposition 1.4. *Let s be a local subgroupoid of Q given by the atlas $\mathcal{U}_s = \{(U_i, H_i)\colon i \in I\}$. Then $glob(s)$ is the intersection of the subgroupoids $glob(\mathcal{V}_s)$ of Q for all refinements $\mathcal{V}_s$ of $\mathcal{U}_s$.*

Proof. Let K be the intersection given in the proposition.

Let S be a subgroupoid of Q on X such that $s \leqslant loc(S)$. Then for all $x \in X$ there is a neighbourhood V of x and $i_x \in I$ such that $x \in U_{i_x}$ and $H_{i_x}|V_x \cap U_{i_x} \leqslant S$. Then $\mathcal{W} = \{(V_x \cap U_{i_x}, H_{i_x}|V_x \cap U_{i_x})\colon x \in X\}$ refines $\mathcal{U}_s$ and $glob(\mathcal{W}) \leqslant S$. Hence $K \leqslant S$, and so $K \leqslant glob(s)$.

Conversely, let $\mathcal{V}_s = \{(V_j, H'_j)\colon j \in J\}$ be an atlas for s which refines $\mathcal{U}_s$. Then for each $j \in J$ there is an $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$, $H'_j = H_{i(j)}|V_j$. Then $s \leqslant loc(glob(\mathcal{V}_s))$. Hence $glob(s) \leqslant glob(\mathcal{V}_s)$ and so $glob(s) \leqslant K$. □

We need the next definition in the following sections.

Definition 1.5. Let s be a local subgroupoid of the groupoid Q on X. An atlas $\mathcal{U}_s$ for s is called *globally adapted* if $glob(s) = glob(\mathcal{U}_s)$.

Remark 1.6. This is a variation on the notion of an r-adaptable family defined by Rosenthal in [18, Definition 4.4] for the case of a local equivalence relation r. He also imposes a connectivity condition on the local equivalence classes.

2. The weak monodromy principle for local subgroupoids

Let s be a local subgroupoid of Q which is given by an atlas $\mathcal{U}_s = \{(U_i, H_i)\colon i \in I\}$, and let $H = glob(s)$, $W(\mathcal{U}_s) = \bigcup_{i \in I} H_i$. Then $W(\mathcal{U}_s) \subseteq H$.

The set $W(\mathcal{U}_s)$ inherits a *pregroupoid* structure from the groupoid H. That is, the source and target maps α, β restrict to maps on $W(\mathcal{U}_s)$, and if $u, v \in W(\mathcal{U}_s)$ and $\beta u = \alpha v$, then the composition uv of u, v in H may or may not belong to $W(\mathcal{U}_s)$. We now follow the method of Brown and Mucuk in [5], which generalizes work for groups in Douady and Lazard [8].

There is a standard construction $M(W(\mathcal{U}_s))$ associating to the pregroupoid $W(\mathcal{U}_s)$ a morphism $\tilde{\imath} : W(\mathcal{U}_s) \to M(W(\mathcal{U}_s))$ to a groupoid $M(W(\mathcal{U}_s))$ and which is universal for pregroupoid morphisms to a groupoid. First, form the free groupoid $F(W(\mathcal{U}_s))$ on the graph $W(\mathcal{U}_s)$, and denote the inclusion $W(\mathcal{U}_s) \to F(W(\mathcal{U}_s))$ by $u \mapsto [u]$. Let N be the normal subgroupoid (Higgins [11], Brown [2]) of $F(W(\mathcal{U}_s))$ generated by the elements $[vu]^{-1}[v][u]$ for all $u, v \in W(\mathcal{U}_s)$ such that vu is defined and belongs to $W(\mathcal{U}_s)$. Then $M(W(\mathcal{U}_s))$ is defined to be the quotient groupoid (loc. cit.) $F(W(\mathcal{U}_s))/N$. The composition $W(\mathcal{U}_s) \to F(W(\mathcal{U}_s)) \to M(W(\mathcal{U}_s))$ is written $\tilde{\imath}$, and is the required universal morphism.

