# The twisted Eilenberg-Zilber Theorem

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The purpose of this paper is to give a simpler proof of a theorem of E.H. Brown [Bro59], that if $F \to E \to B$ is a fibre space, then there is a differential on the graded group $X = C(B) \otimes_\Lambda C(F)$ such that $X$ with this differential is chain equivalent to to $C(E)$ (where $C(E)$ denotes the normalised singular chains of $E$ over a ring $\Lambda$).

We work in the context of (semi-simplicial) twisted cartesian products (thus we assume as do the proofs of the theorem given in [Gug60, Shi62, Szc61] the results of [BGM59] on the relation between fibre spaces and twisted cartesian products). In fact we prove a general result on filtered chain complexes; this result applies to give proofs not only of Brown's theorem but also of a theorem of G. Hirsch, [Hir53]. Our proof is suggested by the formulae (1) of [Shi62, Ch. II, §1].

Let $(X, d)$, $(Y, d)$ be chain complexes over a ring $\Lambda$. Let

$$(Y, d) \xrightarrow{\nabla} (X, d) \xrightarrow{f} (Y, d)$$

be chain maps and let $\Phi : X \to X$ be a chain homotopy such that

$$\text{(1.1)}\quad f\nabla = 1; \qquad \text{(1.2)}\quad \nabla f = 1 + d\Phi + \Phi d; \qquad \text{(1.3)}\quad f\Phi = 0;$$
$$\text{(1.4)}\quad \Phi\nabla = 0; \qquad \text{(1.5)}\quad \Phi^2 = 0; \qquad \text{(1.6)}\quad \Phi d\Phi = -\Phi.$$

Let $X, Y$ have filtrations

$$0 = F^{-1}X \subseteq F^0X \subseteq \cdots \subseteq F^pX \subseteq F^{p+1}X \subseteq \cdots \tag{1}$$
$$0 = F^{-1}Y \subseteq F^0Y \subseteq \cdots \subseteq F^pY \subseteq F^{p+1}Y \subseteq \cdots \tag{2}$$

and let $\nabla, f, \Phi$ all preserve these filtrations.

**Example 1** Let $B, F$ be (semi-simplicial) complexes, let $(X, d) = C(B \times F)$, the normalised chains of $B \times F$, let $(Y, d) = C(B) \otimes_\Lambda C(F)$, and let $\nabla, f, \Phi$ be the natural maps of the Eilenberg-Zilber theorem as constructed explicitly in [EML53]. The relations (1.1)-(1.4) are proved in [EML53] while (1.5), (1.6) are easily proved (cf. [Shi62, p.114]). The filtrations on $X, Y$ are induced by the filtration of $B$ by its skeletons. The fact that $\nabla, f, \Phi$ preserve filtrations is a consequence of naturality of these maps (cf. [Moo56, Ch. 5, p.13]). □

We now wish to compare $C(B \times F)$ with $C(B \times_\tau F)$ where $B \times_\tau F$ coincides with $B \times F$ as a complex except that $\partial_0$ in $B \times_\tau F$ is given by

$$\partial_0(b, x) = (\partial_0 b, \tau(b, x)), \qquad b \in B_p, x \in F_p.$$

Then the filtered groups of $C(B \times F)$ and $C(B \times_\tau F)$ coincide but the latter has a differential $d^\tau$. If $\tau$ satisfies the normalisation condition

$$\tau(s_0 b', x) = \partial_0 x, \qquad b' \in B_{p-1}, x \in F_p$$

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then $d^\tau - d$ lowers filtration in $X$.

Going back to the general case, we suppose $X$ has another differential $d^\tau$ with the property

$$(d^\tau - d)F^pX \subseteq F^{p-1}X, \qquad p \geqslant 0. \tag{3}$$

Our object is to construct a new differential $d^\tau = d_Y^\tau$ on $Y$ and a chain equivalence $(Y, d^\tau) \to (X, d^\tau)$.

We first note that

$$\begin{aligned} \Phi(1+d^\tau\Phi)^r &= (\Phi + \Phi d^\tau\Phi)(1+d^\tau\Phi)^{r-1} \\ &= \Phi(d^\tau - d)\Phi(1+d^\tau\Phi)^{r-1} \qquad \text{by (1.6)} \\ &= \Phi(d^\tau - d)\Phi \ldots \Phi(d^\tau - d)\Phi, \end{aligned}$$

so that (3) implies

$$\Phi(1+d^\tau\Phi)^r F^pX \subseteq F^{p-r}X. \tag{4}$$

But $F^{-1}X = 0$; therefore the map

$$\Phi^\tau = \sum_{r=0}^{\infty} \Phi(1+d^\tau\Phi)^r$$

is well defined. Also from (1.3), (1.4), (1.5) we derive immediately

$$(5.1) \qquad f\Phi^\tau = 0, \qquad (5.2) \quad \Phi^\tau\nabla = 0, \qquad (5.3) \qquad (\Phi^\tau)^2 = 0.$$