There is a unique morphism of groupoids $p : M(W(\mathcal{U}_s)) \to glob(s)$ such that $p\tilde{\imath}$ is the inclusion $i : W(\mathcal{U}_s) \to glob(s)$. It follows that $\tilde{\imath}$ is injective. Clearly, p is surjective if and only if the atlas for s is globally adapted. In this case, we call $M(W(\mathcal{U}_s))$ the *monodromy groupoid* of $W(\mathcal{U}_s)$ and write it $Mon(s, \mathcal{U}_s)$.

Definition 2.1. The local subgroupoid s is called *simply connected* if it has a globally adapted atlas $\mathcal{U}_s$ such that the morphism $p : Mon(s, \mathcal{U}_s) \to glob(s)$ is an isomorphism.

We now relate $Mon(s, \mathcal{U}_s)$ to the extendability of local morphisms to a groupoid K.

Let K be a groupoid with object space X.

Definition 2.2. A *local morphism* $f:\mathcal{U}_s \to K$ consists of a globally adapted atlas $\mathcal{U}_s = \{(U_i, H_j)\colon i \in I\}$ for s and a family of morphisms $f_i : H_i \to K,\ i \in I$ over the inclusion $U_i \to X$ such that for all $i, j \in I$,

$$f_i|(H_i \cap H_j) = f_j|(H_i \cap H_j),$$

and the resulting function $f' : W(\mathcal{U}_s) \to K$ is a pregroupoid morphism.

Theorem 2.3 (Weak monodromy principle). *A local morphism $f:\mathcal{U}_s \to K$ defines uniquely a groupoid morphism $M(f):Mon(s,\mathcal{U}_s) \to K$ over the identity on objects such that $M(f)|H_i = f_i,\ i \in I$. Further, if s is simply connected, then the (f_i) determine a groupoid morphism $glob(s) \to K$.*

Proof. The proof is direct from the definitions. A local morphism f defines a pregroupoid morphism $f' : W(\mathcal{U}_s) \to K$ which therefore defines $M(f):Mon(s,\mathcal{U}_s) \to K$ by the universal property of $W(\mathcal{U}_s) \to Mon(s,\mathcal{U}_s)$. □

In the next section, we will show how to extend this result to the Lie case. This involves discussing the construction of a topology on $Mon(s,\mathcal{U}_s)$ under the given conditions. For this we follow the procedure of Brown–Mucuk in [5] in using the construction and properties of the holonomy groupoid of a locally Lie groupoid. This procedure is in essence due to Pradines, and was announced without detail in [16]. As explained in the preliminary preprint [3] these details were communicated by Pradines to Brown in the 1980s.

3. Local Lie subgroupoids, holonomy and monodromy

The aim of this section is to give sufficient conditions on local subgroupoid s of G for the monodromy groupoid of s to admit the structure of a Lie groupoid, so that the globalization $f:Mon(s,\mathcal{U}_s) \to K$ of a local smooth morphism $f_i : H_i \to K$, $i \in I$, is itself smooth. As explained in the Introduction, our method follows [5] in first constructing a locally Lie groupoid $(glob(s), W(\mathcal{U}_s))$; the holonomy Lie groupoid of this locally Lie groupoid comes with a morphism of groupoids $\psi : Hol(glob(s), W(\mathcal{U}_s)) \to glob(s)$ which is a minimal smooth overgroupoid of $glob(s)$ containing $W(\mathcal{U}_s)$ as an open subspace. From this holonomy Lie groupoid we construct the Lie structure on the monodromy groupoid. We begin therefore by recalling the holonomy groupoid construction.

We consider $\mathcal{C}^r$-manifolds for $r \geqslant -1$. Here a $\mathcal{C}^{-1}$-manifold is simply a topological space and for $r = -1$, a smooth map is simply a continuous map. Thus the Lie groupoids in the $\mathcal{C}^{-1}$ case will simply be the topological groupoids. For $r = 0$, a $\mathcal{C}^0$-manifold is as usual a topological manifold, and a smooth map is just a continuous map. For $r \geqslant 1$, $r = \infty,\ \omega$ the definition of $\mathcal{C}^r$-manifold and smooth map are as usual. We now fix $r \geqslant -1$.

One of the key differences between the cases $r = -1$ or 0 and $r \geqslant 1$ is that for $r \geqslant 1$, the pullback of $\mathcal{C}^r$ maps need not be a smooth submanifold of the product, and so differentiability of maps on the pullback cannot always be defined. We therefore adopt the following definition of Lie groupoid. Mackenzie [15, pp. 84–86] discusses the utility of various definitions of differentiable groupoid.

Recall that if G is a groupoid then the difference map on G is $\delta : G \times_\alpha G \to G$, $(g, h) \mapsto g^{-1}h$.

A *Lie groupoid* is a topological groupoid G such that

(i) the space of arrows is a smooth manifold, and the space of objects is a smooth submanifold of G,
(ii) the source and target maps α, β, are smooth maps and are submersions,
(iii) the domain $G \times_\alpha G$ of the difference map δ is a smooth submanifold of $G \times G$,
(iv) the difference map δ is a smooth map.

The term *locally Lie groupoid* (G, W) is defined later.

The following definition is due to Ehresmann [9].

Definition 3.1. Let G be a groupoid and let $X = O_G$ be a smooth manifold. An *admissible local section* of G is a function $\sigma : U \to G$ from an open set in X such that

(i) $\alpha\sigma(x) = x$ for all $x \in U$;
(ii) $\beta\sigma(U)$ is open in X; and
(iii) $\beta\sigma$ maps U diffeomorphically to $\beta\sigma(U)$.

Let W be a subset of G and let W have the structure of a smooth manifold such that X is a submanifold. We say that (α, β, W) is *locally sectionable* if for each $w \in W$ there is an admissible local section $\sigma : U \to G$ of G such that

(i) $\sigma\alpha(w) = w$,
(ii) $\sigma(U) \subseteq W$ and
(iii) σ is smooth as a function from U to W.

Such a σ is called a *smooth admissible local section*.

The following definition is due to Pradines [16] under the name "*morceau de groupoide différentiables*".

Definition 3.2. A *locally Lie groupoid* is a pair (G, W) consisting of a groupoid G and a smooth manifold W such that:

(G1) $O_G \subseteq W \subseteq G$;
(G2) $W = W^{-1}$;
(G3) the set $W(\delta) = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction of δ to $W(\delta)$ is smooth;
(G4) the restrictions to W of the source and target maps α and β are smooth and the triple (α, β, W) is locally sectionable;
(G5) W generates G as a groupoid.

Note that in this definition, G is a groupoid but does not need to have a topology. The locally Lie groupoid (G, W) is said to be *extendable* if there can be found a topology on G making it a Lie groupoid and for which W is an open submanifold. In general,

(G, W) is not extendable, but there is a holonomy groupoid $Hol(G, W)$ and a morphism $\psi : Hol(G, W) \to G$ such that $Hol(G, W)$ admits the structure of Lie groupoid and is the "minimal" such overgroupoid of G. The construction is given in detail in [1] and is outlined below.

Definition 3.3. A *Lie local subgroupoid* s of a Lie groupoid Q is a local subgroupoid s given by an atlas $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ such that for $i \in I$ each H_i is a Lie subgroupoid of Q.

We know from examples for foliations and hence for local equivalence relations that $glob(s)$ need not be a Lie subgroupoid of Q [6]. Our aim is to define a holonomy groupoid $Hol(s, \mathcal{U}_s)$ which is a Lie groupoid.

We now adapt some definitions from [18].

Definition 3.4. An atlas $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ for a Lie local subgroupoid s of Q is said to be *regular* if the groupoid (α_i, β_i, H_i) is locally sectionable for all $i \in I$. A Lie local subgroupoid s is *regular* if it has a regular atlas.