Next we must prove relations similar to (1.6). In fact we have

$$(6.1) \qquad \Phi^\tau d^\tau \phi^\tau = -\Phi^\tau, \qquad (6.2) \quad \Phi^\tau d^\tau \Phi = -\Phi.$$

These relations are proved by operating on the power series for $\Phi^\tau$; the operations are justified by (4) and the fact that $F^{-1}X = 0$. For example, we prove (6.1):

$$\begin{aligned} \Phi^\tau d^\tau \phi^\tau &= \sum_{r,s=0}^{\infty} \Phi(1+d^\tau\Phi)^r d^\tau \Phi(1+d^\tau\Phi)^s \\ &= \sum_{r,s=0}^{\infty} \left(\Phi(1+d^\tau\Phi)^{r+s+1} - \Phi(1+d^\tau\Phi)^{r+s}\right) \\ &= \sum_{r=0}^{\infty} -\Phi(1+d^\tau\Phi)^r \\ &= -\Phi^\tau. \end{aligned}$$

By (5.3) and (6.1) the deformation operator

$$D^\tau = 1 + d^\tau\Phi^\tau + \Phi d^\tau : X \to X$$

is idempotent. We set

$$\begin{aligned} \nabla^\tau &= D^\tau\nabla : Y \to X, \\ f^\tau &= fD^\tau : X \to Y, \\ d_Y^\tau &= f^\tau d^\tau \nabla^\tau : Y \to Y, \end{aligned}$$

and prove easily from (5.1), (5.2) and (6.1) respectively

$$(7.1) \qquad \nabla^\tau = (1 + \Phi^\tau d^\tau)\nabla, \qquad (7.2) \qquad f^\tau = f(1 + d^\tau\Phi^\tau,$$

$$(7.3) \qquad d_Y^\tau = f(d^\tau + d^\tau \Phi^\tau d^\tau)\nabla = f^\tau d^\tau \nabla = f d^\tau \nabla^\tau, \qquad \text{cf. [Shi62, Ch. II §1.]}$$

The relations given so far are sufficient to prove in turn

$$(8.1) \qquad f^\tau \nabla^\tau = 1, \qquad (8.2) \qquad \nabla^\tau f^\tau = 1 + d^\tau \Phi^\tau + \Phi d^\tau,$$

$$(8.3) \qquad d_y^\tau f^\tau = f^\tau d^\tau, \qquad (8.4) \qquad \nabla^\tau d_Y^\tau = d^\tau \nabla^\tau, \qquad (8.5) \qquad (d_Y^\tau)^2 = 0.$$

Thus $\nabla^\tau : (Y, d_Y^\tau) \to (X, d^\tau)$ is a chain equivalence of chain complexes.

In particular, the construction of $d_Y^\tau$ and $\nabla^\tau$ applies to Example 1.

As another example, we obtain a generalised form of a theorem of G. Hirsch, [Hir53]:

**Example 2** Let $X = C(B) \otimes_\Lambda C(F)$, let $d^\tau$ be the differential on $X$ constructed as above from the twisted cartesian product $B \times_\tau F$. Let $Y = C(B) \otimes_\Lambda H(F)$ and let the homology $H(F)$ be such that the sequence

$$0 \to B(F) \to Z(F) \to H(F) \to 0$$

where $B(F), Z(F)$ denote the boundaries and cycles of $C(F)$, splits over $\Lambda$. This splitting may be used to define chain maps $\nabla' : H(F) \to C(F)$, $f' : C(F) \to H(F)$ and a chain homotopy $\Phi' : C(F) \to C(F)$ satisfying relations of the form (1.1) –(1.6) ($H(F)$ has of course the trivial differential). Let

$$\nabla = 1 \otimes \nabla', \qquad f = 1 \otimes f', \qquad \Phi = 1 \otimes \Phi'.$$

Then $\nabla, f, \Phi$ satisfy the relations (1.1) – (1.6). But on $X$, $(d^\tau - d)F^p X \subseteq F^{p-1}X, p0$. So there is a differential $d^\tau$ on $Y = C(B) \otimes H(F)$ and a chain equivalence $(d_Y^\tau) \to (X, d_X^\tau)$. Composing this with the chain equivalence for Example 1 we obtain a chain equivalence

$$(C(B) \otimes_\Lambda H(F), d^\tau) \to (C(B \times_\tau F), d^\tau). \qquad \square$$

# A Appendix[1]

As explained earlier, the above was written in 1964 for the conference in Sicily, and published in 1967. The result was found by trying to understand the paper [Shi62], and had been stimulated by earlier discussions and correspondence with M.G. Barratt. Later V.K. A. M. Gugenheim went through the same process and published the same argument in [Gug72]. This area has developed extensively, and is now called *Homological Perturbation Theory*, see for example [LS87, BL91], and many others. In conjunction with the theory of twisting cochains, it has proved an important theoretical and computational tool.

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