Definition 3.5. An atlas $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ for a Lie local subgroupoid s is said to be *strictly regular* if

(i) $\mathcal{U}_s$ is globally adapted to s,

(ii) $\mathcal{U}_s$ is regular,

(iii) $W(\mathcal{U}_s)$ has with its topology as a subset of Q the structure of smooth submanifold containing each H_i, $i \in I$, as an open submanifold of $W(\mathcal{U}_s)$ and such that $W(\mathcal{U}_s)(\delta)$ is open in $W(\mathcal{U}_s) \times_\alpha W(\mathcal{U}_s)$.

A Lie local subgroupoid s is *strictly regular* if it has a strictly regular atlas.

Remark 3.6. The main result of [6] is that the local equivalence relation defined by a foliation on a paracompact manifold has a strictly regular atlas.

The following is a key construction of a locally Lie groupoid from a strictly regular Lie local subgroupoid.

Theorem 3.7. *Let Q be a Lie groupoid on X and let $\mathcal{U}_s = \{(H_i, U_i)\colon\ i \in I\}$ be a strictly regular atlas for the Lie local subgroupoid s of Q. Let*

$$G = glob(s), \qquad W(\mathcal{U}_s) = \bigcup_{i \in I} H_i.$$

Then $(G, W(\mathcal{U}_s))$ admits the structure of a locally Lie groupoid.

Proof. (G1) By the definition of G and $W(\mathcal{U}_s)$, clearly $X \subseteq W(\mathcal{U}_s) \subseteq H$.

(G2) In fact, $W(\mathcal{U}_s) = W(\mathcal{U}_s)^{-1}$. Let $g \in W(\mathcal{U}_s)$. Then there is an index $i \in I$ such that $g \in H_i$. Since H_i is a groupoid on U_i, $g^{-1} \in H_i$. So $W(\mathcal{U}_s) = W(\mathcal{U}_s)^{-1}$.

(G3) Since s is strictly regular, by definition, $W(\mathcal{U}_s(\delta))$ is open in $W(\mathcal{U}_s) \times_\delta W(\mathcal{U}_s)$.

We now prove the restriction of δ to $W(\mathcal{U}_s)(\delta)$ is smooth.

For each $i \in I$, H_i is a Lie groupoid on U_i and so the difference map

$$\delta_i : H_i \times_\alpha H_i \to H_i$$

is smooth. Because $H_i \subseteq W(\mathcal{U}_s)$, $i \in I$, using the smoothness of the inclusion map $i_{H_i} : H_i \to W(\mathcal{U}_s)$, we get a smooth map

$$i_{H_i} \times i_{H_i} : H_i \times_\alpha H_i \to W(\mathcal{U}_s) \times_\alpha W(\mathcal{U}_s).$$

The restriction of $W(\mathcal{U}_s)(\delta)$ is also smooth, that is,

$$i_{H_i} \times i_{H_i} : H_i \times_\alpha H_i \to W(\mathcal{U}_s)(\delta)$$

is smooth. Then the following diagram is commutative:

$$\begin{array}{ccc} H_i \times_\alpha H_i & \xrightarrow{\delta} & H_i \\ {\scriptstyle i_{H_i} \times i_{H_i}} \downarrow & & \downarrow {\scriptstyle i_{H_i}} \\ W(\mathcal{U}_s)(\delta) & \xrightarrow[\delta]{} & W(\mathcal{U}_s) \end{array}$$

This verifies (G3), since H_i is open in $W(\mathcal{U}_s)$ and hence $H_i \times_\alpha H_i$ is open in $W(\mathcal{U}_s)(\delta)$.

(G4) We define source and target maps $\alpha_{W(\mathcal{U}_s)}$ and $\beta_{W(\mathcal{U}_s)}$, respectively as follows: if $g \in W(\mathcal{U}_s)$ there exist $i \in I$ such that $g \in H_i$ and we let

$$\alpha_{W(\mathcal{U}_s)}(g) = \alpha_i(g), \qquad \beta_{W(\mathcal{U}_s)}(g) = \beta_i(g).$$

Clearly $\alpha_{W(\mathcal{U}_s)}$ and $\beta_{W(\mathcal{U}_s)}$ are smooth. Since $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ is strictly regular, $(\alpha_i, \beta_i, H_i)_{i\in I}$ is locally sectionable for all $i \in I$. Hence $(\alpha_{W(\mathcal{U}_s)}, \beta_{W(\mathcal{U}_s)}, W(\mathcal{U}_s))$ is locally sectionable.

(G5) Since the atlas $\mathcal{U}_s$ is globally adapted to s, then $G = glob(s)$ is generated by the $\{H_i\}$, $i \in I$, and so is also generated by $W(\mathcal{U}_s)$.

Hence $(glob(s), W(\mathcal{U}_s))$ is a locally Lie groupoid. □

There is a main globalization theorem for a locally topological groupoid due to Aof–Brown [1], and a Lie version of this is stated by Brown–Mucuk [5]; it shows how a locally Lie groupoid gives rise to its holonomy groupoid, which is a Lie groupoid satisfying a universal property. This theorem gives a full statement and proof of a part of Théorème 1 of [16]. We can give immediately the generalization to Lie local subgroupoids.

Theorem 3.8 (Globalisability theorem). *Let s be a Lie local subgroupoid of a Lie groupoid Q, and suppose given a strictly regular atlas $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ for s. Let $(glob(s), W(\mathcal{U}_s))$ be the associated locally Lie groupoid. Then there is a Lie groupoid $Hol = Hol(s, \mathcal{U}_s)$, a morphism $\psi : Hol \to glob(s)$ of groupoids and an embedding $i_s : W(\mathcal{U}_s) \to Hol$ of $W(\mathcal{U}_s)$ to an open neighborhood of O_{Hol} in Hol such that the following conditions are satisfied*:

(i) *ψ is the identity on object, $\psi i_s = id_{W(\mathcal{U}_s)}$, $\psi^{-1}(H_i)$ is open in Hol, and the restriction $\psi_{H_i} : \psi^{-1}(H_i) \to H_i$ of ψ is smooth*;

(ii) *Suppose A is a Lie groupoid on $X = Ob(Q)$ and $\xi : A \to glob(s)$ is a morphism of groupoids such that*:
  (a) *ξ is the identity on objects*;
  (b) *for all i the restriction $\xi_{H_i} : \xi^{-1}(H_i) \to H_i$ of ξ is smooth and $\xi^{-1}(H_i)$ is open in A*;
  (c) *the union of the $\xi^{-1}(H_i)$ generates A*;
  (d) *A is locally sectionable*;
  then there is a unique morphism $\xi' : A \to Hol$ of Lie groupoids such that $\psi\xi' = \xi$ and $\xi' h = i\xi h$ for $h \in \xi^{-1}(H_i)$.

The groupoid *Hol* is called the *holonomy groupoid* $Hol(s, \mathcal{U}_s)$ of the Lie local subgroupoid s and atlas $\mathcal{U}_s$.

We outline the proof of which full details are given in [1]. Some details of part of the construction are needed for Proposition 1.

Proof (Outline). Let $G = glob(s)$ and let $\Gamma(G)$ be the set of all admissible local sections of G. Define a product on $\Gamma(G)$ by

$$(ts)x = (t\beta sx)(sx)$$

for two admissible local sections s and t. If s is an admissible local section then write s^{-1} for the admissible local section $\beta s\mathcal{D}(s) \to G,\ \beta sx \mapsto (sx)^{-1}$. With this product $\Gamma(G)$ becomes an inverse semigroup. Let $\Gamma^r(W)$ be the subset of $\Gamma(G)$ consisting of admissible local sections which have values in W and are smooth. Let $\Gamma^r(G, W)$ be the subsemigroup of $\Gamma(G)$ generated by $\Gamma^r(W)$. Then $\Gamma^r(G, W)$ is again an inverse semigroup. Intuitively, it contains information on the iteration of local procedures.

Let $J(G)$ be the sheaf of germs of admissible local sections of G. Thus the elements of $J(G)$ are the equivalence classes of pairs (x, s) such that $s \in \Gamma(G),\ x \in \mathcal{D}(s)$, and (x, s) is equivalent to (y, t) if and only if $x = y$ and s and t agree on a neighbourhood of x. The equivalence class of (x, s) is written $[s]_x$. The product structure on $\Gamma(G)$ induces a groupoid structure on $J(G)$ with X as the set of objects, and source and target maps $[s]_x \mapsto x,\ [s]_x \mapsto \beta sx$. Let $J^r(G, W)$ be the subsheaf of $J(G)$ of germs of elements of $\Gamma^r(G, W)$. Then $J^r(G, W)$ is generated as a subgroupoid of $J(G)$ by the sheaf $J^r(W)$ of germs of elements of $\Gamma^r(W)$. Thus an element of $J^r(G, W)$ is of the form

$$[s]_x = [s_n]_{x_n} \ldots [s_1]_{x_1},$$

where $s = s_n \ldots s_1$ with $[s_i]_{x_i} \in J^r(W), x_{i+1} = \beta s_i x_i, i = 1, \ldots, n$ and $x_1 = x \in \mathcal{D}(s)$.

Let $\psi : J(G) \to G$ be the final map defined by $\psi([s]_x) = s(x)$, where s is an admissible local section. Then $\psi(J^r(G, W)) = G$. Let $J_0 = J^r(W) \cap \ker\psi$. Then J_0 is a normal subgroupoid of $J^r(G, W)$; the proof is the same as in [1, Lemma 2.2]. The holonomy groupoid $H = Hol(G, W)$ is defined to be the quotient $J^r(G, W)/J_0$. Let $p : J^r(G, W) \to H$ be the quotient morphism and let $p([s]_x)$ be denoted by $\langle s\rangle_x$. Since $J_0 \subseteq \ker\psi$ there is a surjective morphism $\phi : H \to G$ such that $\phi p = \psi$.

The topology on the holonomy groupoid *Hol* such that *Hol* with this topology is a Lie groupoid is constructed as follows. Let $s \in \Gamma^r(G, W)$. A partial function $\sigma_s : W \to Hol$ is defined as follows. The domain of σ_s is the set of $w \in W$ such that $\beta w \in \mathcal{D}(s)$. A smooth admissible local section f through w is chosen and the value $\sigma_s w$ is defined to be

$$\sigma_s w = \langle s \rangle_{\beta w} \langle f \rangle_{\alpha w} = \langle sf \rangle_{\alpha w}.$$

It is proven that $\sigma_s w$ is independent of the choice of the local section f and that these σ_s form a set of charts. Then the initial topology with respect to the charts σ_s is imposed on *Hol*. With this topology *Hol* becomes a Lie groupoid. Again the proof is essentially the same as in Aof–Brown [1].

We now outline the proof of the universal property.

Let $a \in A$. The aim is to define $\xi'(a) \in Hol$.

Since $\xi^{-1}(W)$ generates A we can write $a = a_n \dots a_1$ where $\xi(a_i) \in W$ and hence $\xi(a_i) \in H_{i'}$ for some i'. Since A has enough continuous admissible local sections, we can choose continuous admissible local sections f_i of α_A through a_i, $i = 1, \dots, n$, such that they are composable and their images are contained in $\zeta^{-1}(H_{i'})$. The smoothness of ξ on $\xi^{-1}(W)$ implies that ξf_i is a smooth admissible local section of α through $\xi a_i \in H_{i'}$ whose image is contained in $H_{i'}$. Therefore $\xi f \in \Gamma^c(G, W)$. Hence we can set

$$\xi' a = \langle \xi f \rangle_{\alpha a} \in Hol.$$

The major part of the proof is in showing that ξ' is well defined, smooth, and is the unique such morphism. We refer again to [1].

Remark 3.9. The above construction shows that the holonomy groupoid $Hol(G, W)$ depends on the class $\mathcal{C}^r$ chosen, and so should strictly be written $Hol^r(G, W)$. An example of this dependence is given in Aof–Brown [1].

From the construction of the holonomy groupoid we easily obtain the following extendability condition.

Proposition 3.10. *The locally Lie groupoid (G, W) is extendable to a Lie groupoid structure on G if and only if the following condition holds*:

if $x \in O_G$, and s is a product $s_n \dots s_1$ of local sections about x such that each s_i lies in $\Gamma^r(W)$ and $s(x) = 1_x$, then there is a restriction s' of s to a neighbourhood of x such that s' has image in W and is smooth, i.e., $s' \in \Gamma^r(W)$. (1)

Proof. The canonical morphism $\phi : H \to G$ is an isomorphism if and only if $\ker \psi \cap J^r(W) = \ker \psi$. This is equivalent to $\ker \psi \subseteq J^r(W)$. We now show that $\ker \psi \subseteq J^r(W)$ if and only if the condition (1) is satisfied.

Suppose $\ker \psi \subseteq J^r(W)$. Let $s = s_n \dots s_1$ be a product of admissible local sections about $x \in O_G$ with $s_i \in \Gamma^r(W)$ and $x \in \mathcal{D}_s$ such that $s(x) = 1_x$. Then $[s]_x \in J^r(G, W)$ and $\psi([s]_x) = s(x) = 1_x$. So $[s]_x \in \ker \psi$, so that $[s]_x \in J^r(W)$. So there is a neighbourhood U of x such that the restriction $s|U \in \Gamma^r(W)$.

Suppose the condition (1) is satisfied. Let $[s]_x \in \ker\psi$. Since $[s]_x \in J^r(G, W)$, then $[s]_x = [s_n]_{x_n} \dots [s_1]_{x_1}$ where $s = s_n \dots s_1$ and $[s_i]_{x_i} \in J^r(W)$, $x_{i+1} = \beta s_i x_i$, $i = 1, \dots, n$, and $x_1 = x \in \mathcal{D}(s)$. Since $s(x) = 1_x$, then by (1), $[s]_x \in J^r(W)$. □

In effect, Proposition 1 states that the non-extendability of (G, W) arises from the *holonomically non trivial* elements of $J^r(G, W)$. Intuitively, such an element h is an iteration of local procedures (i.e., of elements of $J^r(W)$) such that h returns to the starting point (i.e., $\alpha h = \beta h$) but h does not return to the starting value (i.e., $\psi h \neq 1$).

The following gives a circumstance in which this extendability condition is easily seen to apply.

Corollary 3.11 (Corollary 4.6 in [5]). *Let Q be a Lie groupoid and let $p : M \to Q$ be a morphism of groupoids such that $p : O_M \to O_Q$ is the identity. Let W be an open subset of Q such that*

(a) $O_Q \subseteq W$;
(b) $W = W^{-1}$;
(c) *W generates Q*;
(d) (α_W, β_W, W) *is smoothly locally sectionable*;

and suppose that $\tilde{\imath} : W \to M$ is given such that $p\tilde{\imath} = i : W \to Q$ is the inclusion and $W' = \tilde{\imath}(W)$ generates M.

Then M admits a unique structure of Lie groupoid such that W' is an open subset and $p : M \to Q$ is a morphism of Lie groupoids mapping W' diffeomorphically to W.

Proof. It is easy to check that (M, W') is a locally Lie groupoid. We prove that condition (1) in Proposition 1 is satisfied (with (G, W) replaced by (M, W')).

Suppose given the data of (1). Clearly, $ps = ps_n \dots ps_1$, and so ps is smooth, since G is a Lie groupoid. Since $s(x) = 1_x$, there is a restriction s' of s to a neighbourhood of x such that $\mathrm{Im}(ps) \subseteq W$. Since p maps W' diffeomorphically to W, then s' is smooth and has image contained in W. So (1) holds, and by Proposition 1, the topology on W' is extendable to make M a Lie groupoid. □

Remark 3.12. It may seem unnecessary to construct the holonomy groupoid in order to verify extendability under condition (1) of Proposition 1. However the construction of the smooth structure on M in the last corollary, and the proof that this yields a Lie groupoid, would have to follow more or less the steps given in Aof and Brown [1] as sketched above. Thus it is more sensible to rely on the general result. As Corollary 3.11 shows, the utility of (1) is that it is a checkable condition, both positively or negatively, and so gives clear proofs of the non-existence or existence of non-trivial holonomy.

Putting everything together gives immediately our main theorem on monodromy.

Theorem 3.13 (Strong monodromy principle). *Let s be a strictly regular Lie local subgroupoid of a Lie groupoid Q, and let $\mathcal{U}_s = \{(U_i, H_i) \colon i \in I\}$ be a strictly regular atlas* *for s. Let $W(\mathcal{U}_s) = \bigcup_{i\in I} H_i$. Then there is a Lie groupoid $M = Mon(s, \mathcal{U}_s)$ and morphism $p : M \to glob(s)$ which is the identity on objects with the following properties*:

(a) *The injections $H_i \to glob(s)$ lift to injections $\eta_i : H_i \to M$ such that*

$$W' = \bigcup_{i\in I} \eta_i(H_i)$$

is an open submanifold of M;

(b) *W' generates M*;

(c) *If K is a Lie groupoid and $f = \{f_i : H_i \to K,\ i \in I\}$ is a smooth local morphism, then there is a unique smooth morphism $M(f) : M \to K$ extending the f_i, $i \in I$.*

Proof. Starting with s we form the locally Lie groupoid $(glob(s), W(\mathcal{U}_s)$ and then its holonomy groupoid $Hol(glob(s), W(\mathcal{U}_s))$. Regarding $W(\mathcal{U}_s)$ as contained in Hol we can form the monodromy groupoid $M = M(W(\mathcal{U}_s))$ with its projection to Hol. By Corollary 3.11 (with $Q = Hol$) M obtains the structure of Lie groupoid.

Conditions (a) and (b) are immediate from this construction of the monodromy groupoid.

In (c), the existence of $M(f)$ follows from the weak monodromy principle. To prove that $M(f)$ is smooth it is enough, by local sectionability, to prove it is smooth at the identities of M. This follows since $p : M \to G$ maps $\tilde{\imath}(W)$ diffeomorphically to W. □

Remark 3.14. We have now formed from a strictly regular Lie local subgroupoid s of the Lie groupoid G a smooth morphism of Lie groupoids

$$\xi : Mon(s, \mathcal{U}_s) \to Hol(s, \mathcal{U}_s)$$

which is the identity on objects so that the latter holonomy groupoid is a quotient of the monodromy groupoid. It also follows from [5, Proposition 2.3] that this morphism is a covering map on each of the stars of these groupoids.

Extra conditions are needed to ensure that ξ is a universal covering map on stars — see [5, Theorem 4.2]. This requires further investigation, for example we may need to shrink W to satisfy the required condition.

This also illustrates that Pradines' theorems in [16] are stated in terms of germs. Again, the elaboration of this needs further work.

Remark 3.15. The above results also include the notion of a Lie local equivalence relation, and a strong monodromy principle for these. We note also that the Lie groupoids we obtain are not étale groupoids. This is one of the distinctions between the direction of this work and that of Kock and Moerdijk [13,14]. It would be interesting to investigate the relation further, particularly with regard to the monodromy principle.

A further point is that a local equivalence relation determines a topos of sheaves of a particular type known as an étendue [14]. What type of topos is determined by a local subgroupoid?

